Proof. It sufficies to prove that D, $D^{\perp}$ and $\{\xi\}$ are parallel distributions with respect to the Levi-Civita connection on M, that is we have

$$\nabla_X PY \in \Gamma(D), \quad \nabla_X QY \in \Gamma(D^{\perp}) \quad \text{and} \quad \nabla_X \xi \in \Gamma(\{\xi\}), \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$. The third relation in (4.2) follows from (1.9). Next, by using (1.3), (1.7) and (1.9) we obtain

$$g(\nabla_X PY, \xi) = -g(PY, \nabla_X \xi) = 0,$$

$$g(\nabla_X PY, QZ) = g(\tilde{\nabla}_X PY, QZ) = g(\varphi \tilde{\nabla}_X PY, \varphi QZ) = g(\tilde{\nabla}_X \varphi PY, \varphi QZ) =$$

$$= g(\nabla_X \varphi PY + h(X, \varphi PY), \varphi QZ) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. Thus $\nabla_X PY \in \Gamma(D)$ for any $X, Y \in \Gamma(TM)$. In a similar way, it follows $\nabla_X QY \in \Gamma(D^{\perp})$ for any $X, Y \in \Gamma(TM)$.

Remarks. 1. The assertion of Theorem 6 is stil true for semi-invariant submanifolds with h satisfying

$$g(h(TM, D), \varphi D^{\perp}) = 0.$$

2. Since any co-Kähler manifold is locally a Riemannian product of a Kähler manifold and a curve, the geometry of semi-invariant submanifolds is closely related to the geometry of CR-submanifolds in a Kähler manifold, (see [1]). However, Theorem 5 is valid only for CR-submanifolds with dim. $D^{\perp} > 1$, (see [2] and [5]).

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FIBER BUNDLES OF FUNCTION ALGEBRAS AND ATTACHED STRUCTURES

BY

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0. **Introduction.** It is known that a tangent vector at a point x_0 of an n-dimensional C^{∞}-manifold M is a mapping

$$X_{x_0} : \mathcal{F}^{\infty}_{x_0}(M) \to \mathbf{R},$$

satisfying

$$a) \qquad X_{x_0}(\lambda f + \mu g) = \lambda X_{x_0}(f) + \mu X_{x_0}(g),$$

$$b) \qquad X_{x_0}(f \cdot g) = f(x_0) X_{x_0}(g) + g(x_0) X_{x_0}(f).$$

In order to simplify the considerations, let $M = \mathbf{R}$. We shall study the family of mappings verifying *a*) and *b*), but we shall replace $\mathcal{F}^{\infty}_{x_0}(\mathbf{R})$ with other function algebras. An interesting example is that obtained by the use of the family:

$$\Omega_2(x_0) = \{f / f \in \mathcal{F}^{\infty}_{x_0}(\mathbf{R}), f(x_0) = f'(x_0) = 0\}.$$

To the point $x_0 \in \mathbf{R}$ corresponds the vector space of all the functions $X_{x_0} : \Omega_2(x_0) \to \mathbf{R}$, which have the properties *a*) and *b*). Let $T^2_{x_0}$ be this vector space.

Theorem 1. *$T^2_{x_0}$ is a two-dimensional real vector space.*

Proof. It is obvious that $T^2_{x_0}$ is a real vector space. In order to show that its dimension is two, we note that for each $f, g \in \Omega_2(x_0)$ and any $X_{x_0} \in T^2_{x_0}$ we have $X_{x_0}(f \cdot g) = 0$ in virtue of *b*) because $f(x_0) = g(x_0) = 0$.

Choosing $f \in \Omega_2(x_0)$, we can write

$$f(x) = \frac{(x-x_0)^2}{2!} \cdot f''(x_0) + \frac{(x-x_0)^3}{3!} \cdot f'''(x_0) + \frac{(x-x_0)^4}{4!} \cdot g(x).$$

We have also $[(x-x_0)^4/4!]g(x) = h_1(x) \cdot h_2(x)$ where $h_1(x) = (x-x_0)^2$, $h_2(x) = (x-x_0)^2 \cdot g(x)/4!$ Therefore $X_{x_0}(h_1 \cdot h_2) = 0$.
Further on

$$X_{x_0}(f) = \frac{f''(x_0)}{2!} \cdot X_{x_0}(f_1) + \frac{f'''(x_0)}{3!} \cdot X_{x_0}(f_2)$$

where $f_1(x) = (x-x_0)^2$ and $f_2(x) = (x-x_0)^3$. One can observe that X_{1x_0}:

$\Omega_2(x_0) \to \mathbf{R}$, defined through $X_{1x_0}(f_1)=2!$ and $X_{1x_0}(f_2)=0$ (i.e. $X_{1x_0}(f)=f''(x_0)$) is an element of $T^2_{x_0}$. $X_{2x_0}:\ \Omega_2(x_0)\to\mathbf{R}$, defined through $X_{2x_0}(f_1)=0$ and $X_{2x_0}(f_2)=3!$ (i.e. $X_{2x_0}(f)=f'''(x_0)$) is an element of $T^2_{x_0}$. The elements of $T^2_{x_0}$ are called the derivatives of the algebra $\Omega_2(x_0)$ at the point x_0. Obviously $\{X_{1x_0},\ X_{2x_0}\}$ is a base for $T^2_{x_0}$.

It is easy to verify that taking the derivatives of the algebra $\Omega_2(x_0)$ at the point $y_0 \neq x_0$ one gets the tangent space of the manifolds $\mathbf{R}$ at y_0.

The elements of the space of all derivatives are the restrictions of the tangent vectors (considered as real functions verifying *a*) and *b*)) to the subalgebra $\Omega_2(x_0)$ of $\mathcal{F}^\infty_{x_0}(\mathbf{R})$. Obviously one can consider the algebras

$$\Omega_k(x_0)=\{f/f\in\mathcal{F}^\infty_{x_0}(\mathbf{R}),\ f(x_0)=f'(x_0)=\ldots=f^{(k-1)}(x_0)=0\}$$

for $k\geqslant 2$. It is immediate that the derivatives at x_0 are generated by

$$\{X_{1x_0},\ X_{2x_0},\ \ldots,\ X_{kx_0}\}\ ;\ X_{1x_0}(f)=f^{(k)}(x_0).\ X_{2x_0}(f)=f^{k+1}(x_0),\ \ldots,\ X_{kx_0}(f)=f^{2k-1}(x_0).$$

Theorem 2. *The real vector space of the derivatives at the point $x_0\in\mathbf{R}$ is k-dimensional. The real vector spaces of the derivatives at the points $y_0\neq x_0$ are one-dimensional.*

The proof is analogous to that of Theorem 1.

1. **The fiber bundle of the derivatives at a point.** As we have seen, the space of derivatives for $\Omega_2(x_0)$ at x_0 is two-dimensional and for $y_0\neq x_0$ it is one-dimensional. In order to have the same dimension, we build up the fiber bundle over $\mathbf{R}$ having the fiber $\Omega_2(x_0)$, i.e. to each $y_0\in\mathbf{R}$ we attach the algebra $\Omega_2(y_0)$. We can easily verify that we obtain a principal fiber bundle (we can call it the fiber bundle of the singularities). We shall designate by Ω_2 the total space.

Let $f\in\mathcal{F}^\infty_{x_0}(\mathbf{R})$. For this f, we shall consider the section s_f: $\mathbf{R}\to\Omega_2$ defined by $s_f(t)=f(x)-f(t)[(x-t)/1!]f'(t)$.

Let $S\Omega_2$ be the set of these sections. Obviously $s_f(t)\in\Omega_2(t)$ and s_f is of class C^∞. In this way we can construct the sheaf of germs of these sections, denote it by $\mathcal{S}\Omega_2$. $\mathcal{S}\Omega_2(t)$ is an algebra. At each point $t\in\mathbf{R}$ we can consider the derivatives of this algebra, i.e. the functions

$$X_t:\ \mathcal{S}\Omega_2(t)\to\mathbf{R},$$

verifying the conditions *a*) and *b*).

For $\lambda,\mu\in\mathbf{R}$ and $\hat{s}_f,\ \hat{s}_g\in\mathcal{S}\Omega_2$ we take

a) $$X_t(\lambda\hat{s}_f(t)+\mu\hat{s}_g(t))=\lambda X_t(s_f(t))+\mu X_t(s_g(t)),$$

b) $$X_t((\hat{s}_f\,.\,\hat{s}_g)(t))=0.$$

At every point t, the set of all derivatives is two-dimensional. In fact, X_t is the derivative for $x=t$ on the fiber $\mathcal{S}\Omega_2(t)=\Omega_2(t)$. Let $L_2(t)$ be the real vector space of derivatives at t. For $X_t\in L_2(t)$ there exist constants $\alpha,\beta\in\mathbf{R}$ such that $X_t=\alpha X_{1t}+\beta X_{2t}$, where $X_{1t}(s_f)=f''(t)$, $X_{2t}(s_f)=f'''(t)$.

Let $\mathcal{L}_2=\{(t,\ X_t)/t\in\mathbf{R},\ X_t\in L_2(t)\}$.

Theorem 3. *$\mathcal{L}_2$ is a principal fiber bundle with base $\mathbf{R}$ and fiber of type $\mathbf{R}^2$.*

Proof. $L_2(t)$ is isomorphic with $\mathbf{R}^2$, $X_{1t}\mapsto(1,0)$ and $X_{2t}\mapsto(0,1)$ being the correspondence between the bases. By this isomorphism, $\mathcal{L}_2$ can be endowed with the product topology. Therefore $\mathcal{L}_2$ becomes a trivial fiber bundle.

Remark. If we consider a variable t, then X_t is not a derivative on the algebra of sections $S\Omega_2$; in fact $X_{1t}(s_f(t))=f''(t)\notin\Omega_2(t)$ (the fiber at t) and $X_{2t}(s_f(t))=f'''(t)\notin\Omega_2(t)$.

If we consider

$$\mathscr{X}_1(s_f)(t)=[s_f''(t)]_{x^2}\,.\,(x-t)^2\ ;\ \mathscr{X}_2(s_f)(t)=[s'''f(t)]_{x^3}\ (x-t)^3$$

then we get derivations on $S\Omega_2$ or on $\mathcal{S}\Omega_2$.

Analogously if we consider the fiber bundle with the fibers $\Omega_k(t)$, we shall denote by Ω_k the total space of this fiber bundle. Let $\mathcal{S}\Omega_k$ be the sheaf of sections built up using the functions $f\in\mathcal{F}^\infty(\mathbf{R})$. The section $s_f:\mathbf{R}\to\Omega_k$ is defined through $s_f(t)=f(x)-f(t)-[(x-t)/1]f'(t)-\ldots-[(x-t)^{k-1}/(k-1)f^{(k-1)}(t)$.

We can also consider the derivatives at each $t\in\mathbf{R}$. Let $L_k(t)$ be the real vector space of the derivatives at $t\in\mathbf{R}$.

We denote by $\mathcal{L}_k=\{(t,\ X_t)/t\in\mathbf{R},\ X_t\in L_k(t)\}$.

Theorem 4. *$\mathcal{L}_k$ is a fiber bundle having base $\mathbf{R}$ and fiber of type $\mathbf{R}^k$.*

The proof is obvious.

In this case we remark also that the elements $X_t\in L_k(t)$ do not generate derivatives on $S\Omega_k$ if t is variable. We can construct for this case the maximal independent derivatives on $S\Omega_k$ denoted with $\mathscr{X}_1,\ \ldots,\ \mathscr{X}_k$ namely

$$\mathscr{X}_1(s_f)(t)=f^{(k)}(t)\,.\,(x-t)^k,\ \mathscr{X}_2(s_f)(t)=f^{(k+1)}(t)\,.\,(x-t)^{k+1},\ \ldots,\ x_n(s_f)(t)=$$
$$=f^{(2k-1)}(t)\,.\,(x-t)^{2k-1}.$$

In this way we get fiber bundles attached to $\mathbf{R}$ having the fiber of type $\mathbf{R}^k$. For $k=1$ we get the tangent fiber bundle.

2. **Connections between the fiber bundles $\mathcal{L}_k(\mathbf{R})$.** At each $t\in\mathbf{R}$ we can consider the fibers of $\mathcal{L}_k(\mathbf{R})$ $\mathcal{L}_m(\mathbf{R})$; let they be $L_k(t)$ and $L_m(t)$ respectively. They are generated by

$$\left\{\frac{d^k}{dx^k},\ \frac{d^{k+1}}{dx^{k+1}},\ \ldots,\ \frac{d^{2k-1}}{dx^{2k-1}}\right\}\ \text{and}\ \left\{\frac{d^m}{dx^m},\ \frac{d^{m+1}}{dx^{m+1}},\ \ldots,\ \frac{d^{2m-1}}{dx^{2m-1}}\right\}.$$

The two spaces have nonempty intersection, namely the subspace generated by $\{d^{h+1}/dx^{h+1},\ \ldots,\ d^{h+r}/dx^{h+r}\}$, where $\{h+1,\ \ldots,\ h+r\}=\{k,\ \ldots,\ 2k-1\}\cap\{m,\ldots,\ 2m-1\}$.

We can define the mappings $\varphi_{k,m}:L_k(t)\to L_m(t)$, $k\leqslant m$ by $\varphi_{k,m}(X)=i_m\circ \mathrm{pr}_{L_k(t)\cap L_m(t)}\,.(X)$.

One verifies immediately that $\varphi_{k,m}\circ\varphi_{m,h}=\varphi_{k,h}$, for every $k\leqslant m\leqslant h$. We have also $\varphi_{k,k}=\mathrm{id}_{L_k(t)}$.

Let $\overline{L}(t)=\dot{\bigcup}_{k\in N} L_k(t)$ (disjoint union). For $X,Y\in\overline{L}(t)$, $X\in\overline{L}_k(t)$ and $Y\in L_m(t)$, with $k\leqslant m$, we shall say that X is equivalent to Y if $\varphi_{k,m}(X)=Y$. We denote this equivalence with θ. The quotient space $\overline{L(t)}/\theta$ will be denoted by $L(t)$; it is the total space of derivatives at t. We have the canonical inclusions $\varphi_k : L_k(t) \to L(t)$.

The vector space $L(t)$ can be defined also using the maps $\psi_{m,k} : L_m(t) \to \to L_k(t)$, $k \leqslant m$ defined by $\psi_{m,k}(X) = i_k \circ \mathrm{pr}_{L_m(t) \cap L_k(t)} (X)$.

Remarks. In the definition of $\Omega_t(x_0)$ the condition $f(x_0)=0$ is not essential. Only the conditions $f'(x_0)=\ldots=f^{(k-1)}(x_0)=0$ $(k \geqslant 2)$ are significant and this since on every function algebra $A \subseteq \mathcal{F}^\infty(\mathbf{R})$, every derivative $X : A \to \mathbf{R}$ has the property : $f \in A$ and $f(x) =$ const. $\Rightarrow X(f)=0$.

3. **Function algebras attached to a vector field.** Let M be a differentiable manifold and $X \in \mathcal{X}(M)$ a vector field on it, such that $X_p \neq 0$ for any $p \in M$.

Let $\Omega_X(p) = \{f / f \in \mathcal{F}_p^\infty(M),\ f(p) = X_p(f) = 0\}$. Obviously $\Omega_X(p)$ is an algebra and the derivatives on this algebra form a real vector space. Moreover it follows that :

Theorem 5. *The space $L_X(p)$ of the derivatives is $2n$-dimensional; where $n = \dim M$, and there exists a coordinate neighbourhood on M such that the space of derivatives has the base*

$$\{\partial/\partial x^2, \ldots, \partial/\partial x^n,\ \partial^2/\partial(x^1)^2,\ \partial^2/\partial x^1 \partial x^2, \ldots, \partial^2/\partial x^1 \partial x^n,\ \partial^3/\partial(x^1)^3\}.$$

Proof. There is a coordinate neighbourhood (U, φ), $p \in U$ such that $X = \partial/\partial x^1$. Using the same computing techniques as in the case $M = \mathbf{R}$, it follows the assertion of the theorem.

Analogously with the case $M = \mathbf{R}$, considering $\Omega_X = \{(p, f) / f \in \Omega_X(p)\}$ one obtains.

Theorem 6. *The triplet (Ω_X, π, M), where π is the first projection, is a principal fiber bundle.*

Further, for each $f \in \mathcal{F}_p^\infty(M)$, we consider the sections $s_f : M \to \Omega_X$ defined by $s_f(p) = f(q) - f(p) - X_p(f)$. Obviously $s_f(p) \in \Omega_X(p)$.

Let us consider the sheaf of germs of this sections ; let $\mathcal{S}\Omega_X$ be the total space. The derivatives on the fibers $\mathcal{S}\Omega_X(p) = \Omega_X(p)$ are $2n$-dimensional vectors in virtue of theorem 5. For the point p let us denote it by $L_{2n}(p)$. Denote $\mathcal{L}_{2n}(X) = \{(p, D_p),\ p \in M,\ D_p \in L_{2n}(p)\}$.

Theorem 7. *The triplet $(\mathcal{L}_{2n}(X), \pi, M)$, where π is the first projection defined on $\mathcal{L}_{2n}(X)$, is a principal fiber bundle with the fiber type $\mathbf{R}^{n+1}$.*

Proof. We choose the coordinate neighbourhoods in which the vector field X can be written $X = \partial/\partial x^1$. In there coordinate neighbourhoods $\mathcal{L}_{2n}(X)$ has the base

$$\partial/\partial x^2, \ldots, \partial/\partial x^n,\ \partial^2/\partial(x^1)^2,\ \partial^2/\partial x^1 \partial x^2, \ldots, \partial^2/\partial x^1 \partial x^n,\ \partial^3/\partial(x^1)^3$$

hence $L_{2n}(p)$ is isomorphic to $\mathbf{R}^{2n}$. If (U, φ) and (V, ψ) are the coordinate neighbourhoods with the local coordinates $(x^1, \ldots, x^n)$ resp. $(y^1, \ldots, y^n)$ such that $X = \partial/\partial x^1$ and $X = \partial/\partial y^1$ in these coordinate neighbourhoods for each $q \in U \cap V$, the induced coordinate transformation for $L_{2n}(p)$ has the matrix

$$\begin{pmatrix}
\frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^2} & 0 & 0 & \cdots & 0 & 0 \\
\frac{\partial y^2}{\partial x^3} & \frac{\partial y^3}{\partial x^3} & \cdots & \frac{\partial y^n}{\partial x^3} & 0 & 0 & & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial y^2}{\partial x^n} & \frac{\partial y^3}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} & 0 & 0 & & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & & 0 & 0 \\
\frac{\partial^2 y^2}{\partial x^1 \partial x^2} & \frac{\partial^2 y^3}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 y^n}{\partial x^1 \partial x^2} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & & \frac{\partial y^n}{\partial x^2} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2 y^2}{\partial x^1 \partial x^n} & \frac{\partial^2 y^3}{\partial x^1 \partial x^n} & \cdots & \frac{\partial^2 y^n}{\partial x^1 \partial x^n} & \frac{\partial y^1}{\partial x^n} & \frac{\partial y^2}{\partial x^n} & & \frac{\partial y^n}{\partial x^n} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

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