Theorem 1. *If a Riemannian manifold (M^n, g) $(n>3)$ admits a semi-symmetric connection whose curvature tensor is a recurrent tensor with respect to the Levi-Civita connection having B as its associated 1-form, then the manifold (M^n, g) is conformally recurrent with B as 1-form of recurrence.*

If, in particular, B is zero, then the curvature tensor of ∇ becomes covariant constant with respect to the Levi-Civita connection and consequently (M^n, g) becomes conformally symmetric[3]. Hence we have the following corollary:

Corollary. If a Riemannian manifold (M^n, g) $(n>3)$ admits a semi-symmetric metric connection whose curvature tensor is covariant constant with respect to the Levi-Civita connection, then (M^n, g) is conformally symmetric.

The case considered in the above corollary was treated by T. Imai [4]. From (2.18) we get

$$(\operatorname{div} C)(X, Y)Z = B(C(X, Y)Z). \tag{2.19}$$

But

$$(\operatorname{div} C)(X, Y)Z = \frac{n-3}{n-2}\Big[(\overline{\nabla}_X S)(Y, Z) - (\overline{\nabla}_Y S)(X, Z) - \frac{1}{2(n-1)}\{g(Y, Z)\overline{\nabla}_X r - g(X, Z)\overline{\nabla}_Y r\}\Big]$$

Hence (2.19) can be written as

$$B(C(X, Y)Z) = \frac{n-2}{n-3}\Big[[(\overline{\nabla}_X S)(Y, Z) - (\overline{\nabla}_Y S)(X, Z) - \frac{1}{2(n-1)}\{g(Y, Z)\overline{\nabla}_X r - g(X, Z)\overline{\nabla}_Y r\}\Big] \tag{2.20}$$

From (2.20) it follows that r is constant if and only if

$$(\overline{\nabla}_X S)(Y, Z) - (\overline{\nabla}_Y S)(X, Z) = (n-2)/(n-3)B(C(X, Y)Z). \tag{2.21}$$

Hence we can state the following theorem:

Theorem 2. *If a Riemannian manifold (M^n, g) $(n>3)$ admits a semi-symmetric metric connection whouse curvature tensor is a recurrent tensor with respect to the Levi-Civita connection having B as the associated 1-form, then (M^n, g) is of constant scalar curvature if and only if* (2.21) *holds.*

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NEW CLASSES OF QUASICONFORMAL MAPPINGS IN R^n (I)

BY

PETRU CARAMAN

To Professor Adolf Haimovici on the occasion of his 70-th birthday

The most important tools in the theory of qc (quasiconformal mappings) in the euclidean n-space R^n are the conformal capacity, the module and the extremal length. They correspond to the p-capacity, the p-module and the p-extremal length for $p=n$. In the case $1<p<\infty (p\neq n)$, one obtains a subclass — the (K, p)-qc — of the class of qc (corresponding to $p=n$). Some years ago [4], we started the generalization of the theory of qc in the infinite dimensional spaces. When we investigated the possibility of extending to such a space the concepts of conformal capacity, module and extremal length, it appeared to us, as the only natural way, to consider something like p-capacity, p-module and p-extremal length with $p=\infty$. But, since the corresponding classes of qc have been studied neither in the finite dimensional case, we think it will be interesting to study them previously in R^n, establishing their inclusion in the class of qc and this will be the topic of the first part (§1) of our paper.

In the second part (§2)., we give four equivalent definitions of the (K, p)-qc. We show that the class (p)-qc — corresponding to a fixed $p\neq n$ and all the $K\in[1, \infty)$ — is contained in the class of the qc and the inclusion is strict. Next, we establish that, if $1<p\leqslant p'\leqslant 2$, the class of (K, p)-qc is contained in the class of (K, p') and that there is no inclusion relation between such classes if p, p' do not satisfy the preceding inequality. Finally, we obtain an analogous result in the case of a class of (K, p)-qc introduced by F. Gehring [9] by means of the p-capacity, i.e. we prove that his class of (K, p)-qc is contained in his class of (K, p')-qc if $2(n-1)\leqslant p\leqslant p'$ and otherwise, there is no inclusion relation.

§1. **(K, ∞)-quasiconformal mappings.** Let us introduce first the concepts of ∞-module, ∞-extremal length and ∞-capacity.

Let Γ be an arc family of R^n and $F(\Gamma)$ the class of admissible functions $\rho\geqslant 0$ Borel measurable and such that $\int_\gamma \rho dH^1 \geqslant 1\ \forall\gamma\in\Gamma$ ($\forall$ means „for every"), where H^1 is linear Hausdorff measure. The ∞*-module* of Γ is

$$M_\infty\Gamma = \inf_{\rho\in F(\Gamma)} \operatorname{ess\,sup}_{R^n} \rho(x), \quad \text{where} \quad \operatorname{ess\,sup}_{R^n} \rho(x) = \inf_E\,[\sup_{R^n - E} \rho(x)],$$

the infimum being taken over all the null-sets (i.e. with n-dimensional Lebesgue measure zero).

In the particular case in which the arcs of Γ are contained in a domain D and ρ satisfies also the additional condition of being continuous in D and bounded in R^n, the corresponding class of admissible functions will be denoted be $F^{\circ D}(\Gamma)$ and the corresponding module by

$$M_\infty^{\circ D}(\Gamma)=\inf_{\rho\in F^{\circ D}(\Gamma)}\sup_{R^n}\rho(x)=\inf_{\rho\in F^{\circ D}(\Gamma)}\sup_{D}\rho(x).$$

The ∞-*extremal length* in the 2 casses will be $\lambda_\infty(T)=1/M_\infty(\Gamma)$, $\lambda_\infty^{\circ D}=1/M_\infty^{\circ D}(\Gamma)$.

In a similar way, we shall define the ∞-capacity. Let Δ be a Jordan domain and $F\subset\Delta$ a closed set, then the ∞-*capacity* is

$$\operatorname{cap}_\infty(\Delta, F, \partial\Delta)=\inf_u \operatorname{ess}\sup_{R^n}|\nabla u(x)|,$$

where ∇u is the gradient of u and the infimum is taken over the class $\mathcal{U}$ of the functions u admissible for $\operatorname{cap}_\infty(\Delta, F, \partial\Delta)$, characterized by the following properties: u satisfies a local Lipschitz condition in Δ, it is continuous in $\bar\Delta$, $0\leq u\leq 1$, $u_{|F}=1$ and $u_{|C\Delta}=0$, where $C\Delta$ means the complement of Δ. If Δ is a fixed ball $B(R)$ centred at the origin O and with a sufficiently large radius R, we use the notation $\operatorname{cap}_\infty F=\operatorname{cap}_\infty[B(R), F, S(R)]$, where $S(R)=\partial B(R)$ (∂E means the boundary of E).

In order to justify the definition of the ∞-module, let us consider for the p-module the definition proposed by F. Gehring [8] in the case of the rings for $p=n$ (not Väisälä's definition [11]), with the additional condition that the admissible functions ρ are supposed to be continuous and bounded in R^n, i.e.

$$(1)\qquad M_p\Gamma=\inf_{\rho\in F^\circ(\Gamma)}\left(\int\rho^p dm\right)^{\frac{1}{p-1}}=\left[\inf_{\rho\in F^\circ(\Gamma)}\left(\int\rho^p dm\right)^{\frac{1}{p}}\right]^{\frac{p}{p-1}},$$

then, letting $p\to\infty$ and taking into account lemma 2 of our paper [5].

$$M_\infty\Gamma=\lim_{p\to\infty}M_p\Gamma=\lim_{p\to\infty}\left[\inf_{\rho\in F^\circ(\Gamma)}\left(\int\rho^p dm\right)^{\frac{1}{p}}\right]^{\frac{p}{p-1}}=\inf_{\rho\in F^\circ(\Gamma)}\sup_{R^n}\rho(x),$$

if the arcs $\gamma\in\Gamma$ join two sets $E_1, E_2\in\partial D$ in D.

In a similar way, if F is a continuum of $B(R)$, then, lemma 3 of [5] yields

$$\operatorname{cap}_\infty F=\lim_{p\to\infty}\operatorname{cap}_p F=\lim_{p\to\infty}\inf_{\mathcal{U}}\left(\int|\nabla u|^p dm\right)^{\frac{1}{p}}=\inf_{\mathcal{U}}\operatorname{ess}\sup_{R^n}|\nabla u(x)|.$$

Let us recall that a *ring* $A\subset R^n$ is a domain homeomorphic to a spherical ring $\{x; a<|x|<b\}$. CA consists of two components C_0, C_1, with C_0 bounded. Let us denote by $\Gamma_A=\Gamma(C_0, C_1, A)$ the family of the arcs joining C_0 and C_1 in A.

Proposition 1. *If $A=\{x; a<|x|<b\}\subset R^n$ and ω_n is the volume of the unit ball, then*

$$M_p\Gamma_A=\frac{(n\omega_n)^{\frac{1}{p-1}}}{\frac{p-1}{p-n}\left(b^{\frac{p-n}{p-1}}-a^{\frac{p-n}{p-1}}\right)}.$$

(lemma 4 of our paper [2]).

Corollary. *In the hypotheses of the preceding proposition,* $M_\infty\Gamma_A=\lim_{p\to\infty}M_p\Gamma_A=1/(b-a)$.

Lemma 1. *If $A\subset R^n$ is a ring, then*

$$(2)\qquad M_\infty^{\circ A}\Gamma_A=\frac{1}{d_A(C_0, C_1)}.$$

Indeed, $\forall\rho\in F^{\circ A}(\Gamma_A)$ and $\forall\gamma\in\Gamma_A$, we have $1\leq\int_\gamma\rho\, dH^1\leq\sup_A\rho H^1(\gamma)$, hence $\sup_A\rho\geq 1/H^1(\gamma)$, but, since this inequality holds $\forall\gamma\in\Gamma_A$, taking the infimum for all such γ, we obtain $\sup_A\rho\geq 1/\inf_\gamma H^1(\gamma)=1/d_A(C_0, C_1)$, where, if $E_1, E_2\subset\bar D$, then $d_D(E_1, E_2)$ means the relative distance between the sets E_1 and E_2, i.e. the infimum of the length of the arcs $\gamma\in\Gamma(E_1, E_2, D)$ and, since this in equality holds $\forall\rho\in F^{\circ A}(\Gamma_A)$, it follows that

$$(3)\qquad M_\infty^{\circ A}\Gamma_A=\inf_{\rho\in F^{\circ A}(\Gamma_A)}\sup_A\rho\geq 1/d_A(C_0, C_1).$$

On the other side, the function ρ, such that $\rho_{|A}(x)=1/d_A(C_0, C_1)$ and $\rho_{|CA}(x)=0$, belongs to $F^{\circ A}(\Gamma_A)$, and then

$$(4)\qquad M_\infty^{\circ A}\Gamma_A=\inf_{\rho\in F^{\circ A}(\Gamma_A)}\sup_A\rho\leq 1/d_A(C_0, C_1).$$

But, (3) and (4) yield (2), as desired.

Let us define now the subclasses of K-qc by means of M_∞, λ_∞ and $\operatorname{cap}_\infty$.

A homeomorphism $f: D\rightleftarrows D^*$ is said to be K-qc ($1\leqq K<\infty$) if for every Jordan domain $\Delta\subset D$ and every closed set $F\subset\Delta$, the following double inequality

$$(5)\qquad M_\infty\Gamma_F/K\leqq M_\infty\Gamma_F^*\leqq KM_\infty\Gamma_F$$

holds for all families $\Gamma_F=\Gamma(F, \partial, \Delta, \Delta)$, where $\Gamma_F^*=f(\Gamma_F)$. By taking λ_∞ instead of M_∞, we obtain an equivalent definition of K qc.

A homeomorphism $f: D\rightleftarrows D^*$ is called K-qc if $\forall\Delta\subset D$ and $\forall F\subset\Delta$ (from above), the following double inequality

$$\frac{\operatorname{cap}_\infty(F, \partial\Delta, \Delta)}{K}\leq\operatorname{cap}_\infty(F^*, \partial\Delta^*, \Delta^*)\leqq K\operatorname{cap}_\infty(F, \partial\Delta, \Delta)$$

holds for all Γ_F.

Arguing as in theorems 2 and 4 of our paper [5], we obtain the following two lemmas:

Lemma 2. *If $D\subset B(R)\subset R^n$, $E_1, E_2\subset\partial D$ and $\Gamma=\Gamma(E_1, E_2, D)$ then*

$$M_\infty \Gamma = M_\infty^{\cdot D} \Gamma. \tag{6}$$

Lemma 3. *If F is a closed subset of a Jordan domain $\Delta \subset R^n$ then*

$$M_\infty \Gamma(F, \partial\Delta, \Delta) = \operatorname{cap}_\infty(F, \partial\Delta, \Delta).$$

Corollary. *The three preceding definitions of K-qc are equivalent (with the same K).*

A homeomorphism $f: D \rightleftarrows D^*$ is said to be K-*qc according to Gehring's metric definition* if $\delta_L(x) = \varlimsup\limits_{r \to 0} \dfrac{L(x, r)}{l(x, r)}$ is bounded in D and $\delta_L(x) \leqq K$ a.e. (almost everywhere) in D (i.e. except a null-set), where $L(x, r) = \sup\limits_{\|x'-x\|=r} \|f(x') - f(x)\|$, $l(x, r) = \inf\limits_{\|x'-x\|=r} \|f(x') - f(x)\|$.

Theorem 1. *A qc according to one of the first three definitions is qc according to Gehring's metric definition and then also according to the other definitions.*

On account of the equivalence of the first three definitions, it is enough to consider only a mapping $f: D \leftrightarrows D^*$ qc for instance according to the first definition. We remind that a qc is a K-qc with non-specified K. For the necessity of the proof, we shall fix such a constant as being K. Let us show that $\delta_L(x)$ is bounded in D. Let $x \in D$, $B(x, r) \subset D$ be an n-dimensional ball centred at x and with the radius r, $L = L(x, r)$, $l = l(x, r)$, $A^* = \{x^* \in D^*;\ l < [\, x^{*\prime} - x^* \,[\ < L\}$ a spherical ring of D^*, $A = f^{-1}(A^*)$ and x^1 a point where the surfaces $f^{-1}[S[(x^*, l)]$ and $S(x, r)$ are tangent. Let us denote by E the intersection of A and the line through the points x, x^1 and by E_1, E_2 the two line segments contained in E and such that E_1 is outside $S(x, r)$ an has an endpoint at x^1, while $E_2 = E - E_1$. Clearly, $d_A(E_1, E_2) \geqq r$. On the other side, it is easy to see that $d_{A^*}(E_1^*, E_2^*) \leqq \pi l$, where $E_k^* = f(E_l)$ $(k = 1, 2)$. Taking into account all this, the K-qc of f and lemma 1, we deduce that

$$\frac{1}{\pi l(x, r)} \leqq \frac{1}{d_{A^*}(E_1^*, E_2^*)} = M_\infty \Gamma^* \leqq K M_\infty \Gamma = \frac{K}{d_A(E_1, E_2)} \leqq \frac{K}{r},$$

implying $l(x, r)/r \geqq 1/K\pi > 0$ and taking the inferior limit for $r \to 0$, we obtain

$$\lambda_f(x) = \varliminf_{x' \to x} | f(x') - f(x) |\, x' - x \,| = \varliminf_{r \to 0} l(x, r)/r \geqq 1/K\pi > 0. \tag{7}$$

But, from the symmetry of the first three definitions, the inverse f^{-1} is K-qc too, so that

$$\frac{1}{K\pi} \leqq \lambda_{f^{-1}}(x^*) = \varliminf_{x^{*\prime} \to x^*} \frac{|f^{-1}(x^{*\prime}) - f^{-1}(x^*)|}{|x^{*\prime} - x^*|} = \lim_{x \to x'} \frac{|x' - x|}{\|f(x') - f(x)\|} \leqq$$

$$\frac{1}{\varlimsup\limits_{x' \to x} \dfrac{|f(x') - f(x)|}{|x' - x|}} = \frac{1}{\Lambda_f(x)}$$

which, together with (7), yields $\delta_L(x) \leqq \Lambda_f(x)/\lambda_f(x)$ $K^2\pi^2$, as desired.

Let us remind that an arc family Γ is said *to minorize a family* Γ_1 if $\forall \gamma_1 \in \Gamma_1$, there exists a $\gamma \in \Gamma$ such that $\gamma \subset \gamma_1$; then we write $\Gamma < \Gamma_1$.

Lemma 4. *If $\Gamma < \Gamma_1$, then $M_\infty \Gamma_1 \leqq M_\infty \Gamma$.*

Indeed, $\Gamma < \Gamma_1$ implies $F(\Gamma) \subset F(\Gamma_1)$, hence

$$M_\infty \Gamma_1 = \inf_{\rho \in F(\Gamma_1)} \operatorname{ess\,sup}_{R^n} \rho(x) \leqq \inf_{\rho \in F(\Gamma)} \operatorname{ess\,sup}_{R^n} \rho(x) = M_\infty \Gamma.$$

Lemma 5. *If the ring A separates the boundary components of the ring A_1, then $M_\infty \Gamma_{A_1} \leqq M_\infty \Gamma_A$.*

This lemma is a direct consequence of the preceding one, since clearly, $\Gamma_A < \Gamma_{A_1}$.

Theorem 2. *A K-qc $f: D \leftrightarrows D^*$ according to one of the first three definitions is K^2-qc according to Gehring's metric definition.*

From the preceding theorem, we deduce that f is qc according to Gehring's metric definition (and then also according to the other equivalent to it). But this implies the differentiability of f with $J(x) \neq 0$ a.e. in D. Let x^0 be such a point and let us consider the spherical ring A bounded by $S(x^0, r)$ and $S(x^0, 2r)$. For simplicity sake and without loss of generality, we may choose the coordinate axes in R^n corresponding to D and to D^* such that $x^0 = f(x^0) = O$ and that f be in a neighbourhood of the form

$$f(x) = \sum_{k=1}^{n} a_k x_k e_k + \varepsilon(O, |x|)\, |x|, \tag{8}$$

where e_k is the unit vector of the axis x_k and $\varepsilon(O, x) \to 0$ as $x \to 0$. But then, the affine transformation

$$f_1(x, O) = f(O) + f'(O)\, x = \sum_{k=1}^{n} a_k x_k e_k \tag{9}$$

will map $S(r)$ into an ellipsoid E_r centred at O and with the semiaxes $a_1(r) \geqslant \geqslant \ldots\ a_n(r) > 0$.

Given $\varepsilon > 0$, for r sufficiently small, if $x \in S(r)$, $y \in S(2r)$, then (8) implies $|f(x) - f_1(x, O)| < \varepsilon |x| = \varepsilon r$, $|f(y) - f_1(y, O)| < \varepsilon |y| = 2\varepsilon r$. Let us suppose that ε has been chosen sufficiently small in order that $a_n - 4\varepsilon > 0$. Then, the ring $A^{*\prime}$ — having as boundary components the elipsoids E'_r, E''_r homothetical to E_r and with the semi-axes equal to $(a_n - \varepsilon)r$ and $2(a_n + \varepsilon)\, r$, respectively — will have its boundary components separated by the ring $A^* = f(A)$.

Next, from the definition of f and on account of the preceding lemma, we deduce that

$$\frac{K}{r} = K M_\infty \Gamma_A \geqq M_\infty \Gamma_{A^*} \geqq M_\infty \Gamma_{A^{*\prime}} = \frac{1}{d_{A^{*\prime}}(C_0^{*\prime}, C_1^{*\prime})} = \frac{1}{d(C_0^{*\prime}, C_1^{*\prime})} = \frac{1}{(a_n + 3\varepsilon) r},$$

where by $C_0^{*\prime}$, $C_1^{*\prime}$, we denoted the connected components of $CA^{*\prime}$ and from this sequence of inequalities, it follows that $a_n + 3\varepsilon \geqslant 1/K$, hence, letting $\varepsilon \to 0$, we obtain $a_n \geqslant 1/k$, hence, on account of theorem 14 of our paper [1],

$$\lambda_f(O) = \inf_s \left| \frac{\partial f(O)}{\partial s} \right| = a_n(O) \geq \frac{1}{K}, \tag{10}$$

and then

$$\lambda_f(x^0) \geq \frac{1}{K}, \tag{11}$$

for almost every x^0 in D. But from the symmetry of the first three definitions it follows that also the inverse f^{-1} is K-qc according to the first three definitions, and, on account of the preceding theorem, is qc according to Gehring's metric definition, hence, f^{-1} will be differentiable with $J(x) \neq 0$ a.e. in D^*. But, from theorem 1, chap. 1, part. II, p. 93 of our monograph [3], we deduce that f and f^{-1} are measurable, allowing us to conclude that almost all the points $x \in D$ will have the property that x and $x^* = f(x)$ are A-points of f and f^{-1}, respectively. If x^0 is such a point, then, from the relations of the form (10) and (11), but with respect to the inverse mapping f^{-1}, we obtain that $\lambda_{f^{-1}}(x^{*0}) = a_n^* \geq 1/K$, which implies

$$\Lambda_f(x^0) = \overline{\lim_{x \to x^0}} \frac{|f(x) - f(x^0)|}{|x - x^0|} = \overline{\lim_{x^* \to x^{*0}}} \frac{|x^* - x^{*0}|}{|f^{-1}(x^*) - f^{-1}(x^{*0})|} =$$

$$= \frac{1}{\underline{\lim}_{x^* \to x^{*0}} \dfrac{|f^{-1}(x^*) - f^{-1}(x^{*0})|}{|x^* - x^{*0}|}} = \frac{1}{\lambda_{f^{-1}}(x^{*0})} \leq K, \tag{12}$$

which, together with (11), yields

$$\delta_L(x^0) = \overline{\lim_{r \to 0}} \frac{L(x^0, r)}{l(x^0, r)} = \overline{\lim_{r \to 0}} \frac{\dfrac{L(x^0, r)}{r}}{\dfrac{l(x^0, r)}{r}} \leq \frac{\overline{\lim}_{r \to 0} \dfrac{L(x^0, r)}{r}}{\underline{\lim}_{r \to 0} \dfrac{l(x^0, r)}{r}} =$$

$$= \frac{\overline{\lim}_{x \to x^0} \dfrac{|f(x) - f(x^0)|}{|x - x^0|}}{\underline{\lim}_{x \to x^0} \dfrac{|f(x) - f(x^0)|}{|x - x^0|}} = \frac{\Lambda_f(x^0)}{\lambda_f(x^0)} \leq K^2.$$

Since the preceding inequalities hold for almost all the points of D, it follows that f is K^2-qc in D according to Gehring's metric definition.

Corollary. *A conformal mapping according to the first three definitions is conformal also according to Gehring's metric definition.*

This is a direct consequence of the preceding theorem since a conformal mapping is 1-qc, and then $K = 1$, hence also $K^2 = 1$.

Theorem 3. *If f is conformal in D according to one of the first three definitions and $x^0 \in D$, then the principal part $f_1(x, x^0)$ of f will be the product (in the functional sense) between a rotation and a translation.*

Indeed, by a convenient rotation and translation, $f_1(x, x^0)$ may be written under the form (9), where, on account of (10), $a_n \geq 1$; but, on the other side, on account of (12) and of theorem 14 of our paper [1], $a_1 = \sup_s |\partial f(O)/\partial s| = \Lambda_f(O) \leq 1$, hence, since $a_1 \geq \ldots \geq a_n$, it follows that $a_1 = \ldots = a_n = 1$ and then $f_1(x, O) = x$, i.e. $f_1(x, O)$ is the identity mapping.

Theorem 4. *The class of qc according to the first three definitions is contained in the class of qc according to Gehring's metric definition (and then also in all the classes obtained by an equivalent definition) and the inclusion is strict.*

The first part of the theorem is a direct consequence of the preceding theorem.

In order to establish the second part, we shall construct a counterexample. Let D be the ring $\{x; 0 < |x| < R\}$ $(R < 1)$ and f the inversion with respect to the unit sphere centred at O (which is a conformal mapping according to Gehring's metric definition). Next, let us consider the ring $A = \{x; r < |x| < 2r < R\}$ and let us calculate the ∞-module for A and for its image $A^* = f(A)$. Clearly,

$$M_\infty \Gamma_A = \frac{1}{2r - r} = \frac{1}{r}, \quad M_\infty \Gamma_{A^*} = \frac{1}{\dfrac{1}{r} - \dfrac{1}{2r}} = 2r.$$

Hence, letting $r \to 0$, $M_\infty \Gamma_A \to \infty$, while $M_\infty \Gamma_{A^*} \to 0$, so that there is no constant $K < \infty$ for which inequality (5) be verified.

Remark. The example from above shows that there exist not only qc, but even conformal mappings (according to Gehring's metric definition and to all the other equivalent to it), which are not qc according to the first 3 definitions.

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MORPHISMS OF H-CONES

BY

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We introduce in this paper a notion of morphism of H-cones, and present some of its properties. §1 contains the construction of a „cone of hyperharmonics" $\tilde{C}$, associated with an H-cone C. Some properties of the cone $\tilde{C}$ are presented, and $\tilde{C}$ is computed when C is the dual of an H-cone or is an H-cone of functions. The last case justifies also the terminology. §2 contains the definition of a morphism of H-cones, as an: additive, increasing and continuous in order from below map from $\tilde{C}_1$ to $\tilde{C}_2$. We establish completely the relations with the notions of morphisms of H-cones, as introduced in [1] and [3]. The main interest of the notion proposed here is that we get a category (i.e. the composition of two morphisms, if it exists, is still a morphism). Also, some results about isomorphisms of H-cones are included.

In § 3, the class of morphisms having adjoints is considered. We prove that this class is stable under composition and taking adjoints and it contains the most important examples of morphisms. Finally, §4 presents the morphisms of H-cones of functions, which are introduced by maps between the base spaces.

All the definitions and results about H-cones, are as in [1].

This work develops the first three parts of [4].

§1. **The cone of hyperharmonics.** We begin with the construction of an ordered convex cone $\tilde{C}$, associated with an H-cone C. When C is the cone of positive superharmonic functions of a $\mathcal{D}$ harmonic space [2], then $\tilde{C}$ is the cone of positive, hyperharmonic functions. It is this cone $\tilde{C}$ which will be used in the definition of the morphisms of H-cones.

Definition 1.1. *Let C be an H-cone. C^0 will denote the set of all maps $\mu : C \to \overline{\mathbf{R}}_+$, such that there exists an upper directed family $(\mu_i)_{i\in I}$ of H-integrals on C, for which:*

$$\mu(s) = \sup_{i\in I} \mu_i(s), \quad \forall s \in C.$$

It is clear that any $\mu \in C^0$ is additive, increasing and continuous in order from below; but not any map $C \to \overline{\mathbf{R}}_+$, having these properties, is in C^0

Proposition 1.2. *With the operations and order defined pointwise (and the convention $0.\infty=0$), C^0 is a complete lattice and an ordered convex cone. C^0 contains C^* as a solid and dense subcone.*