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SEMI-INVARIANT SUBMANIFOLDS OF A CO-KÄHLER MANIFOLD

BY

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1. Introduction. Let $\widetilde{M}$ be a $(2n+1)$-dimensional differentiable manifold of class C^∞ and φ, ξ, η, g be a tensor field of type $(1, 1)$, a vector field, a 1-form and respectively a Riemannian metric on $\widetilde{M}$, satisfying

$$\varphi^2 X = -X + \eta(X)\xi \,;\; \eta(\xi) = 1,\; \varphi\xi = 0,\; \eta\circ\varphi = 0, \tag{1.1}$$

$$\eta(X) = g(X, \xi), \tag{1.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

for any X, Y tangent to $\widetilde{M}$. Then $\widetilde{M}$ is said to have an *almost contact metric structure* (φ, ξ, η, g) and it is called an *almost contact metric manifold*, (see [4]). We denote by $\tilde{\nabla}$ the Levi-Civita connection with respect to the Riemannian metric on $\widetilde{M}$. If we have

$$\tilde{\nabla}_X \varphi = 0, \text{ for any vector field } X \text{ tangent to } \widetilde{M}, \tag{1.4}$$

then we say that $\widetilde{M}$ is a *co-Kähler manifold*.

By using (1.4) and the third equality in (1.1) we obtain $\varphi\tilde{\nabla}_X\xi = 0$. Then by means of (1.3), we have $g(\tilde{\nabla}_X\xi, Y) = g(\varphi\tilde{\nabla}_X\xi, \varphi Y) = 0$, for any Y tangent to $\widetilde{M}$ and satisfying $g(Y, \xi) = 0$. On the other hand, since ξ is a unit vector field on $\widetilde{M}$, we have $g(\tilde{\nabla}_X\xi, \xi) = 0$. Hence on any co-Kähler manifold we have

$$\tilde{\nabla}_X \xi = 0, \text{ for any vector field } X \text{ tangent to } \widetilde{M}. \tag{1.5}$$

We denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on $\widetilde{M}$ and by Δ the orthogonal complementary distribution to $\{\xi\}$ in the tangent bundle $T\widetilde{M}$. From (1.5) follows that $\{\xi\}$ is a parallel distribution, which implies Δ is also a parallel distribution on $\widetilde{M}$. In fact, for any Y from Δ and X tangent to $\widetilde{M}$ we have $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X\xi) = 0$. That is $\tilde{\nabla}_X Y$ belongs to Δ. Hence we have

Proposition 1. *Any co-Kähler manifold $\widetilde{M}$ is locally a Riemannian product $\widetilde{M}_1 \times \tilde{\gamma}$ where $\widetilde{M}_1$ is a Kähler manifold and $\tilde{\gamma}$ is a curve.*

Proof. Since both distributions $\{\xi\}$ and Δ are parallel with respect to the Levi-Civita connection, $\widetilde{M}$ is locally a Riemannian product of a leaf $\widetilde{M}_1$ of Δ and a leaf $\tilde{\gamma}$ of $\{\xi\}$. Moreover, since X belongs to Δ if and only if we have $\eta(X)=0$, from (1.1) and (1.3) it follows that φ defines an almost Hermitian structure on $\widetilde{M}_1$. Finally, from (1.4) we obtain that $\widetilde{M}_1$ is a Kähler manifold. This completes the proof.

A co-Kähler manifold is often called a cosymplectic manifold, (see [4]). Moreover, an almost contact metric manifold $\widetilde{M}$ is a co-Kähler manifold if and only if $\widetilde{M}\times R$ is a Kähler manifold, (see [6]).

Now, let M be a Riemannian manifold isometrically immersed in a co-Kähler manifold $\widetilde{M}$. Then we denote by the same symbol g the Riemannian metric on M and by ∇, $\nabla^{\perp}$ and h the Levi-Civita connection on M, the Riemannian connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$ and respectively the second fundamental form of M. Also for each vector bundle H we denote by $\Gamma(H)$ the module of its differentiable sections. Then we have the equations of Gauss and Weingarten given by

(1.6) $$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y), \text{ and respectively}$$

(1.7) $$\tilde{\nabla}_X N=-A_N X+\nabla^{\perp}_X N,$$

for any $X,\ Y\in\Gamma(TM)$, $N\in\Gamma(TM^{\perp})$, where A_N is the fundamental tensor of Weingarten with respect to the normal section N satisfying

(1.8) $$g(h(X, Y), N)=g(A_N X, Y).$$

Suppose ξ is tangent to the submanifold M. Then by using notations from Proposition 1 we have

Porposition 2. *The submanifold M of a co-Kähler manifold $\widetilde{M}$ is locally a Riemannian product $M_1\times\gamma$, where M_1 is a submanifold of $\widetilde{M}_1$ and γ is a portion of $\tilde{\gamma}$.*

Proof. By using (1.6) and (1.5) we obtain

(1.9) $$\nabla_X\xi=0, \text{ for any } X\in\Gamma(TM).$$

Thus the distribution $\{\xi\}$ on M is parallel with respect to the Levi-Civita connection on M. The complementary orthogonal distribution to $\{\xi\}$ in TM is also parallel with respect to ∇ and the proof follows.

Next, we suppose M be endowed with two orthogonal distributions D and $D^{\perp}$ satisfying

(1.10) $$TM=D\oplus D^{\perp}\oplus\{\xi\},\ \varphi(D_x)=D_x \text{ and } \varphi(D^{\perp}_x)\subset T_xM^{\perp},$$

for any $x\in M$. Then we say that M is a *semi-invariant submanifold* of $\widetilde{M}$. It is not difficult to see that each hypersurface of $\widetilde{M}$ which is tangent to ξ is an example of semi-invariant submanifold of $\widetilde{M}$. From the definition we see that semi-invariant submanifolds of a co-Kähler manifold are corresponding to CR-submanifolds of a Kähler manifold (see [1]). The concept was also considered for submanifolds in Sasakian manifolds (see [3]).

The purpose of the present paper is to obtain results on the geometry of semi-invariant submanifolds in a co-Kähler manifold. In §2 we derive fundamental formulas for later use. In §3 we study the integrability of distributions on M. Finally, in §4 we obtain characterizations for totally umbilical and totally geodesic semi-invariant submanifolds.

2. Basic formulas. Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we denote by P and Q the projection morphisms of TM to D and $D^{\perp}$ respectively. Then for any $X\in\Gamma(TM)$ we have

(2.1) $$X=PX+QX+\eta(X)\xi.$$

Also, for any $N\in\Gamma(TM^{\perp})$ we put

(2.2) $$\varphi N=BN+CN, \text{ where } BN\in\Gamma(D^{\perp}) \text{ and } CN\in\Gamma(TM^{\perp}).$$

Lemma 1. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we have*

(2.3) $$P\nabla_X\varphi PY-PA_{\varphi QY}X=\varphi P\nabla_X Y,$$

(2.4) $$Q\nabla_X\varphi PY-QA_{\varphi QY}X=Bh(X, Y),$$

(2.5) $$h(X, \varphi PY)+\nabla^{\perp}_X\varphi QY=Ch(X, Y)+\varphi Q\nabla_X Y,$$

(2.6) $$\eta(\nabla_X Y)=0 \text{ and } \eta(A_N X)=0,$$

for any $X, Y\in\Gamma(TM)$ and $N\in\Gamma(TM^{\perp})$.

Proof. By using (2.1), (2.2) and the equations of Gauss and Weingarten in (1.4) we obtain

(2.7) $$\begin{aligned}\nabla_X\varphi PY+h(X, \varphi PY)-A_{\varphi QY}X+\nabla_X{}_X\varphi QY=\varphi P\nabla_X Y+\\ +\varphi Q\nabla_X Y+Bh(X, Y)+Ch(X, Y),\end{aligned}$$

for any $X, Y\in\Gamma(TM)$. Taking the components of D, $D^{\perp}$ and $TM^{\perp}$ in (2.7) we obtain (2.3), (2.4) and respectively (2.5). The first equality in (2.6) is valid for $Y=\xi$ taking account of (1.9). Now, we take $Y\in\Gamma(D\oplus D^{\perp})$ and obtain

$$\eta(\nabla_X Y)=g(\nabla_X Y,\xi)\ =-g(Y, \nabla_X\xi)=0.$$

From (1.5) taking account of (1.6), it follows

(2.8) $$h(X, \xi)=0, \text{ for any } X\in\Gamma(TM).$$

Hence by means of (1.2), (1.8) and (2.8) we have

$$\eta(A_N X)=g(A_N X, \xi)=g(h, \xi), N)=0.$$

Thus the proof is complete.

Lemma 2. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we have*

(2.9) $$P\nabla_X BN-PA_{CN}\gamma X+\varphi PA_N X=0,$$

$$Q\nabla_X BN - QA_{CN}X - B\nabla^{\perp}_X N = 0, \tag{2.10}$$

$$h(X, BN) + \nabla^{\perp}_X CN + \varphi QA_N X - C\nabla^{\perp}_X N = 0, \tag{2.11}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

Proof. By using (1.4), (1.6), (1.7) and (2.2) we obtain

$$\begin{aligned}\nabla_X BN + h(X, BN) - A_{CN}X + \nabla^{\perp}_X CN = -\varphi PA_N X - \varphi QA_N X + \\ + B\nabla^{\perp}_X N + C\nabla^{\perp}_X N.\end{aligned} \tag{2.12}$$

Thus the assertion of the lemma follows by taking the components of D, $D^{\perp}$ and $TM^{\perp}$ in (2.12).

Lemma 3. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we have*

$$A_{\varphi QX}QY = A_{\varphi QY}QX, \tag{2.13}$$

for any X, $Y \in \Gamma(TM)$.

Proof. By using (1.3), (1.4), (1.8) and the equations of Gauss and Weingarten we obtain

$$g(A_{\varphi QX}QY, Z) = g(h(QY, Z), \varphi QX) = g(\tilde{\nabla}_Z QY, \varphi QX) = -g(\varphi\tilde{\nabla}_Z QY, QX) =$$
$$= g(A_{\varphi QY}Z, QX) = g(A_{\varphi QY}QX, Z),$$

for any $X, Y, Z \in \Gamma(TM)$. Thus we have the assertion of the lemma.

3. Integrability of distributions on M. Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we have already seen in Proposition 2 that $D \oplus D^{\perp}$ is a parallel distribution with respect to the Levi-Civita connection on M. In this paragraph we study the integrability of all other distributions on M.

First we have

Theorem 1. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then the distribution $D^{\perp}$ is involutive.*

Proof. By using (1.9) we have

$$g([QX, QY], \xi) = g(\nabla_{QX}QY - \nabla_{QY}QX, \xi) = g(QX, \nabla_{QY}\xi) - g(QY, \nabla_{QX}\xi) = 0,$$

for any X, $Y \in \Gamma(TM)$. Next, by using (1.3), (1.4) and Lemma 3 we obtain

$$g([QX, QY], PZ) = g(\varphi([QX, QY]), \varphi PZ) = g(\varphi\tilde{\nabla}_{QX}QY - \varphi\tilde{\nabla}_{QY}QX, \varphi PZ) =$$
$$= g(\tilde{\nabla}_{QX}\varphi QY - \tilde{\nabla}_{QY}\varphi QX, \varphi PZ) = g(-A_{\varphi QY}QX + \nabla^{\perp}_{QX}\varphi QY +$$
$$+ A_{\varphi QX}QY - \nabla^{\perp}_{QY}\varphi QX, \varphi PZ) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. Thus the distribution $D^{\perp}$ is involutive.

Theorem 2. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then the distribution $D^{\perp} \oplus \{\xi\}$ is involutive.*

Proof. By Theorem 1 it suffices to prove $[QX, \xi] \in \Gamma(D^{\perp} \oplus \{\xi\})$ for any $X \in \Gamma(TM)$. By using (1.1), (1.3), (1.4), (1.7) and (2.8) we have

$$g([QX, \xi], PY) = g(\tilde{\nabla}_{QX}\xi - \tilde{\nabla}_{\xi}QX, PY) = -g(\tilde{\nabla}_{\xi}QX, PY) = g(\tilde{\nabla}_{\xi}QX, \varphi^2 PY) =$$
$$= -g(\varphi\tilde{\nabla}_{\xi}QX, \varphi PY) = -g(\tilde{\nabla}_{\xi}\varphi QX, \varphi PY) = g(A_{\varphi QX}\xi, \varphi PY) =$$
$$g(h(\varphi PY, \xi), \varphi QX) = 0,$$

for any X, $Y \in \Gamma(TM)$. Thus we have the assertion of the theorem.

Theorem 3. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if the second fundamental form of M satisfies*

$$h(X, \varphi Y) = h(Y, \varphi X), \tag{3.1}$$

for any X, $Y \in \Gamma(D)$.

Proof. We take X, $Y \in \Gamma(D)$ in (2.5) and obtain

$$h(X, \varphi Y) = Ch(Y, X) + \varphi Q\nabla_X Y. \tag{3.2}$$

Since h is a symmetric bilinear form, from (3.2) it follows

$$h(X, \varphi Y) - h(Y, \varphi X) = \varphi Q([X, Y]). \tag{3.3}$$

Thus $[X, Y] \in \Gamma(D \oplus \{\xi\})$ if and only if (3.1) is satisfied. On the other hand we have

$$g([PX, \xi], QY) = 0, \text{ for any } X, Y \in \Gamma(TM). \tag{3.4}$$

In fact, we have

$$g([PX, \xi], QY) = g(-\tilde{\nabla}_{\xi}PX, QY) = g(PX, \tilde{\nabla}_{\xi}QY),$$

and (3.4) follows by using the proof given for Theorem 2.

Now, taking account that $D \oplus D^{\perp}$ is parallel, from Theorem 3 follows

Theorem 4. *Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then the distribution D is involutive if and only if (3.1) is satisfied.*

4. Totally umbilical and totally geodesic semi-invariant submanifolds. Let M be a semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then we say that M is *totally umbilical* if we have

$$h(X, Y) = g(X, Y)H, \tag{4.1}$$

for any X, $Y \in \Gamma(TM)$, where H is the mean curvature vector of M. If the second fundamental form of M vanishes identically on M, then we say that M is a *totally geodesic submanifold*.

Theorem 5. *Any totally umbilical semi-invariant submanifold of a co-Kähler manifold is a totally geodesic submanifold.*

Proof. We take $X = \xi$ in (2.8) and obtain $h(\xi, \xi) = 0$. Since ξ is a unit vector field, from (4.1) follows $H = 0$, that is M is totally geodesic.

Theorem 6. *Let M be a totally geodesic semi-invariant submanifold of a co-Kähler manifold $\widetilde{M}$. Then M is locally the Riemannian product $M_1 \times M_2 \times \gamma$ where M_1 is a leaf of D, M_2 is a leaf of $D^{\perp}$ and γ is a curvature tangent to ξ.*

Proof. It sufficies to prove that D, $D^{\perp}$ and $\{\xi\}$ are parallel distributions with respect to the Levi-Civita connection on M, that is we have

$$\nabla_X PY \in \Gamma(D),\ \nabla_X QY \in \Gamma(D^{\perp}) \text{ and } \nabla_X \xi \in \Gamma(\{\xi\}), \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$. The third relation in (4.2) follows from (1.9). Next, by using (1.3), (1.7) and (1.9) we obtain

$$g(\nabla_X PY, \xi) = -g(PY, \nabla_X \xi) = 0,$$

$$g(\nabla_X PY, QZ) = g(\tilde{\nabla}_X PY, QZ) = g(\varphi\tilde{\nabla}_X PY, \varphi QZ) = g(\tilde{\nabla}_X \varphi PY, \varphi QZ) =$$
$$= g(\nabla_X \varphi PY + h(X, \varphi PY), \varphi QZ) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. Thus $\nabla_X PY \in \Gamma(D)$ for any $X, Y \in \Gamma(TM)$. In a similar way, it follows $\nabla_X QY \in \Gamma(D^{\perp})$ for any $X, Y \in \Gamma(TM)$.

Remarks. 1. The assertion of Theorem 6 is stil true for semi-invariant submanifolds with h satisfying

$$g(h(TM, D), \varphi D^{\perp}) = 0.$$

2. Since any co-Kähler manifold is locally a Riemannian product of a Kähler manifold and a curve, the geometry of semi-invariant submanifolds is closely related to the geometry of CR-submanifolds in a Kähler manifold, (see [1]). However, Theorem 5 is valid only for CR-submanifolds with dim. $D^{\perp} > 1$, (see [2] and [5]).

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FIBER BUNDLES OF FUNCTION ALGEBRAS AND ATTACHED STRUCTURES

BY

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0. Introduction. It is known that a tangent vector at a point x_0 of an n-dimensional C^{∞}-manifold M is a mapping

$$X_{x_0} : \mathcal{F}^{\infty}_{x_0}(M) \to \mathbf{R},$$

satisfying

$$a) \qquad X_{x_0}(\lambda f + \mu g) = \lambda X_{x_0}(f) + \mu X_{x_0}(g),$$

$$b) \qquad X_{x_0}(f \cdot g) = f(x_0) X_{x_0}(g) + g(x_0) X_{x_0}(f).$$

In order to simplify the considerations, let $M = \mathbf{R}$. We shall study the family of mappings verifying *a*) and *b*), but we shall replace $\mathcal{F}^{\infty}_{x_0}(\mathbf{R})$ with other function algebras. An interesting example is that obtained by the use of the family:

$$\Omega_2(x_0) = \{f / f \in \mathcal{F}^{\infty}_{x_0}(\mathbf{R}), f(x_0) = f'(x_0) = 0\}.$$

To the point $x_0 \in \mathbf{R}$ corresponds the vector space of all the functions $X_{x_0} : \Omega_2(x_0) \to \mathbf{R}$, which have the properties *a*) and *b*). Let $T^2_{x_0}$ be this vector space.

Theorem 1. *$T^2_{x_0}$ is a two-dimensional real vector space.*

Proof. It is obvious that $T^2_{x_0}$ is a real vector space.
In order to show that its dimension is two, we note that for each $f, g \in \Omega_2(x_0)$ and any $X_{x_0} \in T^2_{x_0}$ we have $X_{x_0}(f \cdot g) = 0$ in virtue of *b*) because $f(x_0) = g(x_0) = 0$.

Choosing $f \in \Omega_2(x_0)$, we can write

$$f(x) = \frac{(x-x_0)^2}{2!} \cdot f''(x_0) + \frac{(x-x_0)^3}{3!} \cdot f'''(x_0) + \frac{(x-x_0)^4}{4!} \cdot g(x).$$

We have also $[(x-x_0)^4/4!]g(x) = h_1(x) \cdot h_2(x)$ where $h_1(x) = (x-x_0)^2$, $h_2(x) = (x-x_0)^2 \cdot g(x)/4!$ Therefore $X_{x_0}(h_1 \cdot h_2) = 0$.
Further on

$$X_{x_0}(f) = \frac{f''(x_0)}{2!} \cdot X_{x_0}(f_1) + \frac{f'''(x_0)}{3!} \cdot X_{x_0}(f_2)$$

where $f_1(x) = (x-x_0)^2$ and $f_2(x) = (x-x_0)^3$. One can observe that X_{1x_0}: