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SEMI-INVARIANT PRODUCTS IN SASAKIAN MANIFOLDS

BY

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1. **Preliminaries.** Let $\tilde{M}$ be a $(2n+1)$-dimensional Sasakian manifold with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type (1.1) on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$. As it is well-known, these tensor fields are related by

$$\varphi^2 X = -X + \eta(X)\xi \; ; \; \varphi\xi = 0 \; ; \; \eta(\varphi X) = 0 \; ; \; \eta(\xi) = 1, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(Y)\eta(X) \text{ and} \tag{1.2}$$

$$(\tilde{\nabla}_X \varphi) Y = g(X, Y)\xi - \eta(Y)X, \tag{1.3}$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. Also, from (1.3) we have

$$\tilde{\nabla}_X \xi = -\varphi X \text{ for all } X \text{ tangent to } \tilde{M}. \tag{1.4}$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of $\tilde{M}$ is tangent to M. We denote by TM and $TM^{\perp}$ the tangent bundle to M and respectively the normal bundle to M.

Definition [2]. *The submanifold M of $\tilde{M}$ is called a semi-invariant submanifold if it is endowed with the pair of orthogonal distributions $(D, D^{\perp})$ satisfying the conditions:*

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, *where $\{\xi\}$ denotes the distribution spanned by ξ,*

(ii) *the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$, for each $x \in M$,*

(iii) *the distribution $D^{\perp}$ is anti-invariant by φ, that is, $\varphi(D^{\perp}_x) \subset T_x M^{\perp}$ for each $x \in M$.*

Throughout this paper we denote by $2p$ (resp. q) the dimension of D_x (resp. $D^{\perp}_x$). Then it is easily seen that when $p=0$ (resp. $q=0$) the semi-invariant submanifold is an *anti-invariant submanifold* tangent to ξ (resp. an *invariant submanifold*). A *generic submanifold* is a semi-invariant submanifold characterized by the condition dim $T_x M = q$ for each $x \in M$, [7]. The semi-invariant submanifold M is called a *proper semi-invariant submanifold* if it is neither an invariant nor an anti-invariant submanifold.

We denote by the same symbol g both Riemannian metrics on M and $\tilde{M}$. The projection morphisms of TM to D and $D^\perp$ are denoted respectively by P and Q. For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H. Throughout the paper, all manifolds and morphisms are supposed to be differentiable of class C^∞.

The Gauss and Weingarten formulas for the submanifold M are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{1.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where ∇ is the Levi-Civita connection on M, $\nabla^\perp$ is the linear connection induced by $\tilde{\nabla}$ on $TM^\perp$, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. The fundamental tensor A_N is related to h by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{1.7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. For all $X \in \Gamma(TM)$ we have

$$X = PX + QX + \eta(X)\xi, \text{ where } \eta(X) = g(X, \xi). \tag{1.8}$$

For any $N \in \Gamma(TM^\perp)$ we put

$$\varphi N = tN + fN \tag{1.9}$$

where tN and fN are the tangential and normal components of φN, respectively.

Now, suppose $\tilde{M}(k)$ be a Sasakian space form of constant φ-sectional curvature k. Then, the curvature tensor $\tilde{R}$ of $\tilde{M}(k)$ is given by [5, p. 97]

$$\begin{aligned} \tilde{R}(X, Y)Z = [(k+3)/4]\{g(Y, Z)X - g(X, Z)Y\} + [(k-1)/4]\{\eta(X)\eta(Z)Y - \\ -\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(Z, \varphi Y)\varphi X - \\ -g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\} \end{aligned} \tag{1.10}$$

for any X, Y, Z tangent to $\tilde{M}(k)$.

We denote by $R^{2n+1}(-3)$ the Sasakian space form with constant φ-sectional curvature $k=-3$ and with standard Sasakian structure in the Euclidean space R^{2n+1} (cf. [5], p. 99). By $S^{2n+1}(1)$ we denote a $(2n+1)$-dimensional unit sphere endowed with the natural Sasakian structure of φ-sectional curvature $k=1$, [5, p. 99]. For the second fundamental form h of a submanifold M in $\tilde{M}$, we define the covariant differentiation $\bar{D}$ by

$$(\bar{D}_X h)(Y, Z) = \nabla^\perp_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.11}$$

for any X, Y, $Z \in \Gamma(TM)$.

The equations of Gauss and Codazzi are then given respectively by

$$\begin{aligned} \tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(h(X, W), h(Y, Z)) - \\ -g(h(X, Z), h(Y, W)), \end{aligned} \tag{1.12}$$

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{D}_X h)(Y, Z) - (\bar{D}_Y h)(X, Z) \tag{1.13}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$ and $(\tilde{R}(X, Y)Z)^\perp$ is the normal component of $\tilde{R}(X, Y)Z$.

For a Sasakian manifold $\tilde{M}$, we define the φ-bisectional curvature with respect to two orthonormal vectors X, Y from the contact distribution by

$$\tilde{H}_B(X, Y) = \tilde{R}(X, \varphi X; \varphi Y, Y). \tag{1.14}$$

For a semi-invariant submanifold M in the Sasakian manifold $\tilde{M}$, we denote by ν the orthogonal complementary vector bundle to $\varphi D^\perp$ in $TM^\perp$. Thus, the normal bundle to M has the decomposition

$$TM^\perp = \varphi D^\perp \oplus \nu, \text{ where } g(\varphi D^\perp, \nu) = 0. \tag{1.15}$$

2. **Some basic results.** In this paragraph we recall some basic formulas and results from [2] and [3].

Lemma 2.1, [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

$$P\nabla_X \varphi PY = \varphi P\nabla_X Y + PA_{\varphi QY}X - \eta(Y)PX, \tag{2.1}$$

$$Q\nabla_X \varphi PY = QA_{\varphi QY}X + th(X, Y) - \eta(Y)QX, \tag{2.2}$$

$$\eta(\nabla_X \varphi PY - A_{\varphi QY}X) = g(\varphi X, \varphi Y), \tag{2.3}$$

$$h(X, \varphi PY) + \nabla^\perp_X \varphi QY = fh(X, Y) + \varphi Q\nabla_X Y \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 2.2, [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

$$h(X, \xi) = 0 \text{ and } \nabla_X \xi = -\varphi X \text{ for any } X \in \Gamma(D), \tag{2.5}$$

$$h(Y, \xi) = -\varphi Y \text{ and } \nabla_Y \xi = 0 \text{ for any } Y \in \Gamma(D^\perp), \tag{2.6}$$

$$A_{\varphi X}Y = A_{\varphi Y}X \text{ for any } X, Y \in \Gamma(D^\perp) \text{ and} \tag{2.7}$$

$$h(\xi, \xi) = 0, \qquad \nabla_\xi \xi = 0. \tag{2.8}$$

Concerning the integrability of distributions on M we have the following

Lemma 2.3, [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

(i) *the distribution $D^\perp$ is always integrable,*

(ii) *the distribution D is never integrable,*

(iii) *the distribution $D \oplus \{\xi\}$ is integrable if and only if the second fundamental form of M satisfies*

$$h(X, \varphi Y) = h(\varphi X, Y) \text{ for any } X, Y \in \Gamma(D). \tag{2.9}$$

The semi-invariant submanifold M of $\tilde{M}$ is called $D\oplus\{\xi\}$-*geodesic* if we have :

$$h(X, Y)=0 \text{ for all } X, Y\in\Gamma(D+\{\xi\}). \tag{2.10}$$

Lemma 2.4, [3]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then M is $D\oplus\{\xi\}$-geodesic if and only if the distribution $D\oplus\{\xi\}$ is integrable and each leaf of the distribution $D\oplus\{\xi\}$ is totally geodesic immersed in $\tilde{M}$.*

Lemma 2.5, [3]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then, a leaf $M^\perp$ of the distribution $D^\perp$ is totally geodesic immersed in M if and only if we have*

$$h(X, Y)\in\Gamma(\nu) \text{ for any } X\in\Gamma(D) \text{ and } Y\in\Gamma(D^\perp). \tag{2.11}$$

The semi-invariant submanifold M of $\tilde{M}$ is called $(D, D^\perp)$-*geodesic* if we have

$$h(X, Y)=0 \text{ for all } X\in\Gamma(D) \text{ and } Y\in\Gamma(D^\perp). \tag{2.12}$$

Also, the semi-invariant submanifold M of $\tilde{M}$ is called $D^\perp$-*geodesic* if we have

$$h(X, Y)=0 \text{ for any } X, Y\in\Gamma(D^\perp). \tag{2.13}$$

Then, by using Lemma 2.5 we obtain

Lemma 2.6, [3]. *Let M be a $(D, D^\perp)$-geodesic semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

1. *Each leaf of the distribution $D^\perp$ is totally geodesic immersed in M.*
2. *Each leaf of the distribution $D^\perp$ is totally geodesic immersed in the Sasakian manifold $\tilde{M}$ if and only if the semi-invariant submanifold M is $D^\perp$-geodesic.*

3. Semi-invariant products in Sasakian manifolds. Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$.

Definition. *We say that M is b semi-invariant product if it is locally a Riemannian product $M_1\times M_2$ where M_1 is an invariant submanifold of $\tilde{M}$ and M_2 is an anti-invariant submanifold of $\tilde{M}$ such that the structure vector field ξ of $\tilde{M}$ is normal to M_2.* In [1], A. Bejancu obtains some characterizations of semi-invariant products in a Sasakian manifold and investigates the existence of semi-invariant products in $R^{2n+1}(-3)$.

The purpose of this paper is to obtain new results on semi-invariant products in Sasakian manifolds, specially in S^{2n+1} (1). Similar problems have been studied by B.Y. Chen in [6] for CR-products of CP^n.

Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. We denote by $F=\varphi P$. Then it is easily seen that F is an f-structure on M, that is $F^3+F=0$, [8]. We define

$$(\nabla_X F)Y=\nabla_X FY-F\nabla_X Y, \text{ for any } X, Y\in\Gamma(TM). \tag{3.1}$$

Definition. *The f-structure F is said to be η-parallel if we have*

$$(\nabla_X F)Y=g(PX, PY)\xi-\eta(Y)PX, \tag{3.2}$$

for all $X, Y\in\Gamma(TM)$.

Theorem 3.1. *Let M be an m-dimensional semi-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\tilde{M}$. Then M is a semi-invariant product if and only if the f-structure F is η-parallel.*

Proof. From (3.1) and (3.2) we obtain that the f-*structure* F is η-parallel if and only if we have

$$\nabla_X Y\in\Gamma(D\oplus\{\xi\}) \text{ for any } X\in\Gamma(TM) \text{ and } Y\in\Gamma(D). \tag{3.3}$$

Thus, the assertion of Theorem 3.1 follows from Theorem 1 of [1].

Lemma 3.1. *Let M be a semi-invariant product of a Sasakian manifold $\tilde{M}$. Then, for any unit vectors $X\in\Gamma(D)$ and $Y\in\Gamma(D^\perp)$, we have*

$$\tilde{H}_B(X, Y)=2\|h(X, Y)\|^2-2. \tag{3.4}$$

Proof. Since M is a semi-invariant product, then M is locally the Riemannian product $M_1\times M_2$, where M_1 is an invariant submanifold of $\tilde{M}$ and M_2 is an anti-invariant submanifold normal to the structure vector field ξ. Then M_1 and M_2 are totally geodesic immersed in M, that is (2.9) and (2.11) are satisfied. From (2.9) and (2.11) we get $g(A_{\varphi Z}X, Y)=g(h(X, PY)+h(X, QY), \varphi Z)=0$ for any $X\in\Gamma(D)$, $Z\in\Gamma(D^\perp)$ and $Y\in\Gamma(TM)$, that is we have

$$A_{\varphi Z}X=0 \text{ for any } X\in\Gamma(D) \text{ and } Z\in\Gamma(D^\perp). \tag{3.5}$$

By using (1.11), (2.11) and (3.5) in (1.13), for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^\perp)$ we obtain

$$\begin{aligned}\tilde{R}(X, \varphi X; Z, \varphi Z)&=g(\nabla^\perp_X h(\varphi X, Z), \varphi Z)-g(\nabla^\perp_{\varphi X} h(X, Z), \varphi Z)-\\ &-g(h(\nabla_X\varphi X, Z), \varphi Z)-g(h(\varphi X, \nabla_X Z), \varphi Z)+g(h(\nabla_{\varphi X}X, Z), \varphi Z)+\\ &+g(h(X, \nabla_{\varphi X}Z), \varphi z)=g(\nabla^\perp_X h(\varphi X, Z), \varphi Z)-g(\nabla^\perp_{\varphi X} h(X, Z), \varphi Z)+\\ &+g(h([\varphi X, X], Z), \varphi Z)=g(\nabla^\perp_X h(\varphi X, Z), \varphi Z)-g(\nabla^\perp_{\varphi X} h(X, Z), \varphi Z)+\\ &+\eta([\varphi X, X])g(h(\xi, Z), \varphi Z)\end{aligned} \tag{3.6.}$$

Now, by using Lemma 2.2 we have :

$\eta([\varphi X, X])=g(\varphi X, \nabla_X\xi)-g(X, \nabla_{\varphi X}\xi)=-2g(X, X)=-2$ and $h(\xi, Z)=-\varphi Z$ for any unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^\perp)$. Substituting this in (3.6) we get

$$\tilde{R}(X, \varphi X; Z, \varphi Z)=g(\nabla^\perp_X h(\varphi X, Z), \varphi Z)-g(\nabla^\perp_{\varphi X} h(X, Z), \varphi Z)+2 \tag{3.7}$$

for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^\perp)$.

On the other hand, from (1.3), (1.5), (1.6) and (2.11) we obtain

$$(3.8)\qquad g(\nabla^{\perp}_{X}h(\varphi X, Z), \varphi Z)-g(\nabla^{\perp}_{\varphi X}h(X, Z), \varphi Z)=g(h(X, Z), \tilde{\nabla}_{\varphi X}\varphi Z)-$$
$$-g(h(\varphi X, Z), \tilde{\nabla}_{X}\varphi Z)=g(h(X, Z), \varphi h(\varphi X, Z))-g(h(\varphi X, Z), \varphi h(X, Z))$$

for any $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$.

Also, by using (1.3) and (1.5) we get

$$(3.9)\qquad h(\varphi X, Z)=\varphi h(X, Z) \text{ for any } X\in\Gamma(D) \text{ and } Z\in\Gamma(D^{\perp}).$$

Substituting (3.9) in (3.8) we get

$$(3.10)\qquad g(\nabla^{\perp}_{X}h(\varphi X, Z), \varphi Z)-g(\nabla^{\perp}_{\varphi X}h(X, Z), \varphi Z)=-2\|h(X, Z)\|^{2}$$

for any $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$.

Thus, (3.4) follows from (1.14), (3.7) and (3.10).

From Lemma 3.1 we have the following

Theorem 3.2. *Let $\tilde{M}$ be a Sasakian manifold with φ-bisectional curvature $\tilde{H}_{B}<-2$. Then every semi-invariant product in $\tilde{M}$ is either an invariant submanifold of $\tilde{M}$ or an anti-invariant submanifold tangent to ξ. In particular, there exists no proper semi-invariant products in Sasakian space forms with $\tilde{M}(k)$ with $k<-3$.*

Remark. The last part of Theorem 3.2 has been obtained by A. Bejancu in [1].

Corollary 3.1. *Let $\tilde{M}$ be a Sasakian manifold with φ-bisectional curvature $\tilde{H}_{B}>-2$ and M be a semi-invariant product in $\tilde{M}$. Then we have:*

(i) *M is not a generic semi-invariant submanifold and*

(ii) *M is not a $(D, D^{\perp})$-geodesic semi-invariant submanifold of $\tilde{M}$. In particular, if $\tilde{M}$ is a Sasakian space form with φ-sectional curvature $k>-3$ then the assertion (i) and (ii) are valid too.*

Corollary 3.2. *Every semi-invariant product in $R^{2n+1}(-3)$ is $(D, D^{\perp})$-geodesic.*

Proof. Since M is a semi-invariant product in $R^{2n+1}(-3)$, we have $\tilde{H}_{B}(X, Y)=-2$ for all unit vectors $X\in\Gamma(D)$ and $Y\in\Gamma(D^{\perp})$. Then Lemma 3.1 implies $h(X, Y)=0$ for any $X\in\Gamma(D)$ and $Y\in\Gamma(D^{\perp})$, that is M is $(D, D^{\perp})$-geodesic immersed in $R^{2n+1}(-3)$.

4. **Semi-invariant products in** $S^{2n+1}(1)$. In this section we shall derive the smallest codimension of a semi-invariant product in odd-dimensional unit spheres.

First, we shall give examples of semi-invariant products in $S^{2n+1}(1)$. Let $C^{p+q+pq+1}$ be the complex $(p+q+pq+1)$-dimensional number space and consider the odd-dimensional unit sphere $S^{2p+2q+2pq+1}(1)$ in $C^{p+q+pq+1}$. Then $S^{2p+2q+2pq+1}(1)$ admits a Sasakian structure (φ, ξ, η, g) as follows. Let v be the position vector representing a point of $S^{2p+2q+2pq+1}(1)$ in $C^{p+q+pq+1}$. Then the structure vector field of $S^{2p+2q+2pq+1}(1)$ is given by $\xi=Jv$, where J denotes the almost complex structure of $C^{p+q+pq+1}$. Consider the orthogonal projection

$$\pi : T_{x}(C^{p+q+pq+1}) \to T_{x}(S^{2p+2q+2pq+1}(1))$$

and put $\varphi=\pi\circ J$. We denote by η the 1-form dual to ξ and by g the standard metric tensor field on $S^{2p+2q+2pq+1}(1)$. Then, for any vector field X tangent to $S^{2p+2q+2pq+1}(1)$ we have $\varphi X=JX+\eta(X)v$. Now, we consider $S^{2p+1}(1)$ (resp. $S^{2q+1}(1)$) the unit sphere in C^{p+1} (resp. C^{q+1}) and we restrict the almost complex structure of $C^{p+q+pq+1}$ to C^{p+1} (resp. C^{q+1}). Also, we restrict the Sasakian structure of $S^{2p+2q+2pq+1}(1)$ to $S^{2p+1}(1)$ (resp. to $S^{2q+1}(1)$). Then if v is the position vector representing a point of $S^{2p+1}(1)$ in $C^{p+q+pq+1}$, we have that the structure vector field of $S^{2p+2q+2pq+1}(1)$ is tangent to $S^{2p+1}(1)$. We define a mapping $f_{p,q}: S^{2p+1}\times S^{2q+1}\to S^{2p+2q+2pq+1}$ by $f_{p,q}(z_0, z_1, \dots, z_p;$ $w_0, w_1, \dots, w_q)=(z_0w_0, \dots, z_iw_j, \dots, z_pw_q)$ where $(z_0, z_1, \dots, z_p)\in C^{p+1}$ such that $\sum_{i=0}^{p}|z_i|^2=1$ and $(w_0, w_1, \dots, w_q)\in C^{q+1}$ such that $\sum_{j=0}^{p}|w_j|^2=1$. Then it is easily seen that $(z_0w_0, \dots, z_iw_j, \dots, z_pw_q)$ is in $C^{p+q+pq+1}$ and $\sum_{i=0}^{p}\sum_{j=0}^{q}|z_iw_j|^2=1$. Now, let M_2 be a q-dimensional anti-invariant submanifold in $S^{2q+1}(1)$ such that the structure vector field ξ is normal to M_2, where $S^{2q+1}(1)$ is endowed with the restriction of the Sasakian structure of $S^{2p+2q+2pq+1}(1)$ to $S^{2q+1}(1)$. We denote by $M=S^{2p+1}(1)\times M_2$. Then M is a natural $(2p+q+1)$-dimensional proper semi-invariant product in $S^{2p+2q+2pq+1}(1)$ via $f_{p,q}$, in which $M_1=S^{2p+1}(1)$ is a totally geodesic invariant submanifold and M_2 is an anti-invariant submanifold of $S^{2p+2q+2pq+1}(1)$ normal to ξ.

We shall prove that $2n+1=2(p+q+pq)+1$ is in fact the smallest dimension of $S^{2n+1}(1)$ to admit a semi-invariant product.

Lemma 4.1. *Let M be a semi-invariant product in the Sasakian space form $S^{2n+1}(1)$. Then,*

$$\{h(X_i, Z_a)\}\ ;\ i=1, 2, \dots, 2p\ ;\ a=1, 2, \dots, q$$

are $2pq$ orthonormal vectors in ν_x, where $\{X_1, X_2, \dots, X_{2p}\}$ and $\{Z_1, Z_2, \dots, Z_q\}$ are orthonormal bases for D_x and $D^{\perp}_x$, respectively.

Proof. Since $S^{2n+1}(1)$ is of constant φ-sectional curvature $k=1$, from (1.10) and (1.14) we have $\tilde{H}_{B}(X, Z)=0$ for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. Then Lemma 3.1 implies

$$(4.1)\qquad \|h(X, Z)\|=1$$

for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. From (4.1), by linearity we get

$$(4.2)\qquad g(h(X_i, Z), h(X_j, Z))=0\ ;\ i\neq j.$$

Also, from (3.5) we see that $h(X, Z)\in\Gamma(\nu)$ for any $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. Thus, from (4.2) it follows that, if $q=1$ the lemma is proved. Suppose $q\geqslant 2$. Then, by linearity, (4.1) and (4.2) imply

$$(4.3)\qquad g(h(X_i, Z_a), h(X_j, Z_b))+g(h(X_i, Z_b), h(X_j, Z_a))=0$$

for $i\neq j$ and $a\neq b$; $j, i=1, \dots, 2p$; $a, b=1, \dots, q$ where $\{X_1, \dots, X_{2p}\}$ and $\{Z_1, \dots, Z_q\}$ are orthonormal bases for D_x and $D^{\perp}_x$, respectively. On the other hand, since M is a semi-invariant product, we have that the distributions $D\oplus\{\xi\}$ and $D^{\perp}$ are integrable and their leaves are totally geodesic immersed in M. Then, by using Lemma 2.2 and the Bianchi identity we obtain

(4.4) $R(X, Y; Z, W)=0$, for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$.

Also, from (1.10), taking $k=1$ we have

(4.5) $\tilde{R}(X_i, X_j; Z_a, Z_b)=0$ for all $i, j=1, 2, \ldots, 2p$ and $a, b=1, 2, \ldots, q$.

Therefore, substituting (4.4) and (4.5) in the Gauss equation (1.12), we get

(4.6) $$g(h(X_i, Z_a), h(X_j, Z_b))-g(h(X_i, Z_b), h(X_j, Z_a))=0$$

for all $i, j=1, \ldots, 2p$ and $a, b=1, \ldots, q$.

Then (4.1), (4.2), (4.3) and (4.6) give that $\{h(X_i, Z_a);\ i=1, \ldots, 2p;\ a=1, \ldots, q\}$ are $2pq$ orthonormal vectors in ν_x.

From (1.15) and Lemma 4.1 we obtain the following.

Theorem 4.1. *Let M be a semi-invariant product in $S^{2n+1}(1)$. Then we have*

(4.7) $$n \geq p+q+pq$$

Theorem 4.2. *Let $M=M_1 \times M_2$ be a semi-invariant product in $S^{2n+1}(1)$ with $n=p+q+pq$. Then the invariant submanifold M_1 is totally geodesic immersed in $S^{2n+1}(1)$.*

Proof. Let $M=M_1 \times M_2$ be a semi-invariant product in $S^{2n+1}(1)$ with $n=p+q+pq$. Then for any $X, Y \in \Gamma(D)$, from (2.2) we obtain $th(X, Y)=0$, that is

(4.8) $$h(X, Y) \in \Gamma(\nu).$$

Since $k=1$, from (1.10) we get

(4.9) $\tilde{R}(X, Y; Z, W)=0$, for all $X, Y, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Also, by a direct computation, using Lemma 2.2 we obtain

(4.10) $R(X, Y; Z, W)=g(\nabla_X \nabla_Y Z, W)-g(\nabla_Y \nabla_X Z, W)-g(\nabla_{[X,Y]} Z, W)=0$,

for all $X, Y, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$ Substituting (4.9) and (4.10) in the equation of Gauss we obtain

(4.11) $$g(h(X, W), h(Y, Z))-g(h(X, Z), h(Y, W))=0$$

for any $X, Y, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$. In (4.11) we take $Y=\varphi X$ and obtain

(4.12) $g(h(X, W), h(\varphi X, Z))=g(h(X, Z), h(\varphi X, W))$ for all $X, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

On the other hand, from (4.8) and (2.4) we get

(4.13) $h(X, \varphi Z)=\varphi h(X, Z)$ for all $X, Z \in \Gamma(D)$.

Then, by using (3.9) and (4.13) we obtain

(4.14) $g(h(X, W), h(\varphi X, Z))=-g(h(X, Z), h(\varphi X, W))$, for all $X, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$

Thus, (4.12) and (4.14) imply

(4.15) $g(h(X, Z), h(\varphi X, W))=0$, for any $X, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

In (4.15) we take $Z=\varphi Y$, where $Y \in \Gamma(D)$. Then from (4.15), (3.9) and (4.13) we obtain

(4.16) $g(h(X, Y), h(X, W))=0$ for all $X, Y \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

From (4.16), by linearity, we have

(4.17) $g(h(X, Z), h(Y, W))+g(h(X, W), h(Y, Z))=0$ for all $X, Y, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Combining (4.11) and (4.17) we obtain

(4.18) $g(h(X, Z), h(Y, W))=0$ for any $X, Y, Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Now, since $n=p+q+pq$, Lemma 4.1, (4.8) and (4.18) imply

(4.19) $$h(X, Z)=0 \text{ for all } X, Z \in \Gamma(D).$$

From (4.19) and (2.5), it follows that the semi-invariant product is $D \oplus \{\xi\}$-geodesic immersed in $S^{2n+1}(1)$. Thus, by Lemma 2.4 we have that M_1 as a leaf of the distribution $D \oplus \{\xi\}$ is totally geodesic immersed in $S^{2n+1}(1)$ and the proof is complete.

As an immediate consequence of Theorem 4.2, we have the following.

Theorem 4.3. *Let $\tilde{M}=\tilde{M}_1 \times \tilde{M}_2$ be the Riemannian product of two Sasakian manifolds with* dim. $\tilde{M}_1=2p+1$ *and* dim. $\tilde{M}_2=2q+1$. *Then $\tilde{M}_1 \times \tilde{M}_2$ admits an immersion in $S^{2p+2q+2pq+1}(1)$ such that the Sasakian structures of $\tilde{M}_1$ and $\tilde{M}_2$ be the restrictions of the Sasakian structure of $S^{2p+2q+2pq+1}(1)$ to $\tilde{M}_1$ and $\tilde{M}_2$, respectively if and only if both $\tilde{M}_1$ and $\tilde{M}_2$ are Sasakian space forms of constant φ-sectional curvature $k=1$. Moreover $2p+2q+2pq+1$ is the smallest dimension of $S^{2n+1}(1)$ with admits such immersions and $\tilde{M}_1$ and $\tilde{M}_2$ are both totally geodesic in $S^{2n+1}(1)$.*

Now we state

Theorem 4.4. *Let M be a semi-invariant product in $S^{2n+1}(1)$. Then we have*

(4.20) $$\|h\|^2 \geq 2q(2p+1),$$

*where $2p=$*dim. D_x *and $q=$*dim. $D_x^{\perp}$. *If in (4.20) the equality sign holds, then M_1 and M_2 are both totally geodesic immersed in $S^{2n+1}(1)$, M_1 is a Sasakian space form of constant φ-sectional curvature $k=1$ and M_2 is a real space form of constant sectional curvature 1.*

Proof. Since $S^{2n+1}(1)$ is of constant φ-sectional curvature $k=1$ and M is a semi-invariant product, Lemma 3.1 gives

(4.21) $\|h(X, Z)\|=1$ for any unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

On the other hand, it is known that the length of the second fundamental form h of M is given by

$$\|h\|^2=\sum_{\alpha,\beta=1}^{m}\{g(h(E_\alpha,E_\beta),h(E_\alpha,E_\beta))\} \tag{4.22}$$

where $\{E_1,E_2,\dots,E_{m-1},E_m=\xi\}$ is a local field of orthonormal frames on M. The by using Lemma 2.2, (4.21) and (4.22) we get

$$\|h\|^2=2q(2p+1)+\sum_{i,j=1}^{2p}\|h(X_i,X_j)\|^2+\sum_{a,b=1}^{q}\|h(Z_a,Z_b)\|^2, \tag{4.23}$$

where $\{X_1,\dots,X_{2p}\}$ (resp. $\{Z_1,\dots,Z_q\}$) is an orthonormal basis in D_x (resp. $D^{\perp}_x$). Thus, (4.20) follows from (4.23).

If the equality sign of (4.20) holds, then (4.23) implies

$$h(X,Y)=0 \text{ and } h(Z,W)=0 \tag{4.24}$$

for any $X,Y\in\Gamma(D)$ and $Z,W\in\Gamma(D^{\perp})$, that is, the semi-invariant product M is $D\oplus\{\xi\}$-geodesic and $D^{\perp}$-geodesic immersed in $S^{2n+1}(1)$. Since M_1 and M_2 are both totally geodesic immersed in M, from Lemmas 2.4 and 2.6 we have that M_1 and M_2 are both totally geodesic immersed in $S^{2n+1}(1)$. Also, from (1.10), (1.12) and (4.24) we obtain that M_1 is a Sasakian space form of constant φ-sectional curvature $k=1$ and M_2 is a real space form of constant sectional curvature 1.

Theorem 4.5. *Let M be a minimal semi-invariant product in $S^{2n+1}(1)$. Then the scalar curvature ρ of M satisfies*

$$\rho\leq 4p^2+q^2+2p-q. \tag{4.25}$$

The equality sign holds if and only if $\|h\|^2=2q(2p+1)$.

Proof. By direct computation we have, [4], that the scalar curvature ρ of an m-dimensional minimal semi-invariant submanifold M of a Sasakian space form $\tilde{M}(k)$ is given by

$$\rho=\frac{m(m-1)(k+3)-2(m-3p-1)(k-1)}{4}-\|h\|^2. \tag{4.26}$$

Then (4.25) follows from (4.26) and (4.20). Also, from (4.26) and (4.20) it is clear that the equality sign of (4.25) holds if and only if $\|h\|^4=2q(2p+1)$.

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ON THE PROPERTIES OF A CONSTANT CURVATURE RIEMANNIAN MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

BY

PETRE STAVRE

In the present paper we study the semi-symmetric metric connection whose D-concircular curvature tensor vanishes. We obtain on this way a generalisation of the theorem (2) of K. Yano [13] and of the results from [11].

1. Let L_n be an n-dimensional differentiable manifold of class C^∞, g a Riemannian metric in L_n, and D a semi-symmetric connection in L_n:

$$T(X,Y)=\omega(Y)X-X\omega(Y)\ ;\ \forall X,Y\in\chi(L_n). \tag{1.1}$$

where T is the torsion of D, $T(X,Y)=D_XY-D_YX-[X,Y]$, and ω is a 1-form

$$D_Xg=0;\ \ \forall X. \tag{1.2}$$

We have

$$D_XY=\nabla_XY+\omega(Y)X-g(X,Y)G_*\omega. \tag{1.3}$$

where we denote by ∇ the Levi-Civita connection with respect to g and $G_*\omega$ is defined by.

$$g(G_*\omega,X)=\omega(X)\ ;\ \forall X. \tag{1.4}$$

We denote also by

$$R(X,Y)Z=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z \tag{1.5}$$

the curvature tensor of the connection D, by

$$r(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z \tag{1.6}$$

the curvature tensor of the connection ∇, by ric (X,Z), the Ricci tensor of ∇.

Thus, from (1.2) and (1.3) (1.6), we obtain [13]

$$\begin{aligned}R(X,Y)Z={}&r(X,Y)Z+((D_X\omega)(Z)-(1/2)\,g(X,Z)\omega(G_*\omega))Y-((D_Y\omega)(Z)-\\&-(1/2)g(Y,Z)\omega(G_*\omega))X+g(X,Z)\,(D_YG_*\omega-(1/2)\omega(G_*\omega)Y)-\\&-g(Y,Z)\,(D_XG_*\omega-(1/2)\omega(G_*\omega)X).\end{aligned} \tag{1.7}$$

If A is a tensor field of type (1.1) defined by

$$A(Y)=D_YG_*\omega-(1/2)\omega(G_*\omega)Y, \tag{1.8}$$

and a is the trace of A, then we have, from (1.7),

$$\text{Ric}\,(X,Z)=\text{ric}\,(X,Z)+(n-2)g(AX,Z)+ag(X,Z). \tag{1.9}$$

If R is the trace of R_0, where R_0 is tensor field of type (1.1) defined by

$$g(R_0X,Z)=\text{Ric}\,(X,Z), \tag{1.10}$$