## REFERENCES


1. Chen B. Y. — *Geometry of submanifolds*, Marcel Dekker, New York, 1973
2. Chen S. S., Carmo M. Do. and Kobayashi S. — *Minimal submanifolds of a sphere with second fundamental from of constant length*, Functional analysis and related fields, Springer-Verlag, 1970
3. Hasanis Th. *Characterizations of totally umbilical hypersurfaces*, Proc. Amer. Math. Soc. (3), 81 (1981), 447—450
4. Okumura M. — *Hypersurfaces and a pinching problem on the second fundamental tensor* Amer, J. of Math., XCVI (1974) 207—213
5. Okumura M.—*Submanifolds and the length of the second fundamental tensors*, J. Koorean Math. Soc., 13 (1976), 99—103
6. Omori H. —*Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan, 19 (1967) 205—214
7. Simons J. — *Minimal varieties in Riemannian manifolds*, Ann. of Math., 88 (1968), 105
8. Smyth B.—*Submanifolds of constant mean curvature*, Math. Annalen, 205 (1975), 265—280


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*Department of Mathematics*
*University of Ioannina*
*Ioannina-Greece*

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# METRICAL ALMOST PRODUCT FINSLER STRUCTURES

BY

FRANCISC C. KLEPP

The metrical Finsler structures have been studied by R. Miron and M. Hashiguchi [3]. The notions of almost product Finsler structure and almost product Finsler connection, using the method of R. Miron [4], are introduced by the author of this paper [1]. The set of all almost product Finsler connections, the group of the connection transformations and their invariants, and the integrability conditions were established in [1].

The metrical almost product Finsler structure notion is introduced and the set of all metrical almost product Finsler connections and the $G_{map}$ group of the connection transformations are determined in this paper. For the $G_{map}(N)$ group of the transformations of almost product Finsler connections, having the same non-linear connection $N$, we determine the invariants and establish the integrability conditions of the metrical almost product Finsler structures by means of these invariants.

The notation and terminology of M. Matsumoto [2] are used with slight modifications [4].

1. **Preliminaries.** Let $M$ be on $n$-dimensional differential manifold, $T(M)$ the tangent bundle of $M$ and $N$ a non-linear connection on $T(M)$. Let $(x^i, y^i)$ be the canonic coordinates on $T(M)$. Then: $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$; $i=1, 2, \ldots, n$ is a local base for the distribution defining the non-linear connection $N$, and $\frac{\partial}{\partial y^i}$; $i=1, 2, \ldots, n$ is a local base for the vertical distribution $(TM)^V$. The dual base is given by $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N^i_k dx^k$ satisfies:

$$dx^i\left(\frac{\delta}{\delta x^j}\right)=\delta^i_j\,;\; dx^i\left(\frac{\partial}{\partial y^j}\right)=0\,;\; \delta y^i\left(\frac{\partial}{\partial x^j}\right)=0\,;\; \delta y^i\left(\frac{\delta}{\delta y^j}\right)=\delta^i_j. \tag{1.1}$$

A Finsler connection on $M$ is a triade $F\Gamma=(N, F, C)$, where $N$ is the non-linear connection on $T(M)$, and $F$, $C$ are the $h$- and $v$-connection coefficients respectively [4] given by:

$$\nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = F^i_{jk} \frac{\delta}{\delta x^i}; \qquad \nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = F^i_{jk} \frac{\partial}{\partial y^i}$$
$$\dot{\nabla}_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = C^i_{jk} \frac{\delta}{\delta x^i}; \qquad \dot{\nabla}_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = C^i_{jk} \frac{\partial}{\partial y^i} \tag{1.2}$$

For a Finsler tensor field $K$ of type (1.1) with components $K$, the $h$- and $v$-covariant derivatives are given by:

$$K^i_{j|k} = \frac{\delta K^i_j}{\delta x^k} + F^i_{mk} K^m_j - F^m_{jk} K^i_m; \quad K^i_{j|k} = \frac{\partial K^i_j}{\partial y^k} + C^i_{mk} K^m_j - C^m_{jk} K^i_m. \tag{1.3}$$

If $F\Gamma = (N, F, C)$ and $F\overline{\Gamma} = (\overline{N}, \overline{F}, \overline{C})$ are two Finsler connectiond on $M$, then an unique triade of Finsler tensor Fields $(A, B, D)$ is determines such that:

$$\overline{N}^i_j = N^i_j + A^i_j; \quad \overline{F}^i_{jk} = F^i_{jk} - C^i_{jh} A^h_k + B^i_{jk}; \quad \overline{C}^i_{jk} = C^i_{jk} + D^i_{jk}. \tag{1.4}$$

Reciprocally: If the Finsler connection $F\Gamma = (N, F, C)$ and the triade $(A, B, D)$ of Finsler tensor fields are given, then the connection $F\overline{\Gamma} = (\overline{N}, \overline{F}, \overline{C})$ given by (1.4) is a Finsler connection. The map $F\Gamma \to F\overline{\Gamma}$ defined by (1.4) is called transformation of the Finsler connections. [4].

**Definition 1.1.** *A symmetrical and non-degenerated Finsler tensor field $g_{ij}(x, y)$ of type (0,2) will be called a metrical Finsler structure.*

To such a Finsler tensor field $g = (g_{ij})$ we can associate the Obata operators [4]:

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - g_{sj} g^{ir}); \quad \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + g_{sj} g^{ir}). \tag{1.5}$$

**Definition 1.2.** A Finsler connection $F\Gamma = (N, F, C)$ is called a metrical Finsler connection if:

$$g_{ij|k} = 0; \quad g_{ij|k} = 0. \tag{1.6}$$

**2. Almost product Finsler structures. Definition 2.1.** *An almost product Finsler structure on $M$ is a Finsler tensor field $P$ of type (1, 1) having $P^i_j(x, y)$ components wich satisfy:*

$$P^k_j P^i_k = \delta^i_j. \tag{2.1}$$

It follows that the matrix $(P^i_j)$ is non-singular. To such a Finsler tensor field $P$ we can associate the Obata operators [1]:

$$O^{ij}_{hk} = \frac{1}{2}(\delta^i_h \delta^j_k - P^i_h P^j_k); \quad O^{*ij}_{hk} = \frac{1}{2}(\delta^i_h \delta^j_k + P^i_h P^j_k). \tag{2.2}$$

**Definition 2.2.** *A Finsler connection $F\Gamma = (N, F, C)$ is an almost product Finsler connection relative to $P$, if:*

$$P^i_{j|k} = 0; \quad P^i_{j|k} = 0 \tag{2.3}$$

We obtain [1]:



**Theorem 2.1.** *Denoting by $G_{ap}(N)$ the set of the almost product Finsler connection transformations, having the same nonlinear connection, these form a group given by:*

$$\overline{N}^i_j = N^i_j; \quad \overline{F}^i_{jk} = F^i_{jk} + O^{hi}_{jm} X^m_{hk}; \quad \overline{C}^i_{jk} = C^i_{jk} + O^{hi}_{jm} Y^m_{hk} \tag{2.4}$$

where $X$ and $Y$ are arbitrary Finsler tensor fields.

Determining the invariants of the group $G_{ap}(N)$ we obtain the following Finsler tensor fields [1]: $\tilde{N}^i_{jk}$, $\tilde{R}^i_{jk}$, $\tilde{C}^i_{jk}$, $\tilde{P}^i_{jk}$, $\dot{N}^i_{jk}$, $\dot{R}^i_{jk}$, $\dot{C}^i_{jk}$, $\dot{P}^i_{jk}$, $\ddot{T}^i_{jk}$, $\ddot{S}^i_{jk}$.

Let $N$ be a non-linear connection on $T(M)$ and $P$ an almost product Finsler structure on $M$; then $P$ can be lifted to $T(M)$ by the aid of the distributions $N$ and $T(M)^V$. The most important liftings given in [1] lead to the $\tilde{P}$ almost product structure on $T(M)$, with the values of the coefficients given by the Table 1.

| $\tilde{P}$ | $\overset{1}{P}{}^i_j$ | $\overset{2}{P}{}^i_j$ | $\overset{3}{P}{}^i_j$ | $\overset{4}{P}{}^i_j$ |
|---|---|---|---|---|
| $P^I$ | $P^i_j$ | 0 | 0 | $P^i_j$ |
| $P^{II}$ | $P^i_j$ | 0 | 0 | $-P^i_j$ |
| $P^{III}$ | 0 | $P^i_j$ | $P^i_j$ | 0 |

Table 1

**Definition 2.3.** *The almost product Finsler structure $P$ with the components $P^i_j(x, y)$ is integrable of the type* I, II *and* III *respectively, if the almost product structure $P^I$, $P^{II}$ and $P^{III}$ respectively is integrable.*

**Theorem 2.2.** *The almost product Finsler structure $P$ is integrable of type* I, *II and III respectively if and only if the following invariants of the group $G_{ap}(N)$, listed below, vanish:*

| Type of integrability | Characterization by invariants |
|---|---|
| *I* | $\tilde{N}^i_{jk}=0$; $\tilde{R}^i_{jk}=0$; $\tilde{C}^i_{jk}=0$; $\tilde{P}^i_{jk}=0$; $\dot{N}^i_{jk}=0$; |
| *II* | $\tilde{N}^i_{jk}=0$; $\dot{R}^i_{jk}=0$; $\dot{C}^i_{jk}=0$; $\dot{P}^i_{jk}=0$; $\dot{N}^i_{jk}=0$ |
| *III* | $\ddot{S}^i_{jk}=0$; $\ddot{T}^i_{jk}=0$; |

**3. Metrical almost product Finsler structures. Definition 3.1.** *A metrical almost product Finsler structure on $M$ is a paire of Finsler tensor fields $(P, g)$, where $P$ is an almost product Finsler structure and $g$ is a metrical Finsler structure, so that:*

$$g_{hk} = g_{ij} P^i_h P^j_k. \tag{3.1}$$

Using the Obata operators (2.2) it follows:

$$O^{ij}_{hk} g_{ij} = 0. \tag{3.2}$$

**Proposition 3.1.** *If on the differentiable manifold $M$ there exists an almost product Finsler structure $P$ and a metrical Finsler structure $g$, then on $M$ there exists* *a Finsler structure given by the symmetric tensor* $g_{ij}=O^{*hk}_{ij}\tilde{g}_{hk}$ *and an almost product Finsler structure satisfying relation* (3.2).

At a metrical almost product Finsler structure we can associate a metrical Finsler structure given by : $a_{ij}=P^m_i g_{mj}$, called the induced metrical Finsler structure.

**Definition 3.2.** *A Finsler connection $F\Gamma=(N, F, C)$ is called a metrical almost product Finsler connection, if :*

$$P^i_{j|k}=0\ ;\ P^i_{j|k}=0.\ g_{ij|k}=0\ ;\ g_{ij|k}=0. \tag{3.3}$$

**Theorem 3.1.** *If on a differentiable manifold $M$ having a metrical almost product Finsler structure $(P, g)$ there exists a Finsler connection $F\overset{\circ}{\Gamma}=(\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$, then the following Finsler connection $F\overset{k}{\Gamma}=(\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ is a metrical almost product Finsler connection :*

$$\begin{cases} \overset{k}{N}{}^h_j=\overset{\circ}{N}{}^h_j;\ \overset{k}{F}{}^h_{jk}=\overset{\circ}{F}{}^h_{jk}+\dfrac{1}{4}(P^m_j P^h_{m\overset{\circ}{|}k}+g^{mh}g_{mj\overset{\circ}{|}k}+a^{mh}a_{mj\overset{\circ}{|}k}), \\ \overset{k}{C}{}^h_{jk}=\overset{\circ}{C}{}^h_{jk}+\dfrac{1}{4}(P^m_j P^h_m\overset{\circ}{|}_k+g^{mh}g_{mj}\overset{\circ}{|}_k+a^{mh}a_{mj}\overset{\circ}{|}_k). \end{cases} \tag{3.4}$$

To a metrical almost product Finsler structure $(P, g)$ we associate the Obata operators $\Omega^{ir}_{sj}$ ; $\Omega^{*ir}_{sj}$ of $g$ given by (1.5) and the Obata operators $O^{ir}_{sj}$ ; $O^{*ir}_{sj}$ of $P$ given by (2.2). Using the R. Miron method [4] we obtain :

**Theorem 3.2.** *The set of all metrical almost product Finsler connections on a differentiable manifold having a metrical almost product structure $(P, g)$ and a fixed Finsler connection $F\overset{\circ}{\Gamma}=(\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ is given by :*

$$\begin{cases} N^h_j=\overset{k}{N}{}^h_j-X^h_j\ ;\ F^h_{jk}=\overset{k}{F}{}^h_{jk}+\overset{k}{C}{}^h_{jm}X^m_k+O^{hb}_{mj}\Omega^{mr}_{sb}Y^s_{rk}, \\ C^h_{jk}=\overset{k}{C}{}^h_{jk}+O^{hb}_{sj}\,\Omega^{sr}_{mb}Z^m_{rk}, \end{cases} \tag{3.5}$$

where $F\overset{k}{\Gamma}=(\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ is the metrical almost product Finsler connection given by (3.4) and $X, Y, Z$ are arbitrary Finsler tensor fields.

It results the following Theorems :

**Theorem 3.3.** *The set of transformations $F\Gamma(N)\to F\overline{\Gamma}(N)$ of the all metrical almost product Finsler connections, having the same non-linear connection $N$, is given by :*

$$\overline{N}{}^h_j=N^h_j\ ;\ \overline{F}{}^h_{jk}=F^h_{jk}+O^{hs}_{mj}\Omega^{mr}_{ps}X^p_{rk}\ ;\ C^h_{jk}=\overline{C}{}^h_{jk}+O^{hs}_{mj}\Omega^{mr}_{ps}Y^p_{rk} \tag{3.6}$$

where $X$ and $Y$ are arbitrary Finsler tensor fields.

**Theorem 3.4.** The $G_{map}(N)$ *set of transformations of the metrical almost product Finsler connections having the same non-linear connection $N$, form an Abelian group relative to the product of transformations.*

The $G_{map}(N)$ group is a subgroup of the group $G_{ap}(N)$. Consequently, the invariants $\tilde{N}^i_{jk}$, $\tilde{R}^i_{jk}$, $\tilde{C}^i_{jk}$, $\tilde{P}^i_{jk}$, $\overset{\bullet}{R}{}^i_{jk}$, $\overset{\bullet}{N}{}^i_{jk}$, $\overset{\bullet}{C}{}^i_{jk}$, $\overset{\bullet}{P}{}^i_{jk}$, $\overset{\bullet\bullet}{S}{}^i_{jk}$, $\overset{\bullet\bullet}{T}{}^i_{jk}$ of the group $G_{ap}(N)$ [1] are invariant also for $G_{map}(N)$.



For the study of the integrability conditions of the metrical almost product Finsler structure we consider metrical almost product structures on the tangent bundle $T(M)$ of $M$. Let $\tilde{P}$ be an almost product structure on $T(M)$ given by the Table 1, and let $\tilde{G}$ be a metrical structure on $T(M)$ given by the Table 2 :

| $\tilde{G}$ | 1<br>$g_{ij}$ | 2<br>$g_{ij}$ | 3<br>$g_{ij}$ | 4<br>$g_{ij}$ |
|---|---|---|---|---|
| $G^I$ | $g_{ij}$ | 0 | 0 | $g_{ij}$ |
| $G^{II}$ | $g_{ij}$ | 0 | 0 | $-g_{ij}$ |
| $G^{III}$ | 0 | $g_{ij}$ | $g_{ij}$ | 0 |

Table 2

The $(\tilde{P}, \tilde{G})$ structures are metrical almost product structures on $T(M)$ for any $P^I$, $P^{II}$, $P^{III}$ and $G^I$, $G^{II}$, $G^{III}$ respectively, except the cases $(P^{II}, G^{III})$ and $(P^{III}, G^{II})$.

**Definition 3.3.** *The metrical almost product structure $(\tilde{P}, \tilde{G})$ on $T(M)$ is integrable of type I, II and III respectively, if the almost product structures $P^I$, $P^{II}$ and $P^{III}$ respectively are integrable.*

It results the following Theorems :

**Theorem 3.5.** *The metrical almost product Finsler structure $(P, g)$ is integrable of type I if and only if the following invariant Finsler tensor fields vanish :*

$$\tilde{N}^i_{jk}=0\ ;\ \tilde{R}^i_{jk}=0\ ;\ \tilde{C}^i_{jk}=0\ ;\ \tilde{P}^i_{jk}=0\ ;\ \overset{\bullet}{N}{}^i_{jk}=0. \tag{3.7}$$

**Theorem 3.6.** *The metrical almost product Finsler structure $(P, g)$ is integrable of type II, if and only if the following invariant Finsler tensor fields vanish :*

$$\tilde{N}^i_{jk}=0\ ;\ \overset{\bullet}{R}{}^i_{jk}=0\ ;\ \overset{\bullet}{C}{}^i_{jk}=0\ ;\ \overset{\bullet}{P}{}^i_{jk}=0\ ;\ \overset{\bullet}{N}{}^i_{jk}=0. \tag{3.8}$$

**Theorem 3.7.** *The metrical almost product Finsler structure $(P, g)$ is integrable of the type III, if and only if the following invariant Finsler tensor fields vanish :*

$$\overset{\bullet\bullet}{S}{}^i_{jk}=0\ ;\ \overset{\bullet\bullet}{T}{}^i_{jk}=0. \tag{3.9}$$

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## REFERENCES

1. Klepp F. C. — *Almost product Finsler structures.* An. şt. Univ. Iaşi (1981)
2. Matsumoto M. — *The theory of Finsler Connections.* Publ. of the Study Group of Geometry 5. Dep. of Math. Okayama Univ. (1970)
3. Miron R., Hashiguchi M. — *Metrical Finslerconnections.* Rep. Math. Sci. Kagoshima Univ. (Math., Phys., Chem.), 12, (1979), 21—35
4. Miron R. — *Introduction to the theory of Finsler spaces.* Proc. Nat. Sem. on Finsler spaces. Braşov, 1980, 131—183

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*Polytechnic Institute*
*Department of Mathematics*
*Timişoara, România*