for any $X, Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ and $N \in \Gamma(\varphi D^{\perp})$.

In (4.42) we take $X = bY$ and taking account of (4,37) we find

$$2\|A_N bY\|^2 + \frac{k+3}{2}\|bY\|^2\|N\|^2 = 0 \tag{4.43}$$

for all $Y \in \Gamma(D \oplus \tilde{D} \oplus \{\xi\})$ and $N \in \Gamma(\varphi D^{\perp})$. Thus, from (4.43) it results $k \leq -3$ and the proof is complete.

From Theorem 4.5 we have

Corollary 4.1. *Let M be a $(D \oplus \tilde{D}, D^{\perp})$-geodesic almost semi-invariant submanifold in a Sasakian space form $\tilde{M}(k)$, $k > -3$. If the distribution $D \oplus \tilde{D} \oplus \{\xi\}$ is integrable and the subbundle $\varphi D^{\perp}$ is $D \oplus \tilde{D} \oplus \{\xi\}$-parallel, then we have either M is an anti-invariant submanifold of $\tilde{M}(k)$, or M is an almost invariant submanifold of $\tilde{M}(k)$.*

From (3.6) and Theorem 4.5 we also have

Corollary 4.2. *Suppose M satisfies the conditions of Theorem 4.5. If the leaves of distribution $D \oplus \tilde{D} \oplus \{\xi\}$ are totally geodesic immersed in M then we have $k = -3$.*

REFERENCES

1. Bejancu A. and Papaghiuc N. — *Almost semi-invariant submanifolds of a Sasakian manifold*, to appear in Bull. Math. de la Soc. Sci de la R. S. de Roumanie
2. Bejancu A. and Papaghiuc N. — *Semi-invariant submanifolds of a Sasakian manifold*, An. şt. Univ. Iaşi, tom XXVII, s. I a. 1981, f. 1, p. 163–170
3. Blair D. — *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, 509 (1976), Springer-Verlag
4. Kon M. — *Invariant submanifolds in Sasakian manifolds*, Math. Ann. 219 (1976), p. 277–290
5. Ludden G. D. — Okumura M. and Yano K. — *Anti-invariant submanifolds of almost contact metric manifolds*, Math. Ann., 225 (1977)
6. Papaghiuc N. — *Theorem of decomposition for almost semi-invariant submanifolds in a Sasakian manifold* (to appear)

Received 9. XI. 1981

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXIX, s. I a Matematică, f. 2, 1983

SUBMANIFOLDS AND A PINCHING PROBLEM ON THE SECOND FUNDAMENTAL TENSORS

BY

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1. **Introduction.** M. Okumura in [5] has proved the following: *Let M be an n-dimensional ($n \geq 3$), complete, connected submanifold of an $(n+p)$-dimensional Riemannian manifold of positive constant curvature c, whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If M is immersed without minimal point and for a real δ, $0 < \delta < c$, the second fundamental tensors $A_a (a = 1, \ldots, p)$ satisfy*

$$\sum_{a=1}^{p} \operatorname{trace} A_a^2 < 1/n-1 \sum_{a=1}^{p} (\operatorname{trace} A_a)^2 + 2\delta,$$

then M is umbilical with respect to the curvature normal direction.

For codimension 1 ($p = 1$), the present author in [3] has proved a stronger result. The purpose of the present paper is to generalize in some sense the above result of M. Okumura.

2. **Preliminaries.** Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ of constant curvature c. The Riemannian connections of M and $\overline{M}$ are denoted by ∇ and $\overline{\nabla}$ respectively, whereas the connection of the normal bundle of M in $\overline{M}$ is denoted by D. Let $e_1, \ldots, e_p$ be mutually orthogonal unit normal vectors at a point q of M and extend them to local fields in a neighborhood of q. Thus we have the following equations of Gauss and Weigarten:

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{a=1}^{p} \Sigma g(A_a X, Y) e_a, \tag{2.1}$$

$$\overline{\nabla}_X e_a = -A_a X + D_X e_a, \tag{2.2}$$

where X, Y are tangent vector fields in a neighborhood of M at q, g the Riemannian metric of M induced from that of $\overline{M}$ and A_a the second fundamental tensor with respect to e_a, whereas $D_X e_a$ is the normal part of $\overline{\nabla}_X e_a$ to M. The ambient manifold $\overline{M}$ being of constant curvature c, the curvature tensor K and the scalar curvature R of M are respectively given by

$$K(X,Y)Z=c\{g(Y,Z)X-g(X,Z)Y\}+\sum_{a=1}^{p}\{g(A_aY,Z)A_aX-\\ -g(A_aX,Z)A_aY\},\tag{2.3}$$

$$R=n(n-1)+\sum_{a=1}^{p}(\text{trace } A_a)^2-\sum_{a=1}^{p}\text{trace } A_a^2.\tag{2.4}$$

The mean curvature vector H is defined by

$$H=\frac{1}{n}\sum_{a=1}^{p}(\text{trace } A_a)e_a\tag{2.5}$$

and it is well known that H is independent on the choice of mutually orthogonal unit normal vectors e_a to M. Also, we denote by S the square of the length of the second fundamental tensor of M in $\overline{M}$, that is

$$S=\sum_{a=1}^{p}\text{trace } A_a^2.\tag{2.6}$$

If S vanishes identically, then M is called *totally geodesic submanifold* of $\overline{M}$.

A normal vector field ξ on M is said to be parallel in the normal bundle, or simply parallel, if we have $D_X\xi=0$, for every tangent vector X of M. It is obvious that a normal vector field ξ on M which is parallel has constant length. So, if the mean curvature vector field is parallel then

$$\sum_{a=1}^{p}(\text{trace } A_a)^2=\text{constant}.$$

Now, let M be an n-dimensional submanifold of a Riemannian manifold $\overline{M}$ and ξ a unit normal vector field of M. Let A_ξ be the second fundamental tensor with respect to ξ. Since, A_ξ at every point $q\in M$ is a linear symmetric transformation we have at q.

$$\text{trace } A_\xi^2\geq\frac{1}{n}(\text{trace } A_\xi)^2\tag{2.8}$$

If equality in (2.8) holds everywhere on M, then M is said to be umbilical with respect to the direction ξ.

Now, we assume that the mean curvature H of M in $\overline{M}$ is parallel and non-zero. Then the length of H is a positive constant. We choose an orthonormal frame $e_1,\ldots,e_p$ of normal vectors such that e_1 is the unit vector in the direction of H. Thus trace $A_a=0$, for $a=2,\ldots,p$ and trace $A_1=$ constant >0. Now, we put

$$LX=A_1X-1/n\,(\text{trace } A_1)X.\tag{2.9}$$

By easy computations we find the following relations :

$$\text{trace } L=0,\tag{2.10}$$

$$\text{trace } A_1^2=\text{trace } L^2+1/n\,(\text{trace } A_1)^2,\tag{2.11}$$

$$\text{trace } A_1^3=\text{trace } L^3+3/n\,(\text{trace } A_1)\,\text{trace } L^2+1/n^2\,(\text{trace } A_1)^3.\tag{2.12}$$

Obviously trace $L^2\geq0$ and equality holds everywhere iff M is umbilical with respect to e_1 (i.e. to the mean curvature vector).

3. **Main results.** At first we state two lemmas which are easy consequences of result by Chen-Okumura ([1], p. 55).

Lemma 1. *Let M be an n-dimensional of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ of constant curvature c. If*

$$\sum_{a=1}^{p}\text{trace } A_a^2\leq 1/(n-1)\sum_{a=1}^{p}(\text{trace } A_a)^2+2c,$$

then the sectional curvature of M are nonnegative.

Lemma 2. *Let M be a complete n-dimensional minimal submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ of constant sectional curvature c. Assume that $\sup S\leq nc/[2-1/p]-\varepsilon$, with ε a positive number. Then the sectional curvature of M are bounded below by $c-nc/[2(2-1/p)]+\varepsilon/2$.*

Now, we prove a theorem which in the compact case has been proved by J. Simons in [7].

Theorem 3. *Let M be a complete n-dimensional minimal submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ of constant positive sectional curvature c. Then $\sup S\leq nc/[2-1/p]$ or M is a totally geodesic submanifold of $\overline{M}$.*

Proof. We repeat the same calculations as in [2] and we obtain

$$1/2\,\Delta S\geq S[2-1/p]\left[\frac{nc}{[2-1/p]}-S\right]+\sum_{a=1}^{p}g(\nabla A_a,\nabla A_a),\tag{3.1}$$

where Δ is the Laplacian for the metric of M. It is sufficient to prove the following. If $\sup S<nc/[2-1/p]$, then M must be totally geodesic. In fact, if $\sup S<nc/[2-1/p]$ then there exists a positive number ε such that $S\leq nc/[2-1/p]-\varepsilon$, everywhere M and thus the sectional curvature of M are bounded below (see Lemma 2). Also, $nc/[2-1/p]-S\geq\varepsilon$ and (3.1) gives :

$$1/2\Delta S\geq S\varepsilon\tag{3.2}$$

Since S is bounded above and the sectional curvatures of M are bounded below, we claim that $S=0$, everywhere on M. In fact, if for some point q_1 we had $S(q_1)=b>0$, then from (3.2) we should have $1/2\,\Delta S(q_1)\geq b\varepsilon=$ const >0, and thus for all points q of M such that $S(q)\geq S(q_1)$ ought have $(1/2)\Delta S(q)\geq b\varepsilon=$ const. >0, which contradicts Omori's theorem A' in [6]. So, $S=0$, everywhere on M and the proof is finished.

With respect to Okumura's result we have the following theorem.

Theorem 4. *Let M be an n-dimensional $(n\geq3)$ complete connected submanifold of an $(n+p)$-dimensional Riemannian manifold $\overline{M}$ of positive constant curvature c, whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If $\overline{M}$ is immersed without minimal point and the second fundamental tensors A_a $(a=1,\ldots,p)$ satisfy*

$$\text{(3.3)}\qquad \sum_{\alpha=1}^{p} \operatorname{trace} A_\alpha^2 \leqslant (1/(n-1)) \sum_{\alpha=1}^{p} (\operatorname{trace} A_\alpha)^2 + 2c,$$

then M is umbilical with respect to the mean curvature normal direction.

Remark. It is obvious that theorem 4 is stronger than Okumura's result.

Proof. Since the mean curvature vector field H is parallel we conclude that its length is a positive constant (because M is immersed without minimal point). Now, we choose an orthonormal frame $e_1, \dots, e_p$ of normal vectors such that e_1 is the unit vector in the direction of H. Thus $\operatorname{trace} A_\alpha = 0$ for $\alpha = 2, \dots, p$ and $\operatorname{trace} A_1 = \text{const.} > 0$. So, from (2.11) we conclude that the Laplacian Δf^2 of the function $f^2 = \operatorname{trace} L^2$ satisfies

$$\text{(3.4)}\qquad \Delta f^2 = \Delta(\operatorname{trace} A_1^2).$$

On the other hand, B. Smyth in [8] proved the following formula for a submanifold of parallel non-zero mean curvature vector in a Riemannian manifold of constant curvature c:

$$\text{(3.5)}\qquad \begin{aligned} (1/2)\,\Delta f^2 &= (1/2)\Delta(\operatorname{trace} A_1^2) = nc \operatorname{trace} A_1^2 - c(\operatorname{trace} A_1)^2 + \\ &+ \sum_{\alpha=1}^{p} \{\operatorname{trace} A_\alpha \,.\, \operatorname{trace}(A_1^2 A_\alpha) - (\operatorname{trace} A_1 A_\alpha)^2\} + g(\nabla A_1, \nabla A_1) \end{aligned}$$

or because $\operatorname{trace} A_\alpha = 0$, for $\alpha = 2, \dots, p$

$$\text{(3.6)}\qquad \begin{aligned} (1/2)\Delta f^2 &= nc \operatorname{trace} A_1^2 - c(\operatorname{trace} A_1)^2 + \operatorname{trace} A_1 \,.\, \operatorname{trace} A_1^3 - \\ &(\operatorname{trace} A_1^2)^2 + g(\nabla A_1, \nabla A_1) - \sum_{\alpha=1}^{p} (\operatorname{trace} A_1 A_\alpha)^2. \end{aligned}$$

Now, substituting equations (2.10), (2.11) and (2.12) into (3.5) and repeating the same calculations as M. Okumura in [4] we get

$$\text{(3.7)}\qquad \begin{aligned} (1/2)\Delta f^2 &\geqslant f^2 [cn + (1/n)(\operatorname{trace} A_1)^2 - (n-2)/\sqrt{n(n-1)}(\operatorname{trace} A_1) f - \\ &- f^2] + g(\nabla A_1, \nabla A_1) - \sum_{\alpha=2}^{p} (\operatorname{trace} A_1 A_\alpha)^2. \end{aligned}$$

Moreover, since $\operatorname{trace} A_\alpha = 0$, for $\alpha = 2, \dots, p$ we get easily that:

$$\sum_{\alpha=2}^{p} (\operatorname{trace} A_1 A_\alpha)^2 = \sum_{\alpha=2}^{p} (\operatorname{trace} L A_\alpha)^2 \leqslant \operatorname{trace} L^2 . \sum_{\alpha=1}^{p} \operatorname{trace} A_\alpha^2 = f^2 . \sum_{\alpha=2}^{p} \operatorname{trace} A_\alpha^2.$$

Thus (3.7) gives

$$\text{(3.8)}\qquad \begin{aligned} (1/2)\Delta f^2 \geqslant f^2 \Big[& cn + (1/n)(\operatorname{trace} A_1)^2 - \sum_{\alpha=1}^{p} \operatorname{trace} A_\alpha^2 - \\ & - (n-2)/\sqrt{n(n-1)}\,(\operatorname{trace} A_1) f - f^2 \Big] + g(\nabla A_1, \nabla A_1). \end{aligned}$$

Also, from relations (3.3) and (2.11) we get

$$\text{(3.9)}\qquad f^2 \leqslant 1/[n(n-1)]\,(\operatorname{trace} A_1)^2 + 2c - \sum_{\alpha=1}^{p} \operatorname{trace} A_\alpha^2.$$

Now, since $c > 0$, $n \geqslant 3$ and $\operatorname{trace} A_1 = \text{const.} > 0$ we have

$$0 < \varepsilon_0 = \text{const} = (n-2)\left[\frac{(\operatorname{trace} A_1)^2}{n(n-1)} + c - \frac{\operatorname{trace} A_1}{\sqrt{n(n-1)}} \left(\frac{(\operatorname{trace} A_1)^2}{n(n-1)} + 2c \right)^{1/2} \right] \leqslant$$

$$\leqslant (n-2)\left[\frac{(\operatorname{trace} A_1)^2}{n(n-1)} + c - \frac{\operatorname{trace} A_1}{\sqrt{n(n-1)}} \left(\frac{(\operatorname{trace} A_1)^2}{n(n-1)} + 2c - \sum_{\alpha=2}^{p} \operatorname{trace} A_\alpha^2 \right)^{1/2} \right].$$

Obviously, we have now

$$0 \leqslant f = \left[\frac{(\operatorname{trace} A_1)^2}{n(n-1)} + 2c - \sum_{\alpha=2}^{p} \operatorname{trace} A_\alpha^2 \right]^{1/2} \leqslant - \frac{(n-2)}{2\sqrt{n(n-1)}} \operatorname{trace} A_1 +$$

$$+ \left(\frac{(n-2)^2}{4n(n-1)} (\operatorname{trace} A_1)^2 + cn + \frac{(\operatorname{trace} A_1)^2}{n} - \sum_{\alpha=2}^{p} \operatorname{trace} A_\alpha^2 - \varepsilon_0 \right)^{1/2},$$

and thus

$$f^2 + (n-2)/\sqrt{n(n-1)}(\operatorname{trace} A_1) f + \sum_{\alpha=2} \operatorname{trace} A_\alpha^2 + \varepsilon_0 - cn - (\operatorname{trace} A_1)^{2/n} \leqslant 0,$$

or

$$\text{(3.10)}\quad cn + (\operatorname{trace} A_1)^2/n - \sum_{\alpha=1}^{p} \operatorname{trace} A_\alpha^2 - n - 2/\sqrt{n(n-1)}\,(\operatorname{trace} A_1) f - f^2 \geqslant \varepsilon_0.$$

So, by using (3.10) the inequality (3.8) gives

$$\text{(3.11)}\qquad (1/2)\Delta f^2 \geqslant f^2 \varepsilon_0 + g(\nabla A_1, \nabla A_1) \geqslant f^2 \varepsilon_0.$$

Now, since f^2 is bounded above and the sectional curvature of M are bounded below (see Lemma 1), proceeding as in the proof of Theorem 3 we conclude that $f^2 = 0$, everywhere on M, i.e. M is umbilical with respect to the mean curvature vector normal direction.

Remark. If we omit the assumption „M is immersed without minimal point" we have the following theorem.

Theorem 5. *Let M be an n-dimensional ($n \geqslant 4$) complete, connected submanifold of an $(n+p)$-dimensional Riemannian manifold of positive constant curvature c, whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If the second fundamental tensors $A_\alpha (\alpha = 1, \dots, p)$ satisfy (3.3) then M is umbilical or is totally geodesic.*

Proof. It is an easy consequence of Theorems 3 and 4.

REFERENCES

1. Chen B. Y. — *Geometry of submanifolds*, Marcel Dekker, New York, 1973
2. Chen S. S., Carmo M. Do. and Kobayashi S. — *Minimal submanifolds of a sphere with second fundamental from of constant length*, Functional analysis and related fields, Springer-Verlag, 1970
3. Hasanis Th. *Characterizations of totally umbilical hypersurfaces*, Proc. Amer. Math. Soc. (3), 81 (1981), 447—450
4. Okumura M. — *Hypersurfaces and a pinching problem on the second fundamental tensor* Amer, J. of Math., XCVI (1974) 207—213
5. Okumura M.—*Submanifolds and the length of the second fundamental tensors*, J. Koorean Math. Soc., 13 (1976), 99—103
6. Omori H. —*Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan, 19 (1967) 205—214
7. Simons J. — *Minimal varieties in Riemannian manifolds*, Ann. of Math., 88 (1968), 105
8. Smyth B.—*Submanifolds of constant mean curvature*, Math. Annalen, 205 (1975), 265—280

Received 15.XII.1981

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXIX, s. I a Matematică, f. 2, 1983

METRICAL ALMOST PRODUCT FINSLER STRUCTURES

BY

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The metrical Finsler structures have been studied by R. Miron and M. Hashiguchi [3]. The notions of almost product Finsler structure and almost product Finsler connection, using the method of R. Miron [4], are introduced by the author of this paper [1]. The set of all almost product Finsler connections, the group of the connection transformations and their invariants, and the integrability conditions were established in [1].

The metrical almost product Finsler structure notion is introduced and the set of all metrical almost product Finsler connections and the G_{map} group of the connection transformations are determined in this paper. For the $G_{map}(N)$ group of the transformations of almost product Finsler connections, having the same non-linear connection N, we determine the invariants and establish the integrability conditions of the metrical almost product Finsler structures by means of these invariants.

The notation and terminology of M. Matsumoto [2] are used with slight modifications [4].

1. **Preliminaries.** Let M be on n-dimensional differential manifold, $T(M)$ the tangent bundle of M and N a non-linear connection on $T(M)$. Let (x^i, y^i) be the canonic coordinates on $T(M)$. Then: $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k}$; $i=1, 2, \ldots, n$ is a local base for the distribution defining the non-linear connection N, and $\frac{\partial}{\partial y^i}$; $i=1, 2, \ldots, n$ is a local base for the vertical distribution $(TM)^V$. The dual base is given by $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N_k^i dx^k$ satisfies:

$$dx^i\left(\frac{\delta}{\delta x^j}\right)=\delta_j^i\,;\; dx^i\left(\frac{\partial}{\partial y^j}\right)=0\,;\; \delta y^i\left(\frac{\partial}{\partial x^j}\right)=0\,;\; \delta y^i\left(\frac{\partial}{\delta y^j}\right)=\delta_j^i. \tag{1.1}$$

A Finsler connection on M is a triade $F\Gamma=(N, F, C)$, where N is the non-linear connection on $T(M)$, and F, C are the h- and v-connection coefficients respectively [4] given by: