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MORPHISMS OF *H*-CONES

BY

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We introduce in this paper a notion of morphism of H-cones, and present some of its properties. §1 contains the construction of a „cone of hyperharmonics" $\tilde{C}$, associated with an H-cone C. Some properties of the cone $\tilde{C}$ are presented, and $\tilde{C}$ is computed when C is the dual of an H-cone or is an H-cone of functions. The last case justifies also the terminology. §2 contains the definition of a morphism of H-cones, as an: additive, increasing and continuous in order from below map from $\tilde{C}_1$ to $\tilde{C}_2$. We establish completely the relations with the notions of morphisms of H-cones, as introduced in [1] and [3]. The main interest of the notion proposed here is that we get a category (i.e. the composition of two morphisms, if it exists, is still a morphism). Also, some results about isomorphisms of H-cones are included.

In § 3, the class of morphisms having adjoints is considered. We prove that this class is stable under composition and taking adjoints and it contains the most important examples of morphisms. Finally, §4 presents the morphisms of H-cones of functions, which are introduced by maps between the base spaces.

All the definitions and results about H-cones, are as in [1].

This work develops the first three parts of [4].

§1. **The cone of hyperharmonics.** We begin with the construction of an ordered convex cone $\tilde{C}$, associated with an H-cone C. When C is the cone of positive superharmonic functions of a $\mathcal{D}$ harmonic space [2], then $\tilde{C}$ is the cone of positive, hyperharmonic functions. It is this cone $\tilde{C}$ which will be used in the definition of the morphisms of H-cones.

Definition 1.1. *Let C be an H-cone. C^0 will denote the set of all maps $\mu : C \to \overline{\mathbf{R}}_+$, such that there exists an upper directed family $(\mu_i)_{i\in I}$ of H-integrals on C, for which:*

$$\mu(s) = \sup_{i\in I} \mu_i(s), \quad \forall s \in C.$$

It is clear that any $\mu \in C^0$ is additive, increasing and continuous in order from below; but not any map $C \to \overline{\mathbf{R}}_+$, having these properties, is in C^0

Proposition 1.2. *With the operations and order defined pointwise (and the convention $0.\infty = 0$), C^0 is a complete lattice and an ordered convex cone. C^0 contains C^* as a solid and dense subcone.*

Proof. We have to prove only that C^0 is a complete lattice. Let $\mu_1, \mu_2 \in C^0$ and define:

$$\mu(s) = \sup \{\mu_1(s_1) + \mu_2(s_2) \mid s_1 + s_2 \leqslant s\}$$

for any $s \in C$. Then $\mu = \mu_1 \wedge \mu_2$ in C^0. Indeed, let $s \in C$ be such that $\mu(s) < +\infty$; let $\varepsilon > 0$ and let $(\mu_i)_{i \in I}$ and $(\nu_j)_{j \in J}$ be families from C^*, associated with μ_1 and μ_2. We can find $s_1, s_2 \in C$ such that $s_1 + s_2 \leqslant s$ and : $\mu(s) \leqslant \mu_1(s_1) + \mu_2(s_2) + \varepsilon/3$. Next, we can choose $i \in I$ and $j \in J$ such that: $\mu_1(s_1) \leqslant \mu_i(s_1) + \varepsilon/3$ and $\mu_2(s_2) \leqslant \nu_j(s_2) + \varepsilon/3$. Recalling the construction of $\mu_i \vee \nu_j$ in C^*[1], it follows that : $\mu(s) \leqslant (\mu_i \vee \nu_j)\,(s) + \varepsilon$.

If $\mu(s) = +\infty$, then for each $\varepsilon > 0$ there exists $s_1, s_2 \in C$ such that: $\mu_1(s_1) + \mu_2(s_2) > 2\varepsilon$. We can find then $i \in I$ and $j \in J$ such that $\mu_i(s_1) + \nu_j(s_2) > \varepsilon$, which shows that : $(\mu_i \vee \nu_j)\,(s) > \varepsilon$. Hence, in both cases, $\mu(s) = \sup\limits_{\substack{i \in I\\ j \in J}} (\mu_i \vee \nu_j)\,(s)\ \forall s \in C$, which proves that $\mu \in C^0$. It is clear now that $\mu = \mu_1 \vee \mu_2$ in C^0. Since for any increasing family $(\mu_i)_{i \in I}$ from C^0, we have obviously $\mu \in C^0$, where $\mu(s) = \sup\limits_{i \in I} \mu_i(s)\ \ \forall s \in C$, it follows that C^0 is a sup-complete lattice. Having the smallest element, C^0 is in fact a complete lattice.

We define now : $\tilde{C} = (C^*)^0$. Let $\varepsilon : C \to \tilde{C}$ be the composition of the natural maps : $C \to C^{**} \to (C^*)^0$.

Corollary 1.3. a) *$\tilde{C}$ is an ordered convex cone and a complete lattice. C^{**} is dense in order from below and solid in $\tilde{C}$.* b) *$\forall s, t \in C$ and $\alpha \geqslant 0$, we have : $\varepsilon(s+t) = \varepsilon(s) + \varepsilon(t)$; $\varepsilon(\alpha s) = \alpha \cdot \varepsilon(s)$; $s \leqslant t \Rightarrow \varepsilon(s) \leqslant \varepsilon(t)$.*

c) *For any family $(s_i)_{i \in I}$ from C, we have: $\varepsilon(\wedge s_i) = \wedge\, \varepsilon(s_i)$. If $(s_i)_{i \in I}$ is upper directed and bounded, then. $\varepsilon(\vee s_i) = \vee\, \varepsilon(s_i)$.*

d) *is injective iff C^* separates C. $\varepsilon(C)$ is solid in $\tilde{C}$ iff C is solid in C^{**}. $\varepsilon(C)$ is dense in order from below in $\tilde{C}$ iff C is dense in order from below in C^{**}.*

The greatest element in C^0 is :

$$\mu(s) = \begin{cases} 0 \text{ if } \nu(s) = 0, & \forall \nu \in C^*, \\ +\infty \text{ if not.} \end{cases}$$

Hence, the greatest element in $\tilde{C}$ is :

$$f(\mu) = \begin{cases} 0 & \text{if } \mu = 0, \\ +\infty & \text{if } \mu \neq 0. \end{cases}$$

Proposition 1.4. a) *For each $s \in \tilde{C}$ and any increasing family $(s_i)_{i \in I}$ from $\tilde{C}$, we have $s + \vee s_i = \vee (s + s_i)$.*

b) *If C is an H-cone of functions and C is solid in C^{**}, then for any $s \in \tilde{C}$ and any family $(s_i)_{i \in I}$ from $\tilde{C}$ we have : $s + \wedge s_i = \wedge (s + s_i)$.*

Proof. Since any $s \in \tilde{C}$ is finely l.s.c., it is enough to prove that $\wedge s_i = \widehat{\inf}\, s_i$ (the regularisation being considered with respect to the fine topology). Let $x_0 \in X$ be such that: $(s_i) \wedge (x_0) < \widehat{\inf}\, s_i(x_0)$. If we choose ρ, ρ' such that : $(\vee s_i)\,(x_0) < \rho' < \rho < \widehat{\inf}\, s_i\,(x_0)$, then $u_i \in \tilde{C}$, where $u_i = \rho \wedge s_i$. Hence [1] : $\wedge s_i \geqslant \wedge u_i = \widehat{\inf}\, u_i$. But $\{x \in X \mid u_i(x) > \rho'\}$ is a finely open set ; we get then :

$$\widehat{\inf}\, u_i\ (x_0) \geqslant \rho' > (\wedge s_i)\,(x_0)$$

which is the desired contradiction.

Lemma 1.5. *If C is dense in order from below in C^{**}, then $C^* = C^{***}$.*

Proof. Since C^{**} separates C^*, the natural map $C^* \to C^{***}$ is injective. Let now $\varphi \in C^{***}$, hence $\varphi : C^{**} \to \overline{\mathbf{R}}_+$. Let $\hat{\varphi} : C \to \overline{\mathbf{R}}_+$ be the composition of φ with the natural map $C \to C^{**}$. Now the hypothesis implies : $\hat{\varphi} \in C^*$. It remains to verify that $\tilde{\hat{\varphi}} = \varphi$. For any $s \in C$ we have : $\tilde{\hat{\varphi}}(\tilde{s}) = \tilde{s}(\hat{\varphi}) = \hat{\varphi}(s) = \varphi(\tilde{s})$.

Proposition 1.6. *Suppose that C is dense in C^{**} and that C^* separates C. Then $\tilde{C}^*$ and C^0 are isomorphic, as ordered convex cones and complete lattices.*

Proof. Let $\mu \in C^0$. Using the hypothesis, μ can be prolonged to C^{**} by : $\tilde{\mu}(\alpha) = \sup \{\mu(\tilde{s}) \mid s \in C,\ \tilde{s} \leqslant \alpha\}$. The lemma 1.5. proves that $\tilde{\mu} \in \tilde{C}^*$. Conversely, composing any $\mu \in \tilde{C}^*$ with the natural map $C \to C^{**}$, we get an element from C^0. Since the above correspondences are clearly inverse one another, we have the desired isomorphism.

Corollary 1.7. *If C is dense in C^{**}, then $\tilde{C}^{**}$ can be identified with $\tilde{C}$.*

Proof. Since C^{**} separates C^*, lemma 1.5. shows that we can apply prop. 1.6. in order to obtain the isomorphism between $\tilde{C}^{**}$ and $(C^*)^0 = \tilde{C}$.

Proposition 1.8. *Let C be an H-cone of functions on X and suppose moreover that C is dense in C^{**}. Then $\tilde{C}$ is isomorphic, as an ordered convex cone and a complete lattice, with the cone of all functions $f : X \to \overline{\mathbf{R}}_+$, for which there exists an increasing family $(s_i)_{i \in I}$ from C such that :*

$$f(x) = \sup_{i \in I} s_i(x), \quad \forall x \in X.$$

Proof. Let $\mu \in \tilde{C}$; hence $\mu : C^* \to \overline{\mathbf{R}}_+$ and there exists an increasing family $(\mu_i)_{i \in I}$ from C^{**} such that : $\mu(\nu) = \sup\limits_{i \in I} \mu_i(\nu)$, $\forall \nu \in C^*$. The hypothesis shows that we may suppose that $\mu_i = \tilde{s}_i$, with $\tilde{s}_i \in C$. Let then $f(x) = \mu(\varepsilon_x)$. We have $f(x) = \sup\limits_{i \in I} s_i(x)$, $\forall x \in X$. Conversely, let $f : X \to \overline{\mathbf{R}}_+$ and $(s_i)_{i \in I}$ be an increasing family from C, with $f(x) = \sup s_i(x)$, $\forall x \in X$, for any $\nu \in C^*$, let : $\mu(\nu) = \sup\limits_{i \in I} \nu(s_i)$. Hence $\mu : C^* \to \overline{\mathbf{R}}_+$ and by the construction : $\tilde{s}_i \uparrow \mu$. It is easy to prove that μ is independent on the choice of the family (s_i). Hence, the above correspondences are inverse one another, and give thus the desired identification.

Remark. If C is dense in C^{**} and moreover contains a strictly positive element $u \in C$, then $s \in \tilde{C}$ iff $s \wedge nu \in C$, $\forall n \in \mathbf{N}$. It follows that, if C is a standard H-cone, represented as an H-cone of functions on a set X, then C^{**} coincides with the set of all functions from $\tilde{C}$, which are finite on a finely dense subset of X [1]. Also in the case of a standard H-cone of functions on a set X, let $(V_\lambda)_{\lambda\geq 0}$ be a submarkovian resolvent of kernels on X, such that: V is a proper kernel; any $s\in C$ is V-excessive; for any measurable, positive and bounded f, $V_0 f\in C$. Then $\tilde{C}$ can be identified with the set of all V-excessive functions.

We end by proving that balayage and any H-integral on C extends uniquely to $\tilde{C}$.

Proposition 1.9. *Suppose that C is dense in C^{**}, and that C^* separates C. Then:*

a) *There exists a unique bijection $B\leftrightarrow\tilde{B}$ between the set of balayages on C, and the set of all maps $\tilde{B}:\tilde{C}\to\tilde{C}$, having the properties: $\tilde{B}$ is additive, increasing, contraction ($\tilde{B}s\leq s$), idempotent ($\tilde{B}(\tilde{B}s)=\tilde{B}s$) and continuous in order from below; such that the following diagramm commutes:*

$$\begin{tikzcd} C \arrow[r, "B"] \arrow[d, "\varepsilon"'] & C \arrow[d, "\varepsilon"] \\ \tilde{C} \arrow[r, "\tilde{B}"] & \tilde{C} \end{tikzcd}$$

b) *There exists a unique bijection $\mu\leftrightarrow\tilde{\mu}$ between C^* and the set of all maps $\tilde{\mu}:\tilde{C}\to\overline{\mathbf{R}}_+$ which are: additive, increasing and continuous in order from below, and such that, for each $s\in\tilde{C}$ there exists an increasing family $(s_i)_{i\in I}$ from C, such that $s=\vee s_i$ and $\tilde{\mu}(s_i)<+\infty$, $\forall_i\in I$; such that the following diagramm commutes:*

$$\begin{tikzcd} C \arrow[dr, "\mu"] \arrow[d, "\varepsilon"'] & \\ \tilde{C} \arrow[r, "\tilde{\mu}"'] & \overline{\mathbf{R}}_+ \end{tikzcd}$$

Proof. a) Let B be a balayage on C. For each $s\in\tilde{C}$, let $\tilde{B}(s)\in\tilde{C}$ be defined by: $\tilde{B}(s)\,(\mu)=\tilde{s}(B^*\mu)$, $\forall\mu\in C^*$. Since $s=\vee\alpha_i$, with $\alpha_i\in C^{**}$, it follows that $\tilde{B}s=\vee B^{**}\alpha_i$, hence $\tilde{B}$ is well defined. The properties of $\tilde{B}$ follow obviously from those of B. For any $s\in C$ we have:

$$\tilde{B}(\varepsilon(s))\,(\mu)=\varepsilon(s)\,(B^*\mu)=B^*\mu(s)=\mu(Bs)=\varepsilon(Bs)\,(\mu),$$

hence the diagramm commutes. Conversely, if $\tilde{B}$ is given, from $\tilde{B}s\leq s$ it follows that $\tilde{B}(\varepsilon(s))\in\varepsilon(C)$, $\forall s\in C$. Thus $\tilde{B}/_C$ defines a balayage.

b) Let $\mu\in C^*$ and define, for each $s\in\tilde{C}$: $\tilde{\mu}(s)=s(\mu)$. $\tilde{\mu}$ is well defined and clearly has the indicated properties. Since: $\tilde{\mu}(\varepsilon(s))=\varepsilon(s)\,(\mu)=\mu(s)$ the

diagramm commutes. Finally, $\tilde{\mu}$ being given, μ is obtained as the restriction to $C\simeq\varepsilon(C)$.

From now on, the extension to $\tilde{C}$ of any balayage or H-integral on C will be denoted by the same letter.

§2. **Definition and first results.** We are now in a position to define the morphisms of H-cones. A different notion of morphism can be found in [1] or in [3]. The main advantage of our definition is that we get a category. However, our morphisms are not always maps on a H-cone; this is the case only if one supposes that C^* separates C.

Definition 2.1. *Let C_1, C_2 be H-cones. We define a morphism of H-cones from C_1 to C_2 as a map $\varphi:\tilde{C}_1\to\tilde{C}_2$, with the properties:*

additive: $\varphi(s+t)=\varphi(s)+\varphi(t)$

increasing: $s\leq t\Rightarrow\varphi(s)\leq\varphi(t)$

continuous in order from below: for any increasing family $(s_i)_{i\in I}$ from $\tilde{C}_1$ we have $\varphi(\vee s_i)=\vee\varphi(s_i)$.

Remark. Clearly we have thus a category, the objects being the H-cones and the morphisms those defined above. Accordingly, Hom (C_1, C_2) will stand for the set of all morphisms from C_1 to C_2.

Definition 2.2. *For any two morphisms of H-cones $\varphi_1,\varphi_2\in$ Hom (C_1, C_2) we define:*

$$(\varphi_1+\varphi_2)\,(s)=\varphi_1(s)+\varphi_2(s),\ \forall s\in\tilde{C}_1,$$

$$(\alpha\varphi_1)\,(s)=\alpha\varphi_1(s),\ \ \forall s\in\tilde{C}_1,\ \alpha\geq 0,$$

and the order relation: $\varphi_1\leq\varphi_2$ by $\varphi_1(s)\leq\varphi_2(s)$, $\forall s\in\tilde{C}_1$. Thus, Hom (C_1, C_2) becomes an ordered convex cone, in which 0 is the smallest element, and the morphism defined by: $\varphi(s)=\omega$, $\forall s\in\tilde{C}_1$ (where $\omega\in\tilde{C}_2$ is the greatest element), is the greatest element. One verifies easily that, for any increasing family $(\varphi_i)_{i\in I}$ from Hom (C_1, C_2), $\vee\varphi_i$ exists and: $(\vee\varphi_i)\,(s)=\vee\varphi_i(s)$, $\forall s\in\tilde{C}_1$.

Definition 2.3. *For any morphism $\varphi\in$ Hom (C_1, C_2) let us define the domain of φ as: $D(\varphi)=\{s\in C_1 \mid \varphi(\varepsilon(s))\in\varepsilon(C_2)\}$.* $D(\varphi)$ can be void, for instance if φ is the greatest element of Hom (C_1, C_2).

Definition 2.4. *The morphism $\varphi\in$ Hom (C_1, C_2) is said to be semifinite if $D(\varphi)$ is increasingly dense in C_1.*

Examples. a) For any two H-cones C_1, C_2, the map $s\to 0$ is clearly a semifinite morphism, denoted by 0.

b) If C is standard H-cone, the semifinite morphisms from Hom $(C, \mathbf{R}_+)$ are exactly the H-integrals.

c) The morphisms from $\mathbf{R}_+$ to C are exactly the maps $\lambda\to\lambda s$, with $s\in\tilde{C}$. Such a morphism is semifinite iff $s\in C$.

Remark. The composition of two semifinite morphisms need not be semifinite: let C be an H-cone, $\mu\in C^*$ and $s\in C$ be such that $\mu(s)=+\infty$. Then $\lambda\to\lambda s$ composed with μ is not semifinite.

We prove next that, when we consider standard H-cones, the semifinite morphisms are exactly the morphisms considered in [1] or [3].

Proposition 2.5. *Let C_1, C_2 be H-cones and $C' \subseteq \tilde{C}_1$ an increasingly dense, convex subcone. Let $\varphi_0 : C' \to \tilde{C}_2$ be a map such that:*

$$\varphi_0(s+t) = \varphi_0(s) + \varphi_0(t)\ ;\ s \leqslant t \Rightarrow \varphi_0(s) \leqslant \varphi_0(t).$$

The following are equivalent:

a) *There exists a unique morphism $\varphi \in$ Hom (C_1, C_2) such that $\varphi \mid C' = \varphi_0$.*

b) *For any increasing family $(s_i)_{i\in I}$ from C', such that $\bigvee s_i \in C'$, we have $\varphi_0(\bigvee s_i) = \bigvee \varphi_0(s_i)$.*

Proof. a) $\Rightarrow$ b) is obvious. b) $\Rightarrow$ a). Let:

$$\varphi(s) = \bigvee \{\varphi_0(t) \mid t \in C',\ t \leqslant s\},\quad \forall s \in \tilde{C}_1.$$

φ is a well defined map from $\tilde{C}_1$ to $\tilde{C}_2$, and $\varphi \mid C' = \varphi_0$. Clearly $s \leqslant t \Rightarrow \varphi(s) \leqslant \varphi(t)$; and prop. 1.4. shows that $\varphi(s+t) = \varphi(s) + \varphi(t)$. Let $(s_i)_{i\in I}$ be an increasing family from $\tilde{C}_1$ and let $s = \bigvee s_i$. It remains to prove that:

$$(*)\qquad \bigvee \{\varphi(t) \mid t \in C',\ t \leqslant s\} = \bigvee_{i\in I} \bigvee \{\varphi(u) \mid u \leqslant s_i,\ u \in C'\}$$

Let $t \in C'$, $t \leqslant s$. For each $i \in I$, there exists an increasing family $(u_j)_{j\in J}$ from C' such that $s_i \wedge t = \bigvee u_j$. Since $\bigvee (s_i \wedge t) = s \wedge t = t$, the non-obvious inequality in (*) is proved.

Proposition 2.6. *Let C be a standard H-cone, C_1 an H-cone and $\varphi \in$ Hom (C, C_1) a semifinite morphism. Then $C_0 \subseteq D(\varphi)$.*

Proof. Let $D = \{t_n \in C_0 \mid n \in \mathbf{N}\}$ be a countable, dense in order from below subset. Let $u \in C$ be strictly positive. Since $D(\varphi)$ is increasingly dense, let $s_n \in D(\varphi)$ be such that: $s_n \leqslant t_n \leqslant s_n + n^{-1}u$. Let $(w_i)_{i\in I}$ be an increasing, countable family, which contains every s_n. This family is increasingly dense, hence there exists $\alpha_i > 0$ such that: $v = \sum_{i\in I} \alpha_i w_i \in C^{**}$ and v is strictly positive. Hence, for each $t \in C_0$ there exists $i \in I$ such that $t \leqslant w_i + v$. By [3] it follows that: $t \leqslant 2(w_i + \sum_{k\in K} \alpha_k w_k)$ with $K \subseteq I$ finite. Hence $\varphi(t) \in C_1$ and $t \in D(\varphi)$.

Corollary 2.7. *Let C_1, C_2 be standard H-cones and $\varphi_0 : (C_1)_0 \to C_2$ be a map. The following are equivalent:*

a) $\varphi_0(s+t) = \varphi_0(s) + \varphi_0(t)$ and $s \leqslant t \Rightarrow \varphi_0(s) \leqslant \varphi_0(t)$.

b) *There exists a (unique) semifinite morphism $\varphi \in$ Hom (C_1, C_2) such that $\varphi |_{(C_1)_0} = \varphi_0$.*

In accordance with the general definition, we call isomorphism of H-cones, any morphism $\varphi \in$ Hom (C_1, C_2) for which there exists $\psi \in$ Hom (C_2, C_1) such that $\varphi\circ\psi = 1$ and $\psi\circ\varphi = 1$. Obviously, the morphism φ is an isomorphism iff the map $\varphi : \tilde{C}_1 \to \tilde{C}_2$ is bijective and $\varphi(s) \leqslant \varphi(t) \Rightarrow s \leqslant t$. There exists morphisms which are bijections, but not isomorphisms. For example, let C be a standard H-cone. Then $(C, \preccurlyeq)$ endowed with the specific order, is still an H-cone. The identity map $(C, \preccurlyeq) \to (C, \leqslant)$ is a morphism of H-cones which is a bijection, but not isomorphism.

Proposition 2.8. *Let C be a standard H-cone. The natural map $C \to C^{**}$ is an isomorphism.*

Remark. An H-cone could be isomorphic with a standard H-cone, without being itself a standard H-cone. Indeed, $\mathbf{R}_+^{(\mathbf{N})}$ is an H-cone, isomorphic with the standard H-cone $\mathbf{R}_+^{\mathbf{N}}$; but $\mathbf{R}_+^{(\mathbf{N})}$ is not standard, since it has no strictly positive element. However:

Proposition 2.9. *Let C be an H-cone, C_1 a standard H-cone and suppose that C and C_1 are isomorphic. Then C is a standard H-cone iff C is dense in $\tilde{C}$ and C contains a strictly positive element.*

Proof. Denote $u \in C$ a strictly positive element and $\varphi : \tilde{C} \to \tilde{C}_1$ an isomorphism. Let $s \in (C_1)_0$ be arbitrary. There exists $t \in \tilde{C}$ such that $\varphi(t) = s$. It suffices to prove that t is u-continuous. But:

$$s = \varphi(t) = \varphi(\bigvee (t \wedge nu)) = \bigvee (s \wedge n\varphi(u))$$

shows that there exists $\alpha > 0$ such that $s \leqslant \alpha\varphi(u)$. It follows that $t \leqslant \alpha u$, hence $t \in C$, since C is also solid in $\tilde{C}$. Now let $(t_i)_{i\in I}$ be an increasing family from C, such that $t = \bigvee t_i$. Since $s = \varphi(t) = \bigvee \varphi(t_i)$, we deduce that for any $\varepsilon > 0$, there exists $i \in I$ such that $s \leqslant \varphi(t_i) + \varepsilon\varphi(u)$. It results that $t \leqslant t_i + \varepsilon u$.

Let C, C', C_1, C_1' be H-cones and $\varphi \in$ Hom (C, C_1), $\psi \in$ Hom (C', C_1'). We say that φ and ψ are isomorphic if there exists isomorphisms $C \approx C'$ and $C_1 \approx C_1'$ such that the following diagramm is commutative:

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\varphi"] \arrow[d, phantom, "\wr\wr"] & \tilde{C}_1 \arrow[d, phantom, "\wr\wr"] \\
\tilde{C}' \arrow[r, "\psi"] & \tilde{C}_1'
\end{tikzcd}$$

Let us call a morphism $\varphi \in$ Hom (C_1, C_2) finite if $D(\varphi) = C_1$. The balayages are examples of finite morphisms. It is clear that the composition of two finite morphisms is again finite. Moreover, if φ is finite and ψ is semifinite, then $\varphi\circ\psi$ (provided it is defined), is again semifinite. However, if φ is semifinite and ψ is finite, $\varphi\circ\psi$ could be not semifinite, as the remark after def. 2.4. shows.

With these definitions, one verifies easily that any semifinite morphism of standard H-cones is isomorphic with a finite one. Indeed, if C and C_1 are standard H-cones, then $\varphi \in$ Hom (C, C_1) is semifinite iff there exists a strictly positive $u \in C$, such that $\varphi(u) \in C_1^{**}$. If φ is semifinite, there exists (cf. [3]) $\alpha_n > 0$ such that:

$$\sum_{n=1}^{\infty} \alpha_n \varphi(s_n) \in C_1^{**},\quad \text{where } \{s_n \mid n \in \mathbf{N}\} \text{ is a dense part of } C_0. \text{ We can}$$

take then $u = \sum_{n=1}^{\infty} \alpha_n s_n$. Conversely, if $\varphi(u) \in C_1$, it follows that, taking C' as the subcone of u-bounded elements from C, and $C_1' = C_1^{**}$, then C' is isomorphic with C; C_1' is isomorphic with C_1 and $\varphi |_{C'} : C' \to C_1'$ is clearly finite.

§3. Adjoint morphisms. All the H-cones considered here have the properties: C is dense in C^{**} and C^* separates C_f.

Definition 3.1. *We say that* $\varphi \in \operatorname{Hom}(C_1, C_2)$ *has an adjoint if, for each* $\mu \in \tilde{C}_2^*$ *we have* $C^*(\mu) \in \tilde{C}_1^*$, *where*:

$$\varphi^*(\mu)(s) = \mu(\varphi(s)), \quad \forall s \in \tilde{C}_1.$$

It is obvious then, that $\varphi^* : \tilde{C}_2^* \to \tilde{C}_1^*$ is in fact a morphism of H-cones. Moreover, if it exists, φ^* is uniquely defined by φ, and will be called the adjoint of φ.

Proposition 3.2. [3]. *Let* C_1, C_2 *be standard* H*-cones and* $\varphi \in \operatorname{Hom}(C_1, C_2)$ *be a semifinite morphism. Then* φ *has an adjoint; moreover,* φ^* *is also semifinite.*

Proof. Using. cor. 2.7. for any $\mu \in (C_2^*)_0$ we have $\varphi^*(\mu) \in C_1^*$, hence φ^* exists and is semifinite.

Remark. Using the adjoint, we can give a simpler proof to cor. 2.7. It suffices to observe that, for any morphism φ in the sense of [3] we have:

$$\varphi^*(\vee \mu_i)(s) = (\vee \mu_i)(\varphi(s)) = \sup \mu_i(\varphi(s)) = \sup \varphi^*(\mu_i)(s) = (\vee \varphi^*(\mu_i)(s)$$

(where (μ_i) is an increasing family from $D(\varphi^*)$) and that $\varphi^{**} = \varphi$.

There exists morphisms which are not semifinite, but have an adjoint. For example, the morphism $\lambda \to \lambda s$, from $\mathbf{R}_+$ to C, where $s \in \tilde{C} \setminus C$. On the other hand, there exists morphisms (necessarily not semifinite) which have no adjoint: it suffices to take any morphism from C to $\mathbf{R}_+$, which is not in C^0.

Proposition 3.3. *If* φ *has an adjoint, then* φ^* *also has an adjoint and* $(\varphi^*)^* = \varphi$.

Proof. It suffices to prove that φ is the adjoint of φ^*, which follows from:

$$\varphi(\tilde{s})(\mu) = \mu(\varphi(s)) = \varphi^*(\mu(s)) = \tilde{s}(\varphi^*(\mu)) = \varphi^{**}(s)(\mu)$$

Proposition 3.4. *Suppose that* φ_1, φ_2 *have adjoints and* $\varphi_2 \circ \varphi_1$ *is defined. Then* $\varphi_2 \circ \varphi_1$ *has an adjoint and*: $(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

Proof. It suffices to show that $\varphi_1^* \circ \varphi_2^*$ is the adjoint of $\varphi_2 \circ \varphi_1$. But: $(\varphi_2 \circ \varphi_1)^*(\mu)(s) = \mu((\varphi_2 \circ \varphi_1)(s))$ and: $(\varphi_1^* \circ \varphi_2^*)(\mu)(s) = \varphi_1^*(\varphi_2^*(\mu)(s) = \varphi_2^*(\mu)(\varphi_1(s)) = \mu(\varphi_2(\varphi_1(s)))$.

Proposition 3.5. *Let* C_1, C_2 *be standard* H*-cones. Then the following are equivalent*:

a) φ *is an isomorphism between* C_1 *and* C_2.

b) φ^* *is a finite isomorphism between* C_1^* *and* C_2^*.

Proof. a)$\Rightarrow$b). Let $\mu \in C_2^*$; it suffices to prove that $\varphi^*(\mu)$ is semifinite. Let $s \in C_1$; There exists an increasing family $(t_i)_{i \in I}$ in C_2 such that $\varphi(s) = \vee t_i$, and $\mu(t_i) < \infty$. Since φ is an isomorphism, we have $t_i = \varphi(u_i)$, with $u_i \in C_1$ and $s = \vee u_i$. Hence $\varphi^*(\mu)(u_i)$ is finite.

b)$\Rightarrow$a) $\varphi^* : C_2^* \to C_1^*$ extends clearly to an isomorphism between $\tilde{C}_1 = (\tilde{C}_1^*)^\circ$ and $\tilde{C}_2 = (C_2^*)^\circ$.

It is easy to verify that, if $\varphi, \psi \in \operatorname{Hom}(C_1, C_2)$ have adjoints, then $\varphi + \psi$ also has an adjoint and: $(\varphi + \psi)^* = \varphi^* + \psi^*$. Moreover, $\varphi \leqslant \psi \Leftrightarrow \varphi^* \leqslant \psi^*$ which shows that the lattice operations are preserved under the passage to the adjoint: $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$ $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$. Particularly, φ lies on an extreme ray iff φ^* lies on an extreme ray.

Hence, we have another category, in which the morphisms are those which have an adjoint. Taking only standard H-cones as objects, we have a (contravariant) functor from this category into itself, defined by $F(C) = C^*$; $F(\varphi) = \varphi^*$. This functor is fully faithfull and prop. 2.8. show that it establishes in fact an equivalence between the category just considered and its dual.

§4. **H-maps.** We consider finally the morphisms between H-cones of functions. Since, for each H-cone of functions S, S and $\overline{S}$ are isomorphic, we can suppose that all the H-cones of functions contain the positive constants. Moreover, we suppose that all the H-cones of functions have the property: S is increasingly dense in S^{**}. We recall that, for each H-cone of functions S, S^* separates S.

Definition 4.1. *Let* S, S' *be* H*-cones of functions on* X, X'. *A map* $\varphi : X' \to X$ *is called an* H*-map if for any* $s \in S$ *we have* $s \circ \varphi \in \tilde{S}'$.

Proposition 4.2. *Let* φ *be an* H*-map. Then* $s \to s \circ \varphi$ *is a morphism (denoted also by* φ*) from* S *to* S', *which satisfies*:

$$\varphi(1) = 1; \quad \varphi(s \wedge t) = \varphi(s) \wedge \varphi(t), \quad \forall s, t \in S.$$

The map $\varphi : X' \to X$ *is fine-to-fine continuous and the morphism* φ *is semifinite.*

There exists morphisms (even finite isomorphisms) which are not induced by an H-map. If S is an H-cone of functions on X and $f : X \to (0, +\infty)$ is not $\equiv 1$, then S and $f \cdot S$ are isomorphic, but there is no H-map which induces this isomorphism.

Clearly, the morphisms induced by H-maps are closed under composition.

Proposition 4.3. *Let* φ *be an* H*-map which is finely open. Then, if* $S' = S'^{**}$ φ *induces a finite morphism.*

Proof. For any $s \in S$ the set $A = \{x \in X \mid s(x) < \infty\}$ is finely dense. Hence $\varphi^{-1}(A)$ is also finely dense. But $(s \circ \varphi)(x') < \infty$ for any $x' \in \varphi^{-1}(A)$ hence $s \circ \varphi$ is finite on a finely dense set. Using prop. 1.8. and [3], it follows that $s \circ \varphi \in S'^{**} = S'$.

Proposition 4.4. *Let* S, S' *be standard* H*-cones of functions and* φ *be an* H*-map. Then* φ *is naturally measurable. If* $\mu' \in S'^*$ *is an* H*-measure, and* $\mu'(1) < \infty$, *then* $\varphi^*(\mu')$ *is also an* H*-measure.*

Proof. For any $s \in S_0$, $s \circ \varphi$ is a finite, naturally l.s.c. function on X. Hence φ is naturally measurable. It is clear that $\varphi^*(\mu') \in S^*$ is then representable by the measure m, defined on the naturally Borel sets by: $m(A) = m'(\varphi^{-1}(A))$, where m' is the representing measure for μ'.

Definition 4.5. *Let* S, S' *be two standard* H*-cones of functions on* X, X'. *Let us call quasi-*H*-map any map* $\varphi : X' \setminus A' \to X$, *where* $A' \subseteq X'$ *is a semi-polar set, such that, for each* $s \in S$, *there exists an element from* $\tilde{S}'$, *denoted* $\varphi(s)$, *such that* $\varphi(s) = s \circ \varphi$ *on* $X' \setminus A'$.

Proposition 4.6. *Any quasi-*H*-map defines a morphism between* S *and* S', *which has the properties*:

$$\varphi(1) = 1, \quad \varphi(s \wedge t) = \varphi(s) \wedge \varphi(t), \quad \forall s, t \in S, \quad \forall \mu' \in (S')_0^*, \quad \varphi^*(\mu') \text{ is a } H\text{-measure.}$$

Proof. φ is well defined, since from $s=t$ on $X'\setminus A'$, it follows $\mu(s)=\mu(t)$, $\forall \mu \in S_0^*$ (cf. [1]), hence $s=t$. The other statements are obvious.

Remark. If A' is polar, then $\varphi^*(\mu')$ is an H-measure for any universally bounded μ'.

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ON PRODUCT OF MINIMAL PAIRWISE HAUSDORFF BITOPOLOGICAL SPACES

BY

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1. **Introduction.** M. P. Beri [1] proved that if the non empty product space is minimal Hausdorff, then each component space is also minimal Hausdorff. He left the converse of this theorem as an unsolved one and finally Ikenaga [4] proved it. The concept of minimal pairwise Hausdorff bitopological space was introduced by Raghavan and Reilly [7]. In my paper [6], this concept was further studied in terms of the notion of pairwise almost compactness and pairwise H-closedness.

The purpose of this paper is to study whether the topological product of a family of minimal pairwise Hausdorff spaces is also minimal pairwise Hausdorff and also to study the converse problem.

2. **Preliminaries.** Let $\{(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)/\alpha \in \Lambda\}$, where Λ is an index set, be any family of bitopological spaces. If $X=\prod\limits_{\alpha\in\Lambda} X_\alpha$, then two topologies on X can be constructed in a natural way. Let τ_i, for $i=1, 2$, denote the product topology on X, generated by the family $\{(X_\alpha, \tau_\alpha^i) : \alpha \in \Lambda\}$. We shall call (X, τ_1, τ_2) the product bitopological space generated by the family under consideration and shall write $\tau_i = \prod\limits_{\alpha\in\Lambda} \tau_\alpha^i$ for $i=1, 2$. It is obvious that the projection map $p_\alpha : (X, \tau_1, \tau_2) \to (X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is pairwise open and pairwise continuous i.e., $p_\alpha : (X, \tau_i) \to (X_\alpha, \tau_\alpha^i)$ is open and continuous, for all $\alpha \in \Lambda$ and $i=1, 2$.

In what follows we shall state a few definitions and theorems which will be used in the sequel.

Definition 2.1. [Kelly [5]] *A bitopological (X, τ_1, τ_2) is said to be pairwise Hausdoff if for any two distinct points x, y in X, there exist a τ_1 open neighbourhood V_x of x and a τ_2 — open neighbourhood V_y of y, so that $V_x \cap V_y = \emptyset$.*

Definition 2.2. [Mukherjee [6]] *In a bitopological space (X, τ_1, τ_2) a filter F is called a τ_i open filter (for $i=1, 2$) if F has a base consisting of τ_i open sets.*

Definition 2.3. [Mukherjee [6]] *A bitopological space (X, τ_1, τ_2) is said to be τ_i almost compact with respect to τ_j, for $i, j=1, 2$ and $i\neq j$, if every τ_j open filter has a τ_i cluster point. The space (X, τ_1, τ_2) will be called pairwise almost compact if it is τ_1 almost compact with respect to τ_2 and τ_2 almost compact with respect to τ_1.*

In my paper [6], the following theorem was established.