In our opinion, the main advantages of the approach presented in this paper is that our results yield the inclusion and existence statements for the solutions of equation (1) and BVP (7)—(8) at the same time. Althougth the results of this paper have a superficial similarity with the papers of Ruchinskii [13, 14] and Morchalo [11] it is easily seen that the assertions and the proofs of the results established in this paper are decidedly distinct from those of [11, 13, 14].

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AN EQUIVARIANT SHAPE THEORY

BY

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0. Introduction. Shape theory was first introduced by Karol Borsuk, [2], for compact metric spaces. The notion of „shape" was extended to the compact Hausdorff spaces by S. Mardesic and J. Segal [18], [19], and also by W. Holsztynski [12], and then to metric spaces by K. Borsuk [3] and R. Fox [11]. In 1972 S. Mardesic established shape theory for arbitrary topological spaces [20], and in 1975 K. Morita proved in [23] that the notion of shape map can be described using ANR's inverse homotopy systems. K. Morita also showed that the Čech homology and cohomology constructed by normal open coverings of topological spaces are shape invariants.

În 1980 F. W. Bauer [1] defined shape theory for an arbitrary category K of topological spaces with respect to a fine subcategory P of the category K, as the study of those invariants of a topological space X in K which are determined by the maps from X to the fine spaces of the subcategory P.

Recently we showed [26], [27] and [28], [29] that shape theory can also be extended to other categories by constructing, following the model of Mardesic-Morita, a shape theory for Boolean closure algebras and for spaces with two topologies.

In 1979 Yu. M. Smirnov proved [30] the possibility of the construction of a shape theory in the sense of K. Borsuk for a category K of G-spaces in the following situations: 1) G is locally compact and σ-compact, and K is the category of metrizable G-spaces; 2) G is compact, and K is the category of compact G-spaces.

In this paper we construct an equivariant shape theory for the category of arbitrary G-spaces, the group G being assumed finite.

We will know afterwards that the transformation groups theory includes a series of aspects. In this order G. Bredon [5], [6] and Th. Bröker [7] have constructed for discrete groups, the equivariant homology and cohomology with prescribed coefficients. In this case (of actions of discrete groups), the Cech cohomology based on invariant coverings was developed (see [4, p. 135]). Sören Illmann [14] has studied the equivariant singular homology and cohomology in the Eilenberg-Steenrod axiomatical sense. I. M. James & G. B. Segal [15], [16], T. Matumoto [22], G. Bredon [5] have developed the theory of the equivariant homotopy.

The shape theory can afford, as Yu. M. Smirnov [*op. cit.*] anticipated, new problems and results involving G-spaces. For example, in § 7 we give an application, generalizing some results on equivariant homotopy type of G. Bredon [5], T. Matumoto [22], I. M. James & G. B. Segal [15], [16].

1. **Topological and homotopical preliminaries.**

1.1. **G-spaces and G-maps.** We will refer for some definitions and results to [4]. We recall that, G being a topological group, a topological space X is said to be G-space if there is a continuous map $\theta : G \times X \to X$, $(g, x) \to \theta(g, x) = gx$, such that

(a) $\qquad g(hx) = (gh)\, x, \ \forall g, h \in G, \ \forall x \in X$; $\quad$ (b) $ex = x, \ \forall x \in X$,

where e is the identiy element of G.

Let X, Y be G-spaces ; a continuous map $f : X \to Y$ is called a G-map, or an equivariant map, if

(c) $$f(gx) = gf(x), \ \forall g \in G, \ \forall x \in X.$$

If G is a topological group, then G-spaces and G-maps together form a category which we denote by G-Top.

Consider two G-spaces X, Y. Denote $I = [0,1]$, $X \times I$ is also a G-space by considering the map $G \times (X \times I) \to X \times I$, $(g, (x, t)) \to (gx, t)$. Two G-maps $f_0, f_1 : X \to Y$ are said to be G-homotopic (or equivariant homotopic) if there exists a G-map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$; we write $f_0 \simeq_G f_1$. The relation of G-homotopy is an equivalence relation. We denote the G-homotopy class of f by $[f]_G$. We denote also by $G - \mathcal{H}$ the G-homotopy category of G-spaces whose objects are G-spaces and whose morphisms are G-homotopy classes.

From now on we assume that G is a finite group.

1.2. **Simplicial G-complexes.** According to [4, p. 120], a simplicial G-complex is a simplicial complex K together with a simplicial action of G on K, i.e. $G \times K \to K$, $(g, v) \to gv$, such that gs is a simplex in K if s is, for any element $g \in G$. A simplicial G-complex K is regular if the following are fulfilled :

(A) If v and gv are in the same simplex of G, then $v = gv$,

(B) For any subgroup H of G and $g_0, \ldots, g_n \in H$, if $(v_0, \ \ldots, v_n)$ and $(g_0 v_0, \ldots g_n v_n)$ are simplexes of K, then there exists $g \in H$ so that $gv_i = g_i v_i$.

Let K be a simplicial G-complex. Considering K_{cw} as the space K with the weak topology, every simplicial map $g : K \to K$ induces a homeomorphism $g : K_{cw} \to K_{cw}$, and so K_{cw} is a G-space. Similarly, if we consider K_m as the space K with the metric topology, then K_m is also G-space, because the induced map by a simplicial application is continuous with respect to both topologies [32, p. 149].

A simplicial map between two simplicial G-complexes which is also a G-map is called a G-simplicial map.

For any simplicial G-complex K, the identity map $i : K_{cw} \to K_m$ is a G-simplicial map. If K is locally finite, then i is a G-homeomorphism. We will see that, in general, i is only a G-homotopy equivalence.

For a regular simplicial G-complex K, we can define a simplicial complex K/G in the following way : the vertices in K/G are the orbits $v^* = G(v)$ of the action of G on vertices of K ; simplexes are the subsets $(v_0^*, \ldots, v_n^*)$ so that $(v_0, \ldots, v_n)$ is a simplex in K(v_i is then in the class v_i^*). In this way, we say also that $(v_0, \ldots, v_n)$ is a simplex over the simplex $(v_0^*, \ldots, v_n^*)$. The map $K \to K/G$, $v \to v^*$ is simplicial.

1.3. **G-CW complexes.** Refering to [5] and [22], a $G - CW$ complex is a CW-complex X with an action of G on X via a cellular map and such that :

(d) For any $g \in G$, the fixed-point set of g $X^g = \{x \in X / gx = x\}$ is a subcomplex of X.

If X and Y are $G - CW$ complexes, then a continuous map $f : X \to Y$ is G-cellular if it is cellular and equivariant.

1.4. **G-ANR spaces.** As in [25], G-ANR spaces are separable and metrizable G-spaces X with the property :

(e) For any normal G-space B and invariant closed subspace A of B, each G-map $f : A \to X$ can be extended over an invariant neighbourhood of A in B.

(Some other authors, e.g. Alberto Del Fra [8] consider for B only metrizable G-spaces).

2. **G-spaces having the homotopy type of a G-CW complex**

In this paragraph, we generalize a theorem of S. Mardešić [20, Th. 4.1].

2.1. **Theorem.** *Let G be a finite group and X a G-space. Then the following statements are equivalent :*

(i) X has the G-homotopy type of a simplicial G-complex with the weak topology ;

(ii) X has the G-homotopy type of a simplicial G-complex with the strong (metric) topology ;

(iii) X has the G-homotopy type of a $G - ANR$;

(iv) X has the G-homotopy type of a $G - CW$ complex.

Proof[1]. 2.2. (i)$\Leftrightarrow$(ii).

It's enough to show that for a given simplicial G-complex K, the identity map $i : K_{cw} \to K_m$ is a G-homotopy equivalence.

Firstly, i is a G-map. We will construct then a G-map $i'' : K_m \to K_{cw}$, G-homotopy inverse of i. As in [17, Prop. 4.6], consider the locally finitely open covering $\mathcal{U} = \{U_v\}_{v \in K}$ of K_m, where $U_v = \{x / x(v) > (1/2) \max_w x(w)\}$. Of course $v \in U_v$ and $U_v \subset \mathrm{ost}_K v$. This open covering is also invariant. In fact, if $x \in U_v$ and $g \in G$, since $g : K_m \to K_m$ is defined by $g(\Sigma x(v)\, v) = \Sigma x(v)\, (gv)$, we have that $gU_v = U_{gv}$. It follows that $g\mathcal{U} = \mathcal{U}$. Let $\{p_v\}$ be the

[1] Dr. J. Weinstein informed me that this theorem was also proved partly by S. Araki and M. Murayama, Publ. RIMS, Kyoto Univ. 14 (1978), 203–222 and partly by M. Murayama, Publ. RIMS, Kyoto Univ. 18 (1982), 183–189. The proofs are different from those given by us. In fact, the notion of G-ANR used by M. Murayama is rather different from that which we use here.

partition of unity subordinated to the covering $\mathcal{U}$, defined by:

$$p_v(x) = \frac{\max[(x(v)-(1/2)\max x(w)), 0]}{\Sigma \max[(x(v)-(1/2)\max x(w)), 0]}.$$

By [4, p. 136], a G-partition of unity subordinated to the invariant covering $\mathcal{U}$ can be constructed by taking $q_v(x) = (1/|G|)\sum_g p_{gU_v}g(x) = (1/|G|)\sum_{g\in G} p_{U_{gv}}(gx)$.

Define then $i'' : K_m \to K_{cw}$ by $i''(x) = \sum_v q_v(x)\, v$. The map i'' is continuous and $i''(gx) = \sum_v q_v(gx)\, v$. Then $gi''(x) = \sum_w q_w(x)\,(gw) = \sum_w q_{g^{-1}w}(x)\, w = \sum_v q_v(gx)\, v$, hence $i''(gx) = gi''(x)$. So i'' is a G-map.

If $i' : K_m \to K_{cw}$ is homotopy inverse to i [17, Prop. 4.6], then for any $y \in |K|$, there is a closed simplex $[w_1, \ldots, w_p]$ such that y and $i'(y)$ are contained in its interior. Then, for $x \in |K|$, $i''(x)(v) = (1/|G|)\sum_{g\in G} i'(gx)(gv)$. For g fixed, gx and $i'(gx)$ are in the interior of the simplex $[w_1, \ldots, w_p]$, so $(gx)(w_i) \neq 0$ and $i'(gx)(w_i) \neq 0$, therefore $x(g^{-1}w_i) \neq 0$ and $i''(x)(g^{-1}w_i) \neq 0$, i.e. x and $i''(x)$ are in the interior of the simplex $[g^{-1}w_1, \ldots, g^{-1}w_p]$. We can consider the linear homotopies $F : K_m \times I \to K_m$, $F(x,t) = tx + (1-t)(i \circ i'')(x)$ and $F' : K_{cw} \times I \to K_{cw}$, $F'(x,t) = tx + (1-t)(i'' \circ i)(x)$. Remarking that the action of G on K is simplicial and that i'' is a G-map, it follows that $F : i \circ i'' \simeq_G \mathrm{Id}$ and $F' : i'' \circ i \simeq_G \mathrm{Id}$.

To establish the implication (ii)$\Rightarrow$(iii), we firstly give some lemmas which generalize some known results about ANR spaces.

If (X, A) is a G-pair, A is called neighbourhood G-retract of X if there exist an open G-invariant set U of X which contains A and a G-map $r : U \to A$ such that $ri = \mathrm{Id}$.

2.3. **Lemma.** *A separable and metrizable G-space X, which is also a neighbourhood G-retract of a G-ANR Y, is itself a G-ANR.*

Proof. (The proof is similar to in [13, Prop. 5.2]).

Take $W \supset X$, an open G-invariant set of Y and $r : W \to X$, a G-retraction. Let B be a normal G-space and let A be a closed G-invariant subspace of B, and then $f : A \to X$ a G-map. Denote the inclusion map $i : X \to Y$ and let $f' = i \circ f : A \to Y$. Because Y is G-ANR, there exists an extension $\tilde{f}' : V \to Y$ of f, where V is an open G-invariant neighbourhood of A in B. $\tilde{f}'$ is also a G-map. Consider the set $U = \tilde{f}'^{-1}(W)$ which is open in B and which contains A; since W is G-invariant and $\tilde{f}'$ is a G-map, it follows that U is G-invariant. Define $\tilde{f} : U \to X$ by $\tilde{f}(u) = r(\tilde{f}')(u))$. This is a G-map ($\tilde{f}$ is a composition of two G-maps) and is obviously an extension of f.

2.4. **Lemma.** *Any simplicial G-subcomplex L of a simplicial G-complex K is a neighbourhood G-retract, if K and L are considered with the metric topology.*

Proof. Using the notations from [13, Lemma 10.1 and Lemma 11.1] it is sufficient to show that the set $U = \{z \in K' / \Phi(z) > 0\}$ is G-invariant and the map $r : U \to L'$ is a G-map. The first assertion is true, since $z \in U$, and we write $z = \Sigma z(v_\mu)\, v_\mu$, then $gz = \Sigma z(v_\mu)\,(gv_\mu)$ and so $\Phi(gz) = \Phi(z)$, i.e. $gz \in U$.

If $r(z) = \Sigma r(z)_\mu\, v_\mu$, then $gr(z) = \Sigma r(z)_\mu\,(gv_\mu)$ and $r(gz) = \Sigma [r(gz)]_\mu\, v_\mu$, where

$$r(z)_\mu = \begin{cases} z(v_\mu)/\Phi(z), & v_\mu \in L' \\ 0, & v_\mu \in K' - L' \end{cases} \quad \text{and}$$

$$r(gz)_\mu = \begin{cases} (gz)(v_\mu)/\Phi(gz), & v_\mu \in L' \\ 0, & v_\mu \in K' - L' \end{cases} = \begin{cases} z(g^{-1}v_\mu)/\Phi(z), & v_\mu \in L' \\ 0, & v_\mu \in K' - L'. \end{cases}$$

Thus, $r(gz) = \sum_{v_\mu \in L'} [z(g^{-1}v_\mu)/\Phi(z)]\, v_\mu = \Sigma [z(v_\mu)/\Phi(z)]\,(gv_\mu)$ and since L' is a simplicial G-subcomplex of K', we have $r(gz) = \Sigma[z(v_\mu)/\Phi(z)]\,(gv_\mu) = gr(z)$.

2.5. **Lemma.** *Any complete simplicial G-complex with the metric topology is a G-AR and hence a G-ANR.*

Proof. (see [13, Th. 11.2]). Let K be a complete simplicial G-complex, with the metric topology and let $\{v_\lambda / \lambda \in \Lambda\}$ be the set of vertices of K. Consider the Banach space S consisting of all real functions $s : \Lambda \to R$, such that $\sum_{\lambda\in\Lambda} |s(v_\lambda)|$ is convergent. We may organize S in a G-space by defining $(gs)(v_\lambda) = s(g^{-1}v_\lambda)$, $g \in G$, $s \in S$, $\lambda \in \Lambda$. In fact, if $g, h \in G$, then $[(gh)s](v_\lambda) = s[(h^{-1}g^{-1})v_\lambda] = s(h^{-1}(g^{-1}v_\lambda)) = (hs)(g^{-1}v_\lambda) = g(hs)(v_\lambda)$ and obviously $(es)(v_\lambda) = s(v_\lambda)$. Now define $h : K \to S$ in the following way: if $x \in K$, with $x = \Sigma x(v_\lambda)\, v_\lambda$, then $[h(x)](v_\lambda) = x(v_\lambda)$. If $g \in G$ and x is as above, then $gx = \Sigma x(v_\lambda)(gv_\lambda)[h(gx)](v_\mu) = (gx)(v_\mu) = x(g^{-1}v_\mu)$ and $[gh(x)](v_\mu) = h(x)(g^{-1}v_\mu) = x(g^{-1}v_\mu)$, so $h(gx) = gh(x)$. It follows that h is a G-map and moreover it is also a homeomorphism (isometry) of K on a convex subset $h(K)$ of S (since K is complete).

Now let us show that $h(K)$ is a G-AR space (and this implies the same property for K). For this purpose, let B be a normal G-space and let A be a closed invariant subspace of B. Let $f : A \to h(K) \subset S$ be a G-map. According to D u g u n d j i [10], there exists an extension of f over B, $F' : B \to h(K) \subset S$. Define $F : B \to S$ by

$$F(x)(v_\lambda) = (1/|G|)\sum_{g\in G} F'(gx)(gv_\lambda), \quad \forall x \in B, \ \forall v_\lambda \in K.$$

F is continuous and, by virtue of convexity of $h(K)$, we have $F(x) \in h(K)$. It remains to show that F is a G-map and that it is an extension of f.

For $g_0 \in G$, $x \in B$, $v_\lambda \in K$, we have $F(g_0x)(v_\lambda) =$

$$= (1/|G|)\sum_{g\in G} F'(g(g_0x))(gv_\lambda) = (1/|G|)\sum_{g\in G} F'((gg_0)x)(gv_\lambda) =$$

$$= (1/|G|)\sum_{g'\in G} F'(g'x)((g'g_0^{-1})v_\lambda), \text{ for } gg_0 = g', \text{ and}$$

$(g_0F(x))(v_\lambda) = F(x)(g_0^{-1}v_\lambda) = (1/|G|)\sum_{g'\in G} F'(g'(g_0^{-1}v_\lambda))$, which shows that $F(g_0x) = g_0F(x)$ and hence F is a G-map.

Now, let $a \in A$. Then we have $F(a)(v_\lambda) = (1/|G|)\sum_{g'\in G} F'(ga)(gv_\lambda) = (1/|G|)\sum_{g\in G} f(ga)(gv_\lambda) = (1/|G|)\sum_{g\in G} (gf(a))(gv_\lambda) = (1/|G|)\sum_{g\in G} f(a)(g^{-1}gv_\lambda) = (1/|G|)\sum_{g\in G} f(a)(v_\lambda) = f(a)(v_\lambda)$, so F is an extension of f.

2.6. (ii) $\Rightarrow$ (iii).

Let K be a simplicial G-complex with the associated G-space K_m. Consider the complete simplicial G-complex $F=F(K)$ defined by K and let be the G-space F_m. According to 5.2, F_m is a G-ANR. Then, using 4.2 and 3.2 it follows that K_m is a G-ANR. If Y is a G-space and $\alpha=\{U_\lambda/\lambda\in\Lambda\}$ is an invariant covering of Y, then a G-space X is called a α-G-dominator of Y if there exist two G-maps $\Phi: X\to Y$, $\Psi: Y\to X$ and a G-homotopy $F:\Phi\Psi\simeq_G \mathrm{Id}_Y$ such that $\forall y\in Y$, $\exists\lambda\in\Lambda$ with $F(y,t)\in U_\lambda$, $\forall t\in I$.

We can establish the following

2.7. **Lemma.** *If Y is a G-ANR, then for any invariant covering α of Y, there exists a simplicial G-complex K so that K_{cw} is an α-G-dominator of Y.*

Sketch of the proof. The proof is analogous to that in Th. 6.1. [13, p. 138], where the result of Th. 2.1 [*op. cit.*] is replaced with the following result of Yu. M. S m i r n o v, [31] : for any G-space X there exists a convenient normal space A, such that $\mathcal{C}(G, A)$, with the compact open topology and the natural structure of G-space, is metrizable and that X can be invariantly embedded in $\mathcal{C}(G, A)$ as a closed subspace.

2.8. **Lemma.** *A G-space X has the G-homotopy type of a G-CW complex if and only if X is G-dominated by a $G-CW$ complex.*

Proof. The sufficiency is obviously. As for the necessity, we proceed as in Th. 3.8 [17, p. 127] with the following specifications : if X is a G-space and SX is the singular complex of X, then the space $M(SX)=\bigcup\limits_{s_n\in S_n(X)}(\Delta^n, s_n)$ is a normal G-CW complex [17, §4, Chap. III].

We can define the action of the group G by $g(t, s_n)=(t, gs_n)$, for $(gs_n)(t')=g(s_n(t'))$. Since the cells of $M(SX)$ are the cells of the simplexes (Δ^n, s_n) (i.e. of Δ^n), it follows obviously that the action of G is cellular. Then the action of G is compatible with the equivalence relation $\mathcal{R}$ on $M(SX)$. For example : $g(d_i^* t, s_n)=(d_i^* t, gs_n)$ and $g(t, d_i s_n)=(t, gd_i(s_n))$. But $(gd_i(s_n))(t')=g[(d_i(s_n))(t')]=g[s_n d_i^*(t')]$ and $d_i(gs_n)(t')=(gs_n)(d_i^* t')=g[s_n(d_i^* t')]$, hence $g(d_i^* t, s_n)\,\mathcal{R}\, g(t, d_i s_n)$. It follows that $|SX|$ is a G-CW complex[2].

Now, the map $j_X: |SX|\to X$ defined by $j_X([t, s])=s(t)$ is a G-map[2] : $j_X(g[t, s])=j_X([g(t, s)])=j_X([t, gs])=(gs)(t)=g(s(t))=g(j_X[t, s])$. Using the equivariant J. H. C. W h i t e h e a d Theorem [22, Th. 5.3] it, easily follows that the map $j'=j_{K_{cw}}: |SK_{cw}|\to K_{cw}$ is a G-homotopy equivalence[2]. Now the Lemma follows as in the topological case [17, p. 127].

2.9. (iii) $\Rightarrow$ (iv).

Let X be a G-space G-homotopy equivalent to a G-ANR Y. By 7.2, if we take for example $\alpha=\{Y\}$, Y is then G-dominated by a simplicial G-complex K_{cw} (and hence by a G-CW complex). Now, using 8.2, it follows that Y has the G-homotopy type of the G-CW complex $|SY|$.

2.10. (iv) $\Rightarrow$ (i)

If X is a G-CW complex, then by the same arguments as in 8.2, we can show that $j_X: |S(X)|\to X$ is a G-homotopy equivalence (see [17, Cor. 3.5]). Now, using the proof of Th. 6.1 [17, p. 100] and observing that $|S(X)|$ is a G-CW complex, it follows that this is G-homeomorphic with a simplicial G-complex. This completes the proof of Theorem 1.2.

[2]) See also S. A r a k i and M. M u r a y a m a [*op. cit.*].

3. **An equivariant theorem of K. Morita.** Let G be a finite group and $G-\mathcal{H}$ the G-homotopy category of G-spaces. Let G-$\mathcal{CW}$ be the full subcategory of the category G-$\mathcal{H}$ whose objects are all G-spaces which have the G-homotopy type of a G-CW complex (see also 1.2).

3.1. **Definition.** Let $\mathbf{X}=\{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ *be an inverse system in the category $G-\mathcal{H}$ or $G-\mathcal{CW}$, i.e. let A be a direct set and the G-maps $p_{\alpha\alpha'}: X_{\alpha'}\to X_\alpha$ be defined for all α, α' with $\alpha<\alpha'$, and $[p_{\alpha\alpha'}]_G[p_{\alpha'\alpha''}]_G=[p_{\alpha\alpha''}]_G$, for $\alpha<\alpha'<\alpha''$.*

We will say that the inverse system $\mathbf{X}$ *is G-associated with a G space X if there exist G-maps $p_\alpha: X\to X_\alpha$ for $\alpha\in A$ such that the following hold :*

(i) $[p_{\alpha\alpha'}]_G[p_{\alpha'}]_G=[p_\alpha]_G$, *if* $\alpha<\alpha'$;

(ii) *For any G-map $f: X\to Q$, with $Q\in Ob\,(G$-$\mathcal{CW})$ there exist $\alpha\in A$ and a G-map $f_\alpha: X_\alpha\to Q$, such that $[f]_G=[f_\alpha]_G[p_\alpha]_G$;*

(iii) *For any $\alpha\in A$ and any two G-map f_α, $g_\alpha: X_\alpha\to Q$, where $Q\in Ob\,(G$-$\mathcal{CW})$, if $[f_\alpha]_G[p_\alpha]_G=[g_\alpha]_G[p_\alpha]_G$, then there exists $\alpha'\in A$, with $\alpha\leqslant\alpha_f$ such that $[f_\alpha]_G[p_{\alpha\alpha'}]=[g_\alpha]_G[p_{\alpha\alpha'}]$.*

3.2. **Theorem.** *Let X be a G-space. Then there exists an inverse system in the category G-$\mathcal{CW}$ which is G-associated with X.*

Proof. We recall that an invariant covering $\mathcal{U}$ of a G-space X is called a G-covering if, for any $U\in\mathcal{U}$ and any $g\in G$, $U\cap gU\neq\emptyset$ implies $U=gU$.

Let $\{\mathcal{U}_\alpha/\alpha\in A\}$ be the set of all locally finitely and normal G-coverings of X. If $\mathcal{U}_{\alpha'}$ is a refinement of $\mathcal{U}_\alpha$, we write $\alpha\leqslant\alpha'$. Let $K_\alpha=K(\mathcal{U}_\alpha)$ be the nerve of $\mathcal{U}_\alpha$, i.e. the simplicial complex whose vertices are the elements $U\neq\emptyset$ of the covering $\mathcal{U}_\alpha$ and whose simplexes are the systems $(U_0, \ldots, U_n)$ with $U_0, \ldots, U_n\in\mathcal{U}_\alpha$ and $U_0\cap\ldots\cap U_n\neq\emptyset$. Then K_α has a canonical structure of a simplicial G-complex (the covering $\mathcal{U}_\alpha$ being G-invariant).

Since $\mathcal{U}_\alpha$ are normal coverings, there exist partitions of the unity subordinated to these and, moreover, as we have already seen it can be constructed G-partitions of unity subordinated to these. For $\mathcal{U}_\alpha=\{U_\alpha\}$, let $\{\pi_{U_\alpha}\}$ be partition of the unity subordinated to this. Then, define the mappings $p_\alpha: X\to|K_\alpha|$ by : $p_\alpha(x)=\sum\limits_{U_\alpha\in\mathcal{U}_\alpha}\pi_{U_\alpha}(x)U_\alpha$.

The maps p_α are continuous [9, p. 436]. Moreover, these maps are equivariant : $g\in G$, $x\in X$, we have $p_\alpha(gx)=\sum\limits_{U_\alpha\in\mathcal{U}_\alpha}\pi_{U_\alpha}(gx)\,U_\alpha=\sum\limits_{U_\alpha\in\mathcal{U}_\alpha}\pi_{g^{-1}U}(x)\,U_\alpha=$

$$=\sum_{U_\alpha\in\mathcal{U}_\alpha}\pi_{g^{-1}U_\alpha}(x)\,g(g^{-1}U_\alpha)=g\sum_{U'_\alpha\in\mathcal{U}_\alpha}\pi_{U'_\alpha}(x)\,U'_\alpha=gp_\alpha(x).$$

If we change the partitions of unity, the obtained maps p_α are G-homotopic, since they are contiguous. Hence, for any $\alpha\in A$, we can consider the morphisms $[p_\alpha]_G: X\to|K_\alpha|$ in the category $G-\mathcal{H}$.

If $\alpha\leqslant\alpha'$, there is a map $\mu: \mathcal{U}_{\alpha'}\to\mathcal{U}_\alpha$ so that $\mathcal{U}_{\alpha'}\subseteq\mu(U_{\alpha'})$. We can show that the map μ can be taken to be equivariant [4, p. 138], i.e. $\mu(gU_{\alpha'})=g\mu(U_{\alpha'})$, for any $g\in G$, $U_{\alpha'}\in\mathcal{U}_{\alpha'}$ and A is a directed set with respect to the above relation.

In these conditions, a map μ induces a simplicial map $\bar{p}_{\alpha\alpha'}: K_{\alpha'}\to K_\alpha$ which, in virtue of the equivariance of μ, is itself equivariant and hence induces a G-map $p_{\alpha\alpha'}=|\bar{p}_{\alpha\alpha'}|: |K_{\alpha'}|\to|K_\alpha|$. If we change the map the G-homotopy class of the G-map $\bar{p}_{\alpha\alpha'}$ does not change. In this way we obtain an inverse system $\{|K_{\alpha'}|, [p_{\alpha\alpha'}]_G, A\}$ in the category G-$\mathcal{CW}$ (see 1.2), together with the G-maps p_α so that $[p_{\alpha\alpha'}]_G\,[p_\alpha]_G=[p_\alpha]_G$.

Now, let $Q\in \mathrm{Ob}\,(G\text{-}\mathcal{CW})$. According to Th. 1.2, we can assume that Q is even a simplicial G-complex (we denote identically the simplicial G-complex Q and the G-space corresponding to this). Let $f: X\to Q$ be a G-map and let $\mathcal{W}=\{\mathrm{st}_Q(v)/v$ is a vertex of $Q\}$. Since Q is paracompact, $\mathcal{W}$ is a normal locally finite covering. Also, since Q is a simplicial G-complex, i.e. for any $g\in G$, the induced map $g: Q\to Q$ is simplicial, so $\mathcal{W}$ is G-invariant (we have that $g\,\mathrm{st}_Q(v)=\mathrm{st}_Q(gv)$, for any g and any vertex v). However, this is not a G-covering, for $g\,\mathrm{st}_Q(v)\cap \mathrm{st}_Q(v)\neq\varnothing$ (i.e. $\mathrm{st}_Q(gv)\cap\mathrm{st}_Q(v)\neq\varnothing$) does not imply $gv=v$. But, $\mathrm{st}_Q(gv)\cap\mathrm{st}_Q(v)\neq\varnothing$ iff v and gv are vertices of the same simplex in Q. We can apply Prop. 1.1 of [4, p. 121] which asserts that for the barycentric subdivision Q' it satisfies the condition :

(A) If v is a vertex of Q' and if gv is in the same simplex as v, then $gv=v$.

So, if $\mathcal{W}$ is not a G-covering, we can take for Q the G-simplicial structure of Q' and the associated covering formed by the open stars of the vertices of Q' is a G-covering. Hence, we can assume, without reducing generality, that $\mathcal{W}$ is a G-covering. Consider a G-partition of unity $\{\pi_{\mathrm{st}_Q(v)}\}_{v\in Q}$. Let be $\mathcal{U}=f^{-1}(\mathcal{W})$, which is an open covering of X. This is locally finite and G-invariant as $g(f^{-1}(\mathrm{st}_Q(v))=f^{-1}(\mathrm{st}_Q(gv))$ $(y\in g(f^{-1}(\mathrm{st}_Q(v))\Rightarrow y=gz$, where $z\in f^{-1}(\mathrm{st}_Q(v))\Rightarrow f(z)\in\mathrm{st}_Q(v)\Rightarrow gf(z)\in g(\mathrm{st}_Q(v))=\mathrm{st}_Q(gv)\Rightarrow f(gz)\in\mathrm{st}_Q(gv)\Rightarrow y\in$ $\in f^{-1}(\mathrm{st}_Q(gv))$; reciprocally, $u\in f^{-1}(\mathrm{st}_Q(gv))\Rightarrow f(u)\in\mathrm{st}_Q(gv)=g(\mathrm{st}_Q(v))\Rightarrow$ $\Rightarrow f(u)=gt$, where $t\in\mathrm{st}_Q(v)\Rightarrow g^{-1}(f(u))\in f^{-1}(\mathrm{st}_Q(v))\Rightarrow u\in g(f^{-1}(\mathrm{st}_Q(v)))$.

Moreover, $\mathcal{U}$ is a G-covering, since : $g(f^{-1}(\mathrm{st}_Q(v))\cap f^{-1}(\mathrm{st}_Q(v))\neq\varnothing$ implies $f^{-1}(\mathrm{st}_Q(gv))\cap f^{-1}(\mathrm{st}_Q(v))\neq\varnothing$. So, $\mathrm{st}_Q(gv)\cap\mathrm{st}_Q(v)\neq\varnothing\Rightarrow gv=v$, the covering $\mathcal{W}$ being a G-covering. Finally, $\mathcal{U}$ is a normal covering, so we can define the following partition of unity subordinate to it.
$\pi'_{f^{-1}(\mathrm{st}_Q(v))}(x)=\pi_{\mathrm{st}_Q(v)}(f(x))$, for any $x\in X$ and any vertex v of Q. As a matter of fact, $\pi'_{gf^{-1}(\mathrm{st}_Q(v))}(gx)=\pi'_{f^{-1}(\mathrm{st}_Q(gv))}(gx)=\pi_{\mathrm{st}_Q(gv)}(gx)=\pi_{g\mathrm{st}_Q(v)}(f(gx))=$ $=\pi_{g\mathrm{st}_Q(v)}(g(f(x))=\pi_{\mathrm{st}_Q(v)}(f(x))=\pi'_{f^{-1}(\mathrm{st}_Q(v))}(x)$.

In this way we have obtained the „index" $\mathcal{U}$ and we consider the space $|K(\mathcal{U})|$. Define now the simplicial map $\bar f_{\mathcal{U}}: K(\mathcal{U})\to Q$, $\bar f_{\mathcal{U}}(f^{-1}(\mathrm{st}_Q(v))=v$, which is even G-simplicial and it induces a G-map $|\bar f_{\mathcal{U}}|=f_{\mathcal{U}}: |K(\mathcal{U})|\to Q$.

We have to prove that $[f]_G=[f_{\mathcal{U}}]_G\,[p_{\mathcal{U}}]_G=[f_{\mathcal{U}}\circ p_{\mathcal{U}}]_G$. But the G-space Q is homeomorphic to the G-space of the complex $K(\mathcal{W})$, [32, Th. 26, p. 150] and, in this sense, we have
$(f_{\mathcal{U}}\circ p_{\mathcal{U}})(x)(\mathrm{st}_Q(v))=p_{\mathcal{U}}(x)(f^{-1}(\mathrm{st}_Q(v))=\pi'_{f^{-1}(\mathrm{st}_Q(v))}(x)=\pi_{\mathrm{st}_Q(v)}(f(x))$, and this implies the fact that, $(f_{\mathcal{U}}\circ p_{\mathcal{U}})(x)$ and $f(x)$ are both in the same simplex of Q for any $x\in X$. We can then consider the liniar homotopy $F: X\times I\to$ $\to Q$, $F(x,t)=t(f_{\mathcal{U}}\circ p_{\mathcal{U}})(x)+(1-t)f(x)$, which is obviously a G-homotopy in virtue of the fact that $f_{\mathcal{U}}$, $p_{\mathcal{U}}$, f are G-maps. $(F(gx,t)=t(f_{\mathcal{U}}\circ p_{\mathcal{U}})(gx)+$ $+(1-t)\,g(f(x))=tg(f_{\mathcal{U}}\circ p_{\mathcal{U}})(x)+(1-t)\,g(f(x))$ and the homeomorphism $g: Q\to Q$ is simplicial, hence linear on simplexes. It follows that $F: f\simeq_G f_{\mathcal{U}}\circ p_{\mathcal{U}}$.

As for verifying the condition (iii) from Definition 1.3, let $f_\alpha, g_\alpha: X_\alpha\to Q$, where $Q\in\mathrm{Ob}\,(G\text{-}\mathcal{CW})$, such that $[f_\alpha]_G\,[p_\alpha]_G=[g_\alpha]_G\,[p_\alpha]_G$. Using the G-simplicial approximation theorem [4, Exer. 6, p. 77], we can assume that f_α and g_α are G-simplicial maps and then we can give a similar proof to that of Lemma 4.2 of [24]. Otherwise, there is α', with $\alpha<\alpha'$, such that $[f_\alpha]$ $[p_{\alpha\alpha'}]=[g_\alpha]\,[p_{\alpha\alpha'}]$. Since $X_{\alpha'}$, Q are G-spaces and all the maps considered here are G-maps, we can construct easily a G-homotopy between $f_\alpha\circ p_{\alpha\alpha'}$ and $g_\alpha p_{\alpha\alpha'}$. This ends the proof of Theorem 2.3.

3.3. *Remark.* If we consider a metrizable G-space X with the metric satisfying the condition $\rho(gx,y)=\rho(x,gy)$, $\forall x,y\in X$, $\forall g\in G$, then there exists an equivariant form of Theorem K u r a t o w s k i-W o j d y s l a w s k i. So, other results of K. M o r i t a [23, Theorems 1.4, 1.5] and S. M a r d e s i ć [20], can also be extended.

4. **The category G-SHAPE.** Following S. M a r d e s i ć [20] and K. M o r i t a [23], we will define the category G-SHAPE in the following way. The objects of the category G-SHAPE are all G-spaces (as above, G is a fixed finite group) ; and if X and Y are two G-spaces, then a morphism from X to Y, which is also named G-shaping (or G-shape map), is a map f that associates any morphism $\psi\in(G-\mathcal{H})(Y,Q)$, where $Q\in\mathrm{Ob}\,(G\text{-}\mathcal{CW})$, a G-homotopy class $f(\varphi)\in(G\text{-}\mathcal{H})(X,Q)$ such that, for any $Q'\in\mathrm{Ob}\,(G\text{-}\mathcal{CW})$, $\varphi'\in(G\text{-}\mathcal{H})(Y,Q')$ and $\psi\in(G\text{-}\mathcal{CW})(Q,Q')$, the equality $\psi\varphi=\varphi'$ implies $\psi f(\varphi)=f(\varphi')$.

The compositions of morphisms and the identical G-shaping $1_X: X\to X$, can be defined in obvious way. Similarly to [23, §2] we can obtain the following

4.1. **Theorem.** *The* G-SHAPE *defined above is a category.*

Define now a functor $S_G: G\text{-}\mathcal{H}\to G$-SHAPE as follows: $S_G(X)=X$, for any G-space X, and for $\varphi\in(G\text{-}\mathcal{H})(X,Y)$, $S_G(\varphi)$ is such that $S_G(\varphi)(\psi)=\psi\varphi$, for any $\psi\in(G\text{-}\mathcal{H})(Y,Q)$, where Q is any object of the category G-$\mathcal{CW}$.

4.2. **Theorem.** $S_G: G\text{-}\mathcal{H}\to$ G-SHAPE *is a covariant functor.*

The following theorems can be proved as in [20], [23].

4.3. **Theorem.** *If $Y\in\mathrm{Ob}(G\text{-}\mathcal{CW})$ and $f: X\to Y$ is a G-shaping, then there exists a unique G-homotopy class $\varphi: X\to Y$ such that $S_G(\varphi)=f$.*

4.4. **Theorem.** *Let $\mathcal{C}$ be a category whose objects are all G-spaces and le $S'_G: G\text{-}\mathcal{H}\to\mathcal{C}$ be a covariant functor with the following properties :*

(i) $S'_G(X)=X$, for any $X\in\mathrm{Ob}\,(G\text{-}\mathcal{H})$,

(ii) for any $Q\in\mathrm{Ob}\,(G\text{-}\mathcal{CW})$ and any morphism $f'\in\mathcal{C}(X,Q)$, there is a G-homotopy class $h'\in(G\text{-}\mathcal{H})(X,Q)$ such that $S'_G(h')=f'$.

Then there exists a unique covariant functor $F: \mathcal{C}\to$ G-SHAPE such that $F\circ S'_G=S_G$.

4.5. **Corollary.** *There exist a unique category G-SHAPE (up to isomorphisms) having the same objects as G-$\mathcal{H}$, and a unique functor, (up to equivalences) $S_G: G\text{-}\mathcal{H}\to$ G-SHAPE satisfying the properties (i) (ii) from 4.4, such that the universal property of Theorem 4.4 is true.*

4.6. **Theorem.** *If $f: X\to Y$ is a G-shaping and if $\varphi\in(G\text{-}\mathcal{H})(Y,Q)$, where $Q\in\mathrm{Ob}\,(G\text{-}\mathcal{CW})$, then $S_G(\varphi)(f)=S_G(f(\varphi))$.*

4.7. **Theorem.** *Let X be a G-space and $Y=\{Y_\beta, [q_{\beta\beta'}]_G, B\}$ be an inverse system of the category G-$\mathcal{CW}$ which is G-associated with another G-space. Let $q_\beta: Y\to Y_\beta$, $\beta\in B$, be G-maps satisfying the properties (i)—(iii)* *of Definition 1.3. Then there exists an $(1-1)$ correspondence between G-mapings $f: X \to Y$ and systems of G-homotopy classes $\varphi_\beta \in (G\text{-}\mathcal{H})(X, Y_\beta)$, $\beta \in B$, that verify the conditions: $[q_{\beta\beta'}]_G\, \varphi_{\beta'} = \varphi_\beta$, for any $\beta, \beta' \in B$ $\beta \leqslant \beta'$.*

The correspondence is given by $S_G[q_\beta]_G\, f = S_G(\varphi_\beta)$, $\beta \in B$.

4.8. Let $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ and $\mathbf{Y} = \{Y_\beta, [q_{\beta\beta'}]_G, B\}$ be two inverse systems in the category $G\,\mathcal{CW}$. A G-system map from $\mathbf{X}$ to $\mathbf{Y}$ consists of a map $\varphi: B \to A$ and G-homotopy classes of G maps $f_\beta \in (G\text{-}\mathcal{H})(X_{\varphi(\beta)}, Y_\beta)$, for $\beta \in B$, such that, if $\beta \leqslant \beta'$ in B then $[q_{\beta\beta'}]_G f_{\beta'} [p_{\varphi(\beta')\alpha}]_G = f_\beta [p_{\varphi(\beta)\alpha}]_G$, for some α with $\varphi(\beta) \leqslant \alpha$ and $\varphi(\beta') \leqslant \alpha$. Two G-system maps $\{\varphi, f_\beta, B\}$ and $\{\psi, g_\beta, B\}$ are said to be equivalent if, for any $\beta \in B$, there exists $\alpha \in A$ such that $\varphi(\beta) \leqslant \alpha$, $\psi(\beta) \leqslant \alpha$ and $f_\beta [p_{\varphi(\beta)\alpha}]_G = g_\beta [p_{\psi(\beta)\alpha}]_G$.

4.9. **Theorem.** *Suppose that the inverse systems $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ and $\mathbf{Y} = \{Y_\beta, [q_{\beta\beta'}]_G, B\}$ in the category $G\,\mathcal{CW}$ are G-associated with G-spaces X and Y respectively. Then, there is an $(1-1)$ correspondence between the G-shapings $f: X \to Y$ and the equivalence classes of G-system maps from* **X** *to* **Y**.

4.10. **Theorem.** *Let $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ be an inverse system in the category $G\,\mathcal{CW}$, G-associated with a G-space X. In the category G-SHAPE the object X is the inverse limit of the inverse system $\{X_\alpha, S_G([p_{\alpha\alpha'}]_G), A\}$.*

4.11. **Definition.** *Let X, Y be G-spaces. If there exist two G-shapings $f: X \to Y$, $g: Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$, we say that X and Y have the same G-shape and we denote this by $\mathrm{Sh}_G(X) = \mathrm{Sh}_G(Y)$.*

4.12. **Corollary.** *Two G-spaces which have the same G-homotopy type, have also the same G-shape.*

The proof is based on the fact that the functor S_G preserves isomorphisms.

4.13. **Corollary.** *Two G-CW complexes have the same G-shape if and only if they have the same G-homotopy type.*

Proof. The *necessity* follows by 12.4.
As for *sufficiency*, let X, Y be two G-spaces which have the same G-shape. Then, there exist $f \in$ G-SHAPE (X, Y) and $g \in$ G-SHAPE (Y, X) such that $gf = 1_X$ and $fg = 1_Y$. By 3.4, there exist a unique $h \in (G\text{-}\mathcal{H})(X, Y)$ and a unique $k \in (G\text{-}\mathcal{H})(Y, X)$ such that $S_G(h) = f$ and $S_G(k) = g$. It follows that $S_G(k \circ h) = S_G(k) \circ S_G(h) = gf = 1_X$ and by virtue of the uniqueness, we have $k \circ h = 1_X$. Similarly, $h \circ k = 1_Y$.

5. **The natural functor $*$: G-SHAPE$\to$SHAPE**

For a G-space X, denote by X^* the orbit space X/G. We can prove:

5.1. **Theorem.** *There exists a covariant functor*

$$* : G\text{-SHAPE} \to \text{SHAPE}$$

where SHAPE is the Mardesic-Morita shape category, which associates with every G-space X the orbit space $X^* = X/G$.

Proof. Let X, Y be G-spaces and let $f \in G$-SHAPE (X, Y). Define $f^* \in$ SHAPE (X^*, Y^*) as follows: let $\varphi \in \mathcal{H}(Y^*, Q)$, where Q is a CW-complex. Consider Q as a G-space with respect to the trivial action of G. Let $p': Y \to Y^*$ the natural projection. Then, $\varphi p': Y \to Q$ is a morphism in $G-\mathcal{H}(Y, Q)$: $\varphi p'(gy) = \varphi p'(gy) = \varphi(\hat{y})$ and $g(\varphi p')(y) = g\varphi(\hat{y}) = \varphi(\hat{y})$. Hence there is a morphism $f(\varphi p') \in (G\text{-}\mathcal{H})(X, Q)$. This morphism has the property that $f(\varphi p')(gx) = gf(\varphi p')(x) = f(\varphi p')(x)$, for any $x \in X$, which shows that we can define the map $f^*(\varphi): X^* \to Q$, by $f^*(\varphi)(\hat{x}) = f(\varphi p')(x)$. This is well defined, since if $\hat{y} = \hat{x} \in X^*$, then $f^*(\varphi)(\hat{y}) = f(\varphi p')(gx) = f(\varphi p')(x)$. We can verify easily that f^* is a shaping. In fact, if $Q' \in \mathrm{Ob}\,\mathcal{CW}$ and $\varphi' \in \mathcal{H}(Y^*, Q')$, $\psi \in \mathcal{CW}(Q, Q')$, with $\psi\varphi = \varphi'$ then $\psi(\varphi p') = \varphi' p'$ and this implies that $\psi(f(\varphi p')) = f(\varphi' p')$, and therefore, it follows that $\psi f^*(\varphi) = f^*(\varphi')$. Then, $(gf)^*(\varphi)(\hat{x}) = (gf)(\varphi p')(x)$ and $g^*(f^*(\varphi))(\hat{x}) = g(f^*(\varphi)p')(x) = g(f^*(\varphi))(\hat{x}) = g(f(\varphi p'))(x) = (gf)(\varphi p')(x)$, hence $(gf)^* = g^* f^*$. Clearly, $(1_X)^* = 1_{X^*}$ (identical shaping).

2.5. **Corollary.** *If X, Y are G-spaces such that $\mathrm{Sh}_G(X) = \mathrm{Sh}_G(Y)$, then* $\mathrm{Sh}(X^*) = \mathrm{Sh}(Y^*)$.

6. **Equivariant homology and cohomology related with the equivariant shape theory.** We consider the equivariant Čech homology and cohomology based on locally finite normal open G-coverings, i.e.

$$\check{H}^G_n(X) = \lim_{\overleftarrow{\mathcal{U}}} H_n(K(\mathcal{U})), \quad \check{H}^n_G(X) = \lim_{\overrightarrow{\mathcal{U}}} H^n(K(\mathcal{U})),$$

where $\mathcal{U}$ covers the set of all locally finite normal open G-coverings of the G-space X.

Remark. Usually it is not required for the G-coverings $\mathcal{U}$ the restriction to be locally finite and normal, but K. Morita showed, [24], that it is often useful to consider also this case, which for Hausdorff paracompact spaces coincides with the usual case.

From the Theorems 2.3 and 9.4 we obtain:

6.1. **Theorem.** *If X, Y are G-spaces and $f \in G$-SHAPE (X, Y), then there are natural homomorphisms*

$$f_*: \check{H}^G_n(X) \to \check{H}^G_n(Y), \quad f^*: \check{H}^n_G(Y) \to \check{H}^n_G(X).$$

This theorem implies:

6.2. **Corollary.** *If X, Y are G-spaces such that $\mathrm{Sh}_G(X) = \mathrm{Sh}_G(Y)$, then, for every integer n, there exist isomorphisms*

$$\check{H}^G_n(X) \cong \check{H}^G_n(Y), \quad \check{H}^n_G(X) \cong \check{H}^n_G(Y).$$

Then, using 2.5 and [23, Th. 3.1], we obtain:

6.3. **Corollary.** *If X, Y are G-spaces such that $\mathrm{Sh}_G(X) = \mathrm{Sh}_G(Y)$, then*

$$\check{H}_n(X^*) \cong \check{H}_n(Y^*), \quad \check{H}^n(X^*) \cong \check{H}^n(Y^*),$$

the considered Čech homology and cohomology groups being based on locally finite normal open coverings.

7. **An application: On some results concerning the equivariant homotopy type, [5, Cor. 5.5, Ch. II] [22], [15], [16].** 7.1. Let X be a G-space, with G a finite group and H be a subgroup of G.

Denote by X^H the fixed point set of the G-space X with respect to the subgroup H, i.e. $X^H = \{x \in X / hx = x, \ \forall h \in H\}$.

If $f: X \to Y$ is a G-map, this induces, for every subgroup H of G, a continuous map $f^H = f|X^H : X^H \to Y^H$.

7.2. **Proposition.** [15]. *If $f, g: X \to Y$ are two G-homotopic G-maps, then the maps $f^H, g^H : X^H \to Y^H$ are homotopic, for every subgroup H of the group G.*

Proof. If $F: X \times I \to Y$ is a G-homotopy between the G-maps f, g, then we have $F(hx, t) = hF(x, t)$, $\forall h \in H$, $\forall x \in X$, $\forall t \in I$. From this we deduce that $F(X^H \times I) \subset Y^H$ and therefore $F|X^H \times I : f^H \simeq g^H$.

7.3. **Corollary.** [15]. *If $f: X \to Y$ is a G-homotopy equivalence, then, for any subgroup H of the group G, the map $f^H: X^H \to Y^H$ is a homotopy equivalence.*

G. Bredon, [5, Ch. II] and T. Matumoto, [22], obtained some converse results of 3.7, when G is finite and the G-spaces X, Y are G-CW complexes. Also, in 1978, I. M. James and G.B. Segal proved the following theorem:

7.4. **Theorem,** [15]. *Let G be a compact Lie group, and let X, Y be G-ANR's. Then a G-map $f: X \to Y$ is a G-homotopy equivalence iff the map $f^H : X^H \to Y^H$ is a homotopy equivalence for all closed subgroups H of G.*

As an application of the equivariant shape theory we shall present a new result related with these problems. After all, this is one of the reasons for which we study the equivariant shape theory in the above manner, this construction relating arbitrary G-spaces with G-ANR's.

7.5. **Lemma,** [15]. *If X is a G-ANR space, then for every subgroup H of the group G, the space X^H is an ANR space.*

Proof. Firstly, if X is a separable and metrizable G-space, then X^H is a separable and metrizable space. Then, if B is a separable and metrizable space and A is a closed subspace of B, let $f: A \to X^H \subset X$ be a continuous map. Consider the G-space $G/H \times A$ (the group G being finite, this is a finite topological sum of spaces homeomorphic to A) with the following action: $g(\hat{g}', a) = (\widehat{gg}', a)$, for any $g \in G$, $\hat{g}' \in G/H$, $a \in A$. This is well defined and it is a G-action. Now, the map $f': G/H \times A \to X$, by $f'((\hat{g}, a)) = (gf)(a) = g(f(a))$ is well defined. In fact, if $\hat{g}' = \hat{g} \Rightarrow g' = gh \Rightarrow g'(f(a)) = (gh)(f(a)) = g(h(f(a))$ and since $f(a) \in X^H$, we have $h(f(a)) = f(a)$ and therefore $g'f(a) = gf(a)$. Then, f' is a G-map: $f'(g(\hat{g}', a)) = f'(\widehat{gg}', a) = (gg')f(a) = g(g'(f(a)) = gf'(\hat{g}', a)$.

We can consider the normal G-space $G/H \times B$ (with a G-action defined as above) for which $G/H \times A$ is an invariant closed subspace. Since X is a G-ANR space, the G-map f' has a G-extension $F': G/H \times B \to X$. Then, we can define the map $F: B \to X^H$, by $F(b) = F'(\hat{e}, b)$, for any $b \in B$ where e is the identity element of G. In fact: $hF(b) = hF'(\hat{e}, b) = F'(\hat{h}, b) = F'(\hat{e}, b) = F(b)$, i.e. $F(b) \in X^H$, for any $b \in B$. This is also an extension of the map f', since if $a \in A$, $F(a) = F'(\hat{e}. a) = f'(\hat{e}, a) = f(a)$. By the well-known characterization of ANR's, the statement of 5.7 is true.

7.6. **Corollary.** *If a G-space X has the G-homotopy type of a G-ANR, then, for every subgroup H of the group G, the space X^H has the homotopy type of an ANR space.*

7.7. **Definition.** *A G-space X is called a BMJS-space (Bredon-Matumoto-James-Segal) if for every inverse system $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ in the category G-$\mathcal{CW}$, G-associated with X, and every subgroup H of G, the inverse system $\mathbf{X}^H = \{X^H_\alpha, [p^H_{\alpha\alpha'}], A\}$, in the category $\mathcal{CW}$, (see 6.7), is associated with X^H.*

7.8. **Theorem.** *Let X, Y be BMJS-spaces and $f: X \to Y$ be a G-shaping. Then, for every subgroup H of the group G a shaping $f^H : X^H \to Y^H$, with the following properties, is induced in a natural way:*

(i) *if $g: Y \to Z$ is another G-shaping between BMJS-spaces, then $(gf)^H = g^H f^H$;*

(ii) $(1_X)^H = 1_X H$.

Proof. Let $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ and $\mathbf{Y} = \{Y_\beta, [q_{\beta\beta'}]_G, B\}$ be inverse sistems in the category $G\,\mathcal{CW}$, G-associated with X and Y, respectively. Then, $\mathbf{X}^H = \{X^H_\alpha, [p^H_{\alpha\alpha'}], A\}$ and $\mathbf{Y}^H = \{Y^H_\beta, [q^H_{\beta\beta'}], B\}$ are inverse systems in the category $\mathcal{CW}$ (see Corollary 6.7) and they are G-associated with X^H and Y^H respectively. From Theorem 9.4, the G-shaping f corresponds to an equivalence classes of G-system maps $[\{\varphi, f_\beta, B\}]$ from $\mathbf{X}$ to $\mathbf{Y}$. Every G-map $f_\beta: X_{\varphi(\beta)} \to Y_\beta$ induces a map $f^H_\beta : X^H_{\varphi(\beta)} \to Y^H_\beta$ such that $\{\varphi, f^H_\beta, B\}$ is a system map from $\mathbf{X}^H$ to $\mathbf{Y}^H$. In fact, we consider the map $\varphi: B \to A$ and for every $\beta \in B$ the map f^H_β. If $\beta \leqslant \beta'$, then, from 2.7, the equality $[q_{\beta\beta'}]_G f_{\beta'} [p_{\varphi(\beta')\alpha}]_G = f_\beta [p_{\varphi(\beta)\alpha}]_G$, for some $\alpha \in A$, with $\varphi(\beta) \leqslant \alpha$ and $\varphi(\beta') \leqslant \alpha$, implies that $[q^H_{\beta\beta'}] f^H_{\beta'} [p^H_{\varphi(\beta')\alpha}] = f^H_\beta [p^H_{\varphi(\beta)\alpha}]$. Moreover, the class of the system map $\{\varphi, f^H_\beta, B\}$ depends only on the equivalence class of the G-system map, $\{\varphi, f_\beta, B\}$. In fact, if $\{\psi, g_\beta, B\}$ is equivalent with $\{\varphi, f_\beta, B\}$, then $\{\psi, g^H_\beta, B\}$ is equivalent with $\{\varphi, f^H_\beta, B\}$ since the equality $f_\beta [p_{\varphi(\beta)\alpha}]_G = g_\beta [p_{\psi(\beta)\alpha}]_G$ implies the equality $f^H_\beta [p^H_{\varphi(\beta)\alpha}] = g^H_\beta [p^H_{\psi(\beta)\alpha}]$, for any $\beta \in B$ and $\alpha \in A$, such that $\varphi(\beta) \leqslant \alpha$, $\psi(\beta) \leqslant \alpha$.

Thus, the G-shaping f induces an equivalence class of system maps $[\{\varphi, f^H_\beta, B\}]$ from $\mathbf{X}^H$ to $\mathbf{Y}^H$. By Definition 7.7 and using Theorem 9.4 (for the trivial G-action) this defines a shape map $f^H : X^H \to Y^H$.

If $g: Y \to Z$ is another G-shaping between G-spaces with the property BMJS, let $\mathbf{Z} = \{Z_\gamma, [r_{\gamma\gamma'}]_G, C\}$ be an inverse system in the category G-$\mathcal{CW}$ corresponding to Z and $[\{\psi, g_\gamma, C\}]$ an equivalence class of G-system maps from $\mathbf{Y}$ to $\mathbf{Z}$. Then we can consider the composition $[\{\varphi\psi, g_\gamma f_{\psi(\gamma)}, C\}]$ which corresponds to gf. In fact, from 7.4, 8.4, 9.4 and [20, § 7], we have $S_G(f_\beta[p_{\varphi(\beta)}]_G) = S_G([q_\beta]_G) f$ and $S_G(g_\gamma[q_{\psi(\gamma)}]_G) = S_G([r_\gamma]_G)\, g$ and we deduce that $S_G(g_\gamma f_{\psi(\gamma)}[p_{\varphi\psi(\gamma)}]_G = S_G(g_\gamma)\, S_G(f_{\psi(\gamma)}[p_{\varphi\psi(\gamma)}]_G) = S_G(g_\gamma)\, S_G([q_{\psi(\gamma)}]_G f) = S_G(g_\gamma[q_{\psi(\gamma)}]_G) f = (S_G([r_\gamma]_G)\, g) f = S_G([r_\gamma]_G)(gf)$.

By the definition of the shaping f^H, the equivalence class of system maps corresponding to gf is $[\{\varphi\psi, g^H_\gamma f^H_{\psi(\gamma)}, C\}]$ and this is the composition of $[\{\varphi, f^H_\beta, B\}]$ with $[\{\psi, g^H_\gamma, C\}]$ which proves that $(gf)^H = g^H f^H$.

The assertion (ii) is obvious.

7.9. **Corollary.** *Let X, Y be G-spaces with the property BMJS and $f: X \to Y$ be a G-shaping. If f is a G-shape equivalence then, for any subgroup H of the group G, the induced shape map $f^H : X^H \to Y^H$ is a shape equivalence.*

10.7. Now we will state a converse result of Corollary 9.7.

To this, end we recall from [21] that if $f: X \to Y$ is a shaping, then we can suppose that the spaces X, Y have associated systems, in the category $\mathcal{CW}$, indexed by the same set A, i.e. $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}], A\}$ and

$\mathbf{Y}=\{Y_\alpha, [q_{\alpha\alpha'}], A\}$, respectively. Moreover, from 2.2 of [21], we can take a system map corresponding to the shaping f of the type $\{1_A, f_\alpha, A\}$ with $f_\alpha : X_\alpha \to Y_\alpha$. If f is a shape equivalence, then there exists a system map $\{1_A, g_\alpha, A\}$ such that, for every $\alpha \in A$ there are indices $\alpha', \alpha'' \in A$ with $\alpha \leqslant \alpha'$, $\alpha \leqslant \alpha''$ the relations $g_\alpha f_\alpha [p_{\alpha\alpha'}] = [p_{\alpha\alpha'}]$ and $f_\alpha g_\alpha [q_{\alpha\alpha''}] = [q_{\alpha\alpha''}]$ being satisfied.

11.7. **Definition.** *If the indices $\alpha', \alpha'' \in A$ can be choisen such that $p_{\alpha\alpha'}, q_{\alpha\alpha''}$ are homotopy equivalences, we refer to such shaping as to a special shape equivalence.*

12.7. **Theorem.** *Let X, Y, be G-spaces having the property BMJS and $f: X \to Y$ be a G-shaping.*

If for every subgroup H of G the induced shap map $f^H : X^H \to Y^H$ is a special shape equivalence, then f is a G-shape equivalence.

Proof. Using an equivariant version of the result 2.2 of [21], we can suppose that the G-spaces X, Y have G-associated inverse systems, in the category G-$\mathcal{CW}$, both indexed by the same set A, i.e. $\mathbf{X} = \{X_\alpha, [p_{\alpha\alpha'}]_G, A\}$ and $\mathbf{Y}=\{Y_\alpha, [q_{\alpha\alpha'}]_G, A\}$, respectively. Moreover, by the same result, we can take a G-system map corresponding to the G-shaping f (by Theorem 9.4) of the type $\{1_A, f_\alpha, A\}$ with $f_\alpha : X_\alpha \to Y_\alpha$ the G-map defined by the condition $S_G(f_\alpha [p_\alpha]_G) = S_G([q_\alpha]_G) f$ and such that if $\alpha \leqslant \alpha'$ in A, then $[q_{\alpha\alpha'}]_G f_{\alpha'} = f_\alpha [p_{\alpha\alpha'}]_G$.

In these conditions, for every subgroup H of G, the inverse systems $\mathbf{X}^H = \{X^H_\alpha, [p^H_{\alpha\alpha'}], A\}$ and $\mathbf{Y}^H = \{Y^H_\alpha, [q^H_{\alpha\alpha'}], A\}$ are associated to X^H and Y^H, respectively (Definition 7.7) and the shape map $f^H : \mathbf{X}^H \to \mathbf{Y}^H$ is defined by the class of G-system maps $[\{1_A, f^H_\alpha, A\}]$, (Theorem 8.7).

If we suppose that f^H is a special shape equivalence, then there is a G-system map $\{1_A, g^H_\alpha, A\}$ such that for every $\alpha \in A$ there exist $\alpha' \in A$, $\alpha'' \in A$ with $\alpha \leqslant \alpha'$, $\alpha \leqslant \alpha''$ and $g^H_\alpha f^H_\alpha [p^H_{\alpha\alpha'}] = [p^H_{\alpha\alpha'}]$, $f^H_\alpha g^H_\alpha [q^H_{\alpha\alpha''}] = [q^H_{\alpha\alpha''}]$, where $p^H_{\alpha\alpha'}$ and $q^H_{\alpha\alpha''}$ are homotopy equivalence. It follows that $f^H_\alpha : X^H_\alpha \to y^H_\alpha$ is a homotopy equivalence. Now we use Theorem 4.7 by which we have that $f_\alpha : X_\alpha \to Y_\alpha$ are G-homotopy equivalences, for any $\alpha \in A$. If g_α is a G-homotopy inverse to f_α, then, because $[p_{\alpha\alpha}]_G = [1_{X\alpha}]_G$, $[q_{\alpha\alpha}] = [1_{Y\alpha}]_G$, for any $\alpha \in A$, we have $g_\alpha f_\alpha [p_{\alpha\alpha}]_G = [p_{\alpha\alpha}]_G$, $f_\alpha g_\alpha [q_{\alpha\alpha}]_G = [q_{\alpha\alpha}]_G$, which proves that f is a G-shape equivalence.

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