$\omega, \nu))^T$ is the kernel of the operators $H^{\pm}(\varphi)(z)$. Using both the expressions of the matrix of fundamental solutions $\Gamma(z, y; \omega, \nu)$ [5] and (3.3), we can show that K_I and K_{II} are singular kernels of class $G^*(2, \alpha, \alpha)$ (see [1], ch. IV, §3, p. 150), and hence (6.1), (6.2) are singular integral equations.

In the same way as in the general case presented in [1], we can prove that $L^{\pm}(\psi)$ and $H^{\pm}(\varphi)$ are operators of normal type, and therefore they assume a regularization in $L_p^{(8)}(S)$ — the linear space of functions having the form $V=(v_1, v_2, \ldots, v_8)$, summed over S in the p-th power, the norm being

$$\|V\| = \left\{\int_S \left(\sum_1^8 |v_i|^2\right)^{p/2} dS_y\right\}^{1/p}, \quad p > 1. \tag{6.5}$$

It follows that the Fredholm theorems are valid for the singular integral equations (6.1) and (6.2) in the space $L_p^{(8)}(S)$. On the other hand, taking into account that $S \in L_1(\alpha)$, and that $L^{\pm}(\psi)$, $H^{\pm}(\varphi)$ are operators of normal type, if $F_{i,s}$, $G_{i,s} \in C^{0,\beta}(S)$, then any solution of (6.1) and (6.2), of class $L_p^{(8)}(S)$ belongs also to $C^{0,\beta}(S)$ class.

Therefore, the existence of the solutions of the problems (II. I. I)$^{\pm}$ and (I.II.II)$^{\pm}$ is reduced to the study of the singular integral equations (6.1) and (6.2). This study can be made in the same manner as in the case of the classical thermoelasticity [4].

REFERENCES

1. Kupradze, V. D., Gegelia, T. G., Baseleisvili, M. O., Burculadze T. V. — *Trehmernye zadaci matematiceskoi teorii uprugostii termouprugosti*, Izd. „Nauka", Moskva (1976)
2. Nowacki, W. — *Couple Stresses in the Theory of Thermoelasticity*. III, Bull. Acad. Polon. Sci., Sér. Sci. Techn., 14, 8 (1966)
3. Ieşan, D. — *Unele probleme fundamentale ale teoriei liniare a mediilor elastice de tip Cosserat, Probleme actuale în mecanica solidelor*, vol. I. Sub redacţia lui P. P. Teodorescu. Edit. Acad. R.S.R., Bucureşti (1975)
4. Kupradze, V. D., Burculadze, T. V. — *Granicinye zadaci termouprugosti*, Diff. Uravn., 5, 1 (1969)
5. Crăciun, I. — *Potential Method in Micropolar Thermoelasticity*, Rev. Roum. Sci. Techn. Sér. Mec. Appl., 22, 1 (1977)
6. Crăciun, I. — *Radiation Conditions and Uniqueness Theorems in Micropolar Thermoelasticity*, Rev. Roum. Sci. Techn., Sée. Méc. Appl., 23, 2 (1978)
7. Crăciun, I. — *Green Formula for an Infinite Domain with a Cavity and Integral Representation of a Regular Solution in Micropolar Thermoelasticity*, Bul. Inst. Polit. Iaşi, XXIV (XXVIII), s. I a, fasc. 1—2, fasc. 1—2 (1978)
8. Agniasvili, M. R. — *Nekotorye kraevye zadaci termomomentnoi teorii uprugosti*, Raboty po zadaciam mehaniki splosnyh sred, Inct. Prikl. Mat. Tbilisi, G.U.T. (1976).

Received 10.XI.1981

Departecment of Mathematics
Politehnic Institute of Iaşi
R. S. România

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXIX, s. I a Matematică, f. 2, 1983

ON THE DEFORMATION OF ALMOST CYLINDRICAL BEAMS

BY
S. CHIRIŢĂ

1. **Introduction.** The problem of deformation of almost cylindrical beams in the theory of elasticity was initiated by Panov [1], Rukhadze [2] and Borş [3] and it was developed in various papers (see, e.g. [4]—[8]).

In the present paper we give a method, more simple than those known, for solving the problem of deformation of an almost cylindrical beam subjected to body forces and lateral boundary tractions which are polynomials in the axial coordinate. The ends of the beam are subjected to appropriate stress resultants. We consider the case of an almost cylindrical beam made of an isotropic and inhomogeneous elastic material. The Lamé coefficients are assumed to be independent of the axial coordinate.

Our method presents a new treatment of the problem even for inhomogeneous elastic bodies. Moreover, the present paper points out the contribution due to the form of the beam in the solution. In fact, the method presents the solution in terms of the solutions of two problems, $\mathfrak{P}_0$ and $\mathfrak{P}_1$. The solution of the problem $\mathfrak{P}_0$ represents the solution of the initial formulated problem as though the beam would be a cylindrical beam. The solution of the problem $\mathfrak{P}_1$ is the solution of an Almansi type problem for a cylindrical beam. This last solution is due to the form of the beam. Both solutions can be determined by using the method described in [9], [10].

The proposed method is then used in order to study the Saint-Venant's problem for a frustrum of cone made of an isotropic and homogeneous elastic material.

2. **Statement of the problem.** Throughout this paper a rectangular Cartesian coordinate system Ox_k $(k=1, 2, 3)$ is used. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts — unless otherwise specifed — to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate.

We consider *an almost cylindrical beam* limited by two planes $x_3=0$ and $x_3=l$, and a lateral surface F, given by

$$f(x_1(1-\varepsilon x_3),\ x_2(1-\varepsilon x_3))=0, \tag{1}$$

where ε is a small parameter, the square and higher powers of which can be neglected. Let B be the interior of the beam and Σ the generic cross-section of the beam. We assume that Σ is a C^1-smooth domain [11]. We denote by Σ_{x_3} the cross-section located in the plane $x_3=$const. The x_3-axis of our coordinate system will be directed along the line passing through the centroids of the cross-sections of the beam.

In what follows we consider the linear theory of elasticity for inhomogeneous and isotropic materials. Let u_i denote the components of the displacement vector field. The components of the strain tensor are given by

(2) $$2e_{ij}=u_{i,j}+u_{j,i}, \text{ in } \overline{B}.$$

The constitutive equations for an isotropic and inhomogeneous elastic medium are

(3) $$t_{ij}=\lambda e_{ss}\delta_{ij}+2\mu e_{ij}, \text{ in } \overline{B},$$

where t_{ij} are the components of the stress tensor and λ and μ are the Lamé moduli. We assume that $\lambda=\lambda(x_1, x_2)$, $\mu=\mu(x_1, x_2)$, in $\overline{B}$. The functions λ and μ are assumed to belong to C^∞.

The relations between t_{ij} and e_{mn} must be reversible, so that we can write

$e_{ij}=[(1+\nu)/E]t_{ij}-(\nu/E)t_{ss}\delta_{ij}$, in $\overline{B}$, where $(E=\mu(3\lambda+2\mu)/(\lambda+\mu)$, $\nu=\lambda/[2(\lambda+\mu)]$.

The equations of equilibrium are

(4) $$t_{ji,j}+f_i=0, \text{ in } B,$$

where f_i are the components of body force vector.

On the lateral surface of the almost cylindrical beam we have the following conditions

(5) $$t_{ji}n_j=t_i, \text{ on } F,$$

where n_i are the direction cosines of the exterior normal to F.

In this paper, we assume that the body forces and the tractions applied on the lateral surface are polynomials of the r-degree in the coordinate x_3, namely,

(6) $$f_i=\sum_{k=0}^{r}F_{ik}(x_1, x_2)x_3^k, \quad t_i=\sum_{k=0}^{r}T_{ik}(x_1, x_2)x_3^k,$$

where F_{ik} and T_{ik} are prescribed functions which are supposed to belong to C^∞.

Let the loading applied on the end located at $x_3=0$ be statically equivalent to a force $R(R_i)$ and a moment $M(M_i)$. Thus, for $x_3=0$, we have the following conditions

(7) $$\int_{\Sigma_0}t_{3i}dA=-R_i, \quad \int_{\Sigma_0}x_\alpha t_{3\beta}dA=\varepsilon_{3\alpha\beta}M_\beta, \quad \int_{\Sigma_0}\varepsilon_{3\alpha\beta}x_\alpha t_{3\beta}dA=-M_3,$$

where ε_{ijk} denotes the alternating symbol.

The load of the beam over its end located in the plane $x_3=l$ is distributed in a such way that the equilibrium conditions of a rigid body are fulfilled. Thus, the loading applied on the end located in $x_3=l$ is given by

(8) $$\int_{\Sigma_l}t_{3i}dA=-R_i-\int_B f_i dV-\int_F t_i da,$$
$$\int_{\Sigma_l}\varepsilon_{ijk}x_j t_{3k}dA=-M_i-\int_B\varepsilon_{ijk}x_j f_k dV-\int_F\varepsilon_{ijk}x_j t_k da.$$

Our problem consists in the solving of equations (2), (3) and (4), with the conditions (5), (7) and (8), assuming that f_i and t_i are given by (6).

3. **The method of solving.** Let us denote by $\mathcal{B}$ the interior of the cylindrical beam limited by the planes $x_3=0$ and $x_3=l$, and the lateral cylindrical surface $\mathcal{F}$, given by

(9) $$f(x_1, x_2)=0.$$

Let us denote by Ω the generic cross-section of the cylindrical beam $\mathcal{B}$, and let L be the boundary of Ω. Let Ω_{x_3} be the cross-section Ω located in the plane $x_3=$ const. Obviously, Ω_0 coincide with Σ_0.

Let (y_1, y_2, y_3) be a generical point of the cylindrical surface $\mathcal{F}$. It is easy to see that when (y_1, y_2, y_3) describes the cylindrical surface $\mathcal{F}$ given by (9), the point (x_1, x_2, x_3) given by

(10) $$x_1=y_1(1+\varepsilon y_3), \quad x_2=y_2(1+\varepsilon y_3), \quad x_3=y_3,$$

describes the surface F.

Without loss of generality, we can consider a parametrical representation of the surface F in the form

(11) $$y_1=\varphi_1(\theta_1), \quad y_2=\varphi_2(\theta_1), \quad y_3=\theta_2, \quad 0\leqslant\theta_2\leqslant l, \quad \theta_1\in L.$$

Then the surface F is given by

(12) $$x_1=(1+\varepsilon\theta_2)\varphi_1(\theta_1), \quad x_2=(1+\varepsilon\theta_2)\varphi_2(\theta_1), \quad x_3=\theta_2, \quad 0\leqslant\theta_2\leqslant l, \quad \theta_1\in L.$$

By using the relations (11) and (12), we can establish the following formulae

(13) $$n_\alpha=N_\alpha, \quad n_3=-\varepsilon x_\rho N_\rho,$$

where N_α are the direction cosines of the outward normal vector to the cylindrical surface $\mathcal{F}$.

According to the hypotheses of Section 2, concerning the form of the beam (1), we seek the solution of our problem in the form

(14) $$u_i(x_1, x_2, x_3)=u_i^{(0)}(x_1, x_2, x_3)+\varepsilon v_i(x_1, x_2, x_3),$$

where $u_i^{(0)}$ and v_i are unknown functions.

If we substitute the relation (14) into (2), we deduce

(15) $$e_{ij}=e_{ij}^{(0)}+\varepsilon\varkappa_{ij},$$

where

(16) $$2e_{ij}^{(0)}=u_{i,j}^{(0)}+u_{j,i}^{(0)},$$ (17) $$2\varkappa_{ij}=v_{i,j}+v_{j,i}.$$

From the relations (3) and (15), it follows

(18) $$t_{ij}=t_{ij}^{(0)}+\varepsilon\tau_{ij},$$

where

(19) $$t_{ij}^{(0)}=\lambda e_{ss}^{(0)}\delta_{ij}+2\mu e_{ij}^{(0)},$$ (20) $$\tau_{ij}=\lambda\varkappa_{ss}\delta_{ij}+2\mu\varkappa_{ij}, \text{ in } \mathcal{B}.$$

The equilibrium equations (4) and the relation (18) lead to the following equations

(21) $$t_{ji,j}^{(0)}+f_i=0, \text{ in } \mathcal{B},$$ (22) $$\tau_{ji,j}=0, \text{ in } \mathcal{B}.$$

Let (y_1, y_2, y_3) be a point of the lateral surface $\mathcal{F}$. By using the relation (10), we obtain, by Taylor's expansion,

(23) $$\begin{aligned}&t_{ij}^{(0)}(x_1, x_2, x_3)=t_{ij}^{(0)}(y_1(1+\varepsilon y_3), y_2(1+\varepsilon y_3), y_3)=t_{ij}^{(0)}(y_1, y_2, y_3)+\\&+\varepsilon y_3 y_\rho t_{ij,\rho}^{(0)}(y_1, y_2, y_3), \quad \tau_{ij}(x_1, x_2, x_3)=\tau_{ij}(y_1(1+\varepsilon y_3), y_2(1+\\&+\varepsilon y_3), y_3)=\tau_{ij}(y_1, y_2, y_3)+\varepsilon y_3 y_\rho\tau_{ij,\rho}(y_1, y_2, y_3),\\&t_i(x_1, x_2, x_3)=t_i(y_1(1+\varepsilon y_3), y_2(1+\varepsilon y_3), y_3)=t_i(y_1, y_2, y_3)+\varepsilon y_3 y_\rho t_{i,\rho}(y_1, y_2, y_3).\end{aligned}$$

Taking into account the relations (13) and (23), from (18) and (5), we get the following boundary conditions

(24) $$t_{\alpha i}^{(0)}N_\alpha=t_i, \text{ on } \mathcal{F},$$ (25) $$\tau_{\alpha i}N_\alpha=p_i, \text{ on } \mathcal{F},$$

where

$$p_i = x_\rho N_\rho t^{(0)}_{3i} + x_3 x_\rho (t_{i,\rho} - t^{(0)}_{\alpha i,\rho} N_\alpha), \text{ on } \mathcal{F}. \tag{26}$$

From the conditions (7) and by using the relation (18), we obtain, for $x_3 = 0$,

$$\int_{\Omega_0} t^{(0)}_{3i} d\Omega = -R_i, \quad \int_{\Omega_0} x_\alpha t^{(0)}_{33} d\Omega = \varepsilon_{3\alpha\beta} M_\beta, \quad \int_{\Omega_0} \varepsilon_{3\alpha\beta} x_\alpha t^{(0)}_{3\beta} d\Omega = -M_3, \tag{27}$$

$$\int_{\Omega_0} \tau_{3i} d\Omega = 0, \quad \int_{\Omega_0} x_\alpha \tau_{33} d\Omega = 0, \quad \int_{\Omega_0} \varepsilon_{3\alpha\beta} x_\alpha \tau_{3\beta} d\Omega = 0. \tag{28}$$

Now, we make the transformation $z_\alpha = x_\alpha(1 - \varepsilon x_3)$, $z_3 = x_3$, into the relations (8), so that, in the assumed approximation, we find

$$\begin{gathered} \int_{\Omega_l} t^{(0)}_{3i} d\Omega = -R_i - \int_{\mathcal{B}} f_i d\mathcal{V} - \int_{\mathcal{F}} t_i d\omega, \\ \int_{\Omega_l} \varepsilon_{ijl} z_j t^{(0)}_{3k} d\Omega = -M_i - \int_{\mathcal{B}} \varepsilon_{ijk} z_j f_k d\mathcal{V} - \int_{\mathcal{F}} \varepsilon_{ijl} z_j t_l d\omega, \end{gathered} \tag{29}$$

and

$$\int_{\Omega_l} (\tau_{3i} + 2l t^{(0)}_{3i} + l z_\rho t^{(0)}_{3i,\rho}) d\Omega = -\int_{\mathcal{B}} (2z_3 f_i + z_3 z_\rho f_{i,\rho}) d\mathcal{V} - \int_{\mathcal{F}} (z_3 t_i + z_3 z_\rho t_{i,\rho}) d\omega, \tag{30}$$

$$\int_{\Omega_l} \varepsilon_{ijk} (z_j \tau_{3k} + z_j t^{(0)}_{3k,\rho} z_\rho z_3 + 2 z_3 z_j t^{(0)}_{3k} + \delta_{j\rho} z_\rho z_3 t^{(0)}_{3k}) d\Omega = -\int_{\mathcal{B}} \varepsilon_{ijk} (2 z_3 z_j f_k + \delta_{j\rho} z_\rho z_3 f_k +$$

$$+ z_3 z_j z_\rho f_{k,\rho}) d\mathcal{V} - \int_{\mathcal{F}} \varepsilon_{ijk} (z_j z_3 t_k + \delta_{j\rho} z_\rho z_3 t_k + z_3 z_j z_\rho t_{k,\rho}) d\omega.$$

If we substitute f_i from (21) into (30) and we use the notation (26), we can write the relation (30) in the following form

$$\int_{\Omega_l} \tau_{3i} d\Omega = -\int_{\mathcal{F}} p_i d\omega, \quad \int_{\Omega_l} \varepsilon_{ijk} z_j \tau_{3k} \, d\Omega = -\int_{\mathcal{F}} \varepsilon_{ijk} z_j p_k d\omega. \tag{31}$$

Let us denote by $\mathfrak{P}_0$ the problem for the cylindrical beam $\mathfrak{B}$, defined by the equations (16), (19) and (21) and by the conditions (24) and (27). The equilibrium conditions of a rigid body for the cylindrical beam $\mathfrak{B}$ are identically with the conditions (29). It is easy to see that the problem $\mathfrak{P}_0$ is similar to that initially stated into Section 2, for the almost cylindrical beam. The solution of the problem $\mathfrak{P}_0$ for the cylindrical beam $\mathfrak{B}$, can be determined by using the method described in [9, 10]. As it is known, the $u_i^{(0)}(x_1, x_2, x_3)$ and $t^{(0)}_{ij}(x_1, x_2, x_3)$ are polynomials of x_3, whose coefficients are functions of x_1 and x_2.

Let us denote by $\mathfrak{P}_1$ the problem for cylindrical beam $\mathfrak{B}$, defined by the equations (17), (20) and (22) and by the conditions (25) and (28). It is worthy to remark that the equilibrium conditions of a rigid body for the cylindrical beam $\mathfrak{B}$ are satisfied on the basis of the relations (31). In view of the foregoing discussions concerning the problem $\mathfrak{P}_0$ and by taking into account the notation (26), it follows that the problem $\mathfrak{P}_1$ is an Almansi type problem. Therefore, the solution of the problem $\mathfrak{P}_1$ can be determined by using a method similar to that described in [9, 10].

4. Saint-Venant's problem for a frustrum of cone. In this section, we will use our method in order to study the Saint-Venant problem for a frustrum of cone made of an isotropic and homogeneous elastic material.

Throughout this section B denotes the interior of a frustrum of the right circular cone, bounded by plane ends perpendicular to the cone axis. The axis Ox_3 of the coordinate system is taken along the axis of the cone. The plane ends, which are circles of radii $r_1, r_2 (r_1 > r_2)$ are in the planes $x_3 = 0$ and $x_3 = l$, respectively. If θ is the angle between the axis of the cone and the generators, then the equation of the lateral surface F of the beam is

$$x_1^2 + x_2^2 = r_1^2 (1 - x_3 \operatorname{tg} \theta / r_1)^2, \quad 0 \leq x_3 \leq l. \tag{32}$$

It is clear that if the angle θ is a small parameter, then one can take $\varepsilon = -(\operatorname{tg} \theta)/r_1 = -(r_1 - r_2)/(l r_1)$. It is easy to verify that the equation (32) may be written in the form

$$(x_1(1 - \varepsilon x_3))^2 + (x_2(1 - \varepsilon x_3))^2 = r_1^2, \quad 0 \leq x_3 \leq l. \tag{33}$$

The cylindrical surface $\mathcal{F}$ is given by $x_1^2 + x_2^2 = r_1^2$, $0 \leq x_3 \leq l$.

In what follows we suppose that the almost cylindrical beam is made of an isotropic and homogeneous material, so that the elastic coefficients λ and μ are constants. Further, we assume that the body forces and the lateral boundary tractions are absent. Therefore, we have

$$f_i(x_1, x_2, x_3) = t_i(x_1, x_2, x_3) = 0.$$

Let the loading applied on $x_3 = 0$ be statically equivalent to a force $R(R_i)$ and a moment $M(M_i)$. In this case, the problem $\mathfrak{P}_0$ is the Saint-Venant problem for the circular cylindrical beam $\mathfrak{B}$ given by (33).

The solution of the problem $\mathfrak{P}_0$ is

$$\begin{aligned} u_1^{(0)} &= -\frac{1}{2} a_1 x_3^2 - \frac{1}{6} b_1 x_3^3 - a_4 x_2 x_3 - \frac{1}{2} \nu (a_1 + b_1 x_3)(x_1^2 - x_2^2) - \\ &\qquad - \nu (a_2 + b_2 x_3) x_1 x_2 - \nu a_3 x_1, \\ u_2^{(0)} &= -\frac{1}{2} a_2 x_3^2 - \frac{1}{6} b_2 x_3^3 + a_4 x_1 x_3 - \nu (a_1 + b_1 x_3) x_2 x_1 - \frac{1}{2} \nu (a_2 + b_2 x_3) \\ &\qquad (x_2^2 - x_1^2) - \nu a_3 x_2, \\ u_3^{(0)} &= (a_1 x_1 + a_2 x_2 + a_3) x_3 + \frac{1}{2} (b_1 x_1 + b_2 x_2) \left[x_3^2 + \left(\nu + \frac{3}{2} \right) r_1^2 - \frac{1}{2} (x_1^2 + x_2^2) \right], \end{aligned} \tag{34}$$

where

$$a_\alpha = \frac{4}{\pi r_1^4 E} \varepsilon_{3\alpha\beta} M_\beta, \quad a_3 = -\frac{1}{\pi r_1^2 E} R_3, \quad a_4 = -\frac{2}{\pi r_1^4 \mu} M_3, \quad b_\alpha = -\frac{4}{\pi r_1^4 E} R_\alpha.$$

Further, we have

$$t^{(0)}_{\alpha\beta} = 0, \quad t^{(0)}_{33} = E[a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2) x_3]. \tag{35}$$

$$t_{31}^{(0)}=\mu\left\{b_1\left[\left(\frac{\nu}{2}+\frac{3}{4}\right)(r_1^2-x_1^2)+\left(\frac{\nu}{2}-\frac{1}{4}\right)x_2^2\right]-b_2\left(\nu+\frac{1}{2}\right)x_1x_2-a_4x_2\right\},$$

$$t_{32}^{(0)}=\mu\left\{-b_1\left(\nu+\frac{1}{2}\right)x_1x_2+b_2\left[\left(\frac{\nu}{2}+\frac{3}{4}\right)\right](r_1^2-x_2^2)+\left(\frac{\nu}{2}-\frac{1}{4}\,x_1^2\right]+a_4x_1\right\}.$$

From the relations (26) and (35), we obtain

$$p_\alpha=\mu r_1\left[\left(\nu+\frac{1}{2}\right)(-b_1x_2+b_2x_1)+a_4\right]\varepsilon_{33\alpha}x_3,$$

$$p_3=Er_1(a_1x_1+a_2x_2+a_3)+\mu(3\nu+7/2)r_1(b_1x_1+b_2x_2)x_3.$$

Since p_i are polynomials of x_3 it follows that the problem $\mathfrak{A}_1$ is an Almansi type problem. The solution of the problem $\mathfrak{A}_1$ is obtained by using the method described in [9, 10]. Thus, we get

(36) $$v_1=-1/2c_1x_3^2-1/6d_1x_3^3-1/24e_1x_3^4-1/2d_2x_2x_3^2-1/2\nu(c_1+d_1x_3+$$
$$+(1/2)e_1x_3^2)\,(x_1^2-x_2^2)-\nu(c_2+d_2x_3+(1/2)e_2x_3^2)x_1x_2-\nu d_3x_1x_3+V_1(x_1,x_2),$$
$$v_2=-(1/2)c_2x_3^2-(1/6)d_2x_3^3-(1/24)e_2x_3^4+(1/2)d_4x_1x_3^2-\nu(c_1+d_1x_3+$$
$$+(1/2)e_1x_3^2)x_1x_2-(1/2)\nu(c_2+d_2x_3+(1/2)e_2x_3^2)\,(x_2^2-x_1^2)-\nu d_3x_2x_3+V_2(x_1,x_2),$$
$$v_3=(c_1x_1+c_2x_2)x_3+\frac{1}{2}(d_1x_1+d_2x_2+d_3)x_3^2+\frac{1}{6}(e_1x_1+e_2x_2)x_3^3+$$
$$+W(x_1,x_2)x_3+V_3(x_1,x_2).$$

where

$$W(x_1,x_2)=(b_1x_1+b_2x_2)\left[2(x_1^2+x_2^2)+\frac{7c-6}{2(1-c)}r_1^2\right],$$

$$V_3(x_1,x_2)=(a_1x_1+a_2x_2)\,(x_1^2+x_2^2-r_1^2)+a_3(x_1^2+x_2^2),$$

$$V_\alpha(x_1,x_2)=\frac{1}{2}a_4\varepsilon_{3\beta\alpha}x_\beta(x_1^2+x_2^2)+b_\rho x_\rho x_\alpha U+\varepsilon_{3\alpha\beta}x_\beta\varepsilon_{3\gamma\sigma}b_\gamma x_\sigma V.$$

$$U=\frac{13c-9}{24(1-c)}(x_1^2+x_2^2)+\frac{(7c-6)(3c-2)}{12(1-c)^2}r_1^2,\quad V=\frac{-7c+3}{24(1-c)}(x_1^2+x_2^2)-\frac{c(7c-6)}{12(1-c)^2}r_1^2,$$

$$c=\mu/(\lambda+2\mu),\ e_\alpha=-8b_\alpha,\ d_\alpha=-4a_\alpha,\ d_3=-2a_3,\ d_4=-4a_4,\ c_\alpha=$$
$$=r_1^2b_\alpha\,\frac{23c^2-36c+23}{3(1-c)\,(3-4c)}.$$

Therefore, the solution of Saint-Venant's problem for the frustrum of cone is $u_i=u_i^{(0)}+\varepsilon v_i$, where $u_i^{(0)}$ are given by (34) and v_i are given by (36).

5. Further comments on the solution. We return now to a discussion on the solution (14). We introduce the new variables z_i, so that

(37) $$x_\alpha=z_\alpha(1+\varepsilon z_3),\quad x_3=z_3.$$

The lateral surface F becomes the cylindrical surface $\tilde{F}$, given by $f(z_1,z_2)=0$. Let $\tilde{\Sigma}$ be the generic cross-section of the cylindrical beam $\tilde{B}$, bounded by $\tilde{F}$. It is easy to establish the following formulae

(38) $$\frac{\partial}{\partial x_i}=\frac{\partial}{\partial z_i}+\varepsilon\left[-z_3\frac{\partial}{\partial z_i}+\delta_{i3}\left(z_3\frac{\partial}{\partial z_3}-z_\rho\frac{\partial}{\partial z_\rho}\right)\right],\quad n_\alpha=\tilde{N}_\alpha,\quad n_3=-\varepsilon z_\rho\tilde{N}_\rho,$$

where $\tilde{N}_\alpha$ are the direction cosines of the vector normal to the cylindrica surface $\tilde{F}$.

In view of the hypotheses made in Section 2, if we retain only the terms of order ε, by using (37), we can write the solution (14) in the form

(39) $$u_i(x_1,x_2,x_3)=u_i^{(0)}(z_1,z_2,z_3)+\varepsilon[v_i(z_1,z_2,z_3)+z_3z_\rho u_{i,\rho}^{(0)}(z_1,z_2,z_3)].$$

In the above relation and in the remaining of this paper, the indices after comma denote partial differentiation with respect to the variables z_i.

Similarly, if we use the relation (37), we get

(40) $$f_i(x_1,x_2,x_3)=f_i(z_1,z_2,z_3)+\varepsilon z_3z_\rho f_{i,\rho}(z_1,z_2,z_3),$$
$$t_i(x_1,x_2,x_3)=t_i(z_1,z_2,z_3)+\varepsilon z_3z_\rho t_{i,\rho}(z_1,z_2,z_3),$$

and

(41) $$\lambda(x_1,x_2)=\lambda(z_1,z_2)+\varepsilon z_3z_\rho\lambda_{,\rho}(z_1,z_2),\ \mu(x_1,x_2)=\mu(z_1,z_2)+\varepsilon z_3z_\rho\mu_{,\rho}(z_1,z_2).$$

Taking into account the relations (38) and (39), we find

(42) $$\frac{\partial u_i}{\partial x_j}=u_{i,j}^{(0)}+\varepsilon[v_{i,j}+z_3z_\rho(u_{i,j}^{(0)})_{,\rho}].$$

A substitution of the relations (41) and (42) into the relations (2) and (3) gives

(43) $$t_{ij}(x_1,x_2,x_3)=t_{ij}^{(0)}(z_1,z_2,z_3)+\varepsilon[\tau_{ij}(z_1,z_2,z_3)+z_3z_\rho t_{ij,\rho}^{(0)}(z_1,z_2,z_3)].$$

The equilibrium equations (4) and the relations $(38)_1$, $(40)_1$ and (43) lead to the following equations

(44) $$t_{ji,j}^{(0)}+f_i(z_1,z_2,z_3)=0,\ \text{in } \tilde{B},$$

(45) $$\tau_{ji,j}=0,\ \text{in } \tilde{B}.$$

In view of the relations $(38)_2$, $(40)_2$ and (43), from the conditions on the lateral surface F, we obtain the following boundary conditions

(46) $$t_{\alpha i}^{(0)}\tilde{N}_\alpha=t_i(z_1,z_2,z_3),\ \text{on } \tilde{F}.$$

(47) $$\tau_{\alpha i}\tilde{N}_\alpha=p_i(z_1,z_2,z_3),\ \text{on } \tilde{F}.$$

From the conditions (7), we obtain, for $z_3=0$,

(48) $$\int_{\tilde{\Sigma}_0}t_{3i}^{(0)}d\sigma=-R_i,\quad\int_{\tilde{\Sigma}_0}z_\alpha t_{33}^{(0)}d\sigma=\varepsilon_{3\alpha\beta}M_\beta,\quad\int_{\tilde{\Sigma}_0}\varepsilon_{3\alpha\beta}z_\alpha t_{3\beta}^{(0)}d\sigma=-M_3.$$

$$\int_{\tilde{\Sigma}_0} \tau_{3i} d\sigma = 0, \quad \int_{\tilde{\Sigma}_0} z_\alpha \tau_{33} d\sigma = 0, \quad \int_{\tilde{\Sigma}_0} \varepsilon_{3\alpha\beta} z_\alpha \tau_{\beta 3} d\sigma = 0. \tag{49}$$

It is easy to see that the problem defined by the relations (44), (46) and (48) is the problem $\mathfrak{P}_0$ for the cylindrical beam $\bar{B}$. The problem defined by (45), (47) and (49) is the problem $\mathfrak{P}_1$ for the cylindrical beam $\tilde{B}$. Therefore, $u_i^{(0)}(z_1, z_2, z_3)$ is the solution of the problem $\mathfrak{P}_0$, while $v_i(z_1, z_2, z_3)$ is the solution of the problem $\mathfrak{P}_1$.

The above discussion on the determination of the solution of the problem allows to compare our results with those known. It may be of interest to mention that the transformation (37) was used in earlier papers on the subject. However, the form of our solution (14) or (39) leads to the results that are more simple than those known (to compare with [1]—[8]).

The advantage of the method described in Section 3 lies in the following

a) it is proved in a simple manner the way of the determination of the solution;

b) the solution is presented in a natural form; this fact leads to an elegant treatment of the problem;

c) there is advoided the transformation (37); the treatment of the problem is made in the initial system of Cartesian coordinates.

REFERENCES

1. Panov D. Yu. — *On the torsion of almost cylindrical beams* (in Russian), Prikl. Math. Mech., 2 (1938), 159—180
2. Rukhadze A. K. — *The problem of bending of almost prismatic beams* (in Russian), Soobsc. Akad. Nauk Gruzin. SSR, 1 (1940), 577—582
3. Borș C. I. — *Bending of almost cylindrical beams composed of more orthotropic materials*, (in Romanian). St. Cerc. St., Fil. Iași, Acad. R.P.R., 11 (1960), 175—185
4. Borș C. I. — *Theory of elasticity for anisotropic bodies*, (in Romanian), Ed. Academiei, Bucharest, 1970.
5. Borș C. I. — *Saint-Venant's problem for orthotropic almost cylindrical beams*, Rev. Roum. Sci. Techn. — Mec. Appl., 16 (1971), 1275—1284
6. Chiriță S. — *Generalized Saint-Venant's problem for heterogeneous anisotropic almost cylindrical beams*, Bul. Inst. politehn. Iași, Matematică, 22 (1976), 101—106
7. Borș C. I. and Crăciun I. — *The general problem of the orthotropic almost cylindrical beams*, An. șt. Univ. Iași. Matematică, 22 (1976), 79—86
8. Khatiashvili G. M. — *Saint-Venant's problem for a composed anisotropic body similar to prismatic body*, (in Russian), Trudy Vyčisl. Centra Akad. Nauc Gruzin. SSR 4 (1964), 71—90
9. Ieșan D. — *Saint-Venant's problem for inhomogeneous and anisotropic elastic bodies*, J. Elasticity, 6 (1976), 277—294
10. Chiriță S. and Ieșan D. — *Theory of loaded elastic cylinders*, An. șt. Univ. Iași, Matematică, 23 (1977), 169—182
11. Fichera G. — *Existence theorems in elasticity*, in *Handbuch der Physik*, vol. VI a/2, Berlin, Springer-Verlag, 1972.

Received 3.II.1982

Seminarul matematic
Universitatea „Al. I. Cuza"
6600, Iași, România

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXIX, s. I a Matematică, f. 2, 1983

RECENZII

D. E. SARASON, N. A. FRIEDMAN (Editors): *P. R. Halmos, Selecta, Research Contributions*, Springer-Verlag, New York, Heidelberg, Berlin, 1983, XXVIII+458 p., US $ 36.40

Volumul conține 37 dintre lucrările lui P. R. Halmos. Lucrările selectate se referă la teoria probabilităților și teoria măsurii (aceste lucrări au fost publicate pînă în jurul anului 1950) și lucrări care se ocupă de teoria operatorilor (publicate după 1950). De asemenea, volumul conține două expuneri ale lui P. R. Halmos (zece probleme din teoria spațiilor Hilbert și zece ani de studiu al spațiilor Hilbert) și două articole introductive; unul referitor la studiile lui P. R. Halmos din teoria ergodică (datorat lui N. A. Friedman) și altul referitor la activitatea lui P. R. Halmos în domeniul teoriei operatorilor (datorat lui D. E. Sarason). Lucrările lui P. R. Halmos selectate pentru prezentul volum sînt prezentate în ordine cronologică.

V. Popescu

D. E. SARASON, L. GILLMAN (Editors): *P. R. Halmos, Selecta, Expository Writing*, Springer-Verlag, New York, Heidelberg Berlin, 1983, XX+304 p., US $ 23.30

Volumul conține 27 dintre lucrările lui P. R. Halmos și un articol despre activitatea sa (datorat lui D. J. Albers). Cele 27 expuneri sînt grupate în patru capitole; Cap. I conține probleme tehnice (transformări măsurabile, entropie în teoria ergodică progrese recente din teoria ergodică, ce spune teorema spectrală, o idee despre spații Hilbert, spații Hilbert finit dimensionale). Cap. II conține nouă articole ce au apărut în American Mathematical Monthly, Cap. III conține opt articole cu sfaturi și observații privind studiul matematicilor, iar Cap. IV conține patru articole „populare" (despre N. Bourbaki, matematica — o artă creativă, legenda lui John von Neumann, despre matematici aplicate).

V. Popescu

C. REID: *Neyman-from life*, Springer-Verlag, New York, Heidelberg, Berlin, 1982, IV+298 p., US $ 19.60

Cartea a fost scrisă la sugestia profesorului E. L. Lehmann de la Facultatea de Statistică a Universității Berkeley din California. Profesorul J. Neyman a fost distins cu Medalia Națională de Știință. A fost profesor la Facultatea de Statistică a Universității Berkeley din California și director al Laboratorului de Statistică. Profesorul J. Neyman a adus contribuții deosebite și în alte domenii cum ar fi: agricultură, bacteriologie, biologie, cosmologie. Volumul conține o descriere a vieții și operei profesorului J. Neyman; s-a născut la 16 aprilie 1894 la Bender — „poarta Basarabiei", a făcut Facultatea de Științe la Harkov, s-a căsătorit în 1920 cu Olga Solodovnikova, după o perioadă de ședere în Polonia, Franța, și Anglia, s-a stabilit în Statele Unite ale Americii, a murit la 5 august 1981 (în California).

V. Popescu

H. RADEMACHER: *Higher Mathematics from an Elementary Point of View*. Edited by D. Goldfeld. Notes by G. Crane. Birkhäuser, Boston, Basel, Stuttgart, 1982, III+138 p., sFr. 58

Cartea conține o versiune elaborată a lecțiilor predate de către H. Rademacher la Universitatea Stanford în 1947. În primul rînd sînt incluse rezultate asupra unor subiecte din