On the other hand, it is known that the length of the second fundamental form h of M is given by

$$\|h\|^2=\sum_{\alpha,\beta=1}^{m}\{g(h(E_\alpha,E_\beta),h(E_\alpha,E_\beta))\} \tag{4.22}$$

where $\{E_1,E_2,\ldots,E_{m-1},E_m=\xi\}$ is a local field of orthonormal frames on M. The by using Lemma 2.2, (4.21) and (4.22) we get

$$\|h\|^2=2q(2p+1)+\sum_{i,j=1}^{2p}\|h(X_i,X_j)\|^2+\sum_{a,b=1}^{q}\|h(Z_a,Z_b)\|^2, \tag{4.23}$$

where $\{X_1,\ldots,X_{2p}\}$ (resp. $\{Z_1,\ldots,Z_q\}$) is an orthonormal basis in D_x (resp. $D_x^{\perp}$). Thus, (4.20) follows from (4.23).

If the equality sign of (4.20) holds, then (4.23) implies

$$h(X,Y)=0 \text{ and } h(Z,W)=0 \tag{4.24}$$

for any $X,Y\in\Gamma(D)$ and $Z,W\in\Gamma(D^{\perp})$, that is, the semi-invariant product M is $D\oplus\{\xi\}$-geodesic and $D^{\perp}$-geodesic immersed in $S^{2n+1}(1)$. Since M_1 and M_2 are both totally geodesic immersed in M, from Lemmas 2.4 and 2.6 we have that M_1 and M_2 are both totally geodesic immersed in $S^{2n+1}(1)$. Also, from (1.10), (1.12) and (4.24) we obtain that M_1 is a Sasakian space form of constant φ-sectional curvature $k=1$ and M_2 is a real space form of constant sectional curvature 1.

Theorem 4.5. *Let M be a minimal semi-invariant product in $S^{2n+1}(1)$. Then the scalar curvature ρ of M satisfies*

$$\rho\leqslant 4p^2+q^2+2p-q. \tag{4.25}$$

The equality sign holds if and only if $\|h\|^2=2q(2p+1)$.

Proof. By direct computation we have, [4], that the scalar curvature ρ of an m-dimensional minimal semi-invariant submanifold M of a Sasakian space form $\tilde{M}(k)$ is given by

$$\rho=\frac{m(m-1)(k+3)-2(m-3p-1)(k-1)}{4}\|h\|^2. \tag{4.26}$$

Then (4.25) follows from (4.26) and (4.20). Also, from (4.26) and (4.20) it is clear that the equality sign of (4.25) holds if and only if $\|h\|^4=2q(2p+1)$.

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ON THE PROPERTIES OF A CONSTANT CURVATURE RIEMANNIAN MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

BY

PETRE STAVRE

In the present paper we study the semi-symmetric metric connection whose D-concircular curvature tensor vanishes. We obtain on this way a generalisation of the theorem (2) of K. Yano [13] and of the results from [11].

1. Let L_n be an n-dimensional differentiable manifold of class C^{∞}, g a Riemannian metric in L_n, and D a semi-symmetric connection in L_n:

$$T(X,Y)=\omega(Y)X-X\omega(Y)\ ;\ \forall X,Y\in\chi(L_n), \tag{1.1}$$

where T is the torsion of D, $T(X,Y)=D_XY-D_YX-[X,Y]$, and ω is a 1-form

$$D_Xg=0;\ \ \forall X. \tag{1.2}$$

We have

$$D_XY=\nabla_XY+\omega(Y)X-g(X,Y)G_*\omega, \tag{1.3}$$

where we denote by ∇ the Levi-Civita connection with respect to g and $G_*\omega$ is defined by

$$g(G_*\omega,X)=\omega(X)\ ;\ \forall X. \tag{1.4}$$

We denote also by

$$R(X,Y)Z=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z \tag{1.5}$$

the curvature tensor of the connection D, by

$$r(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z \tag{1.6}$$

the curvature tensor of the connection ∇, by ric (X,Z), the Ricci tensor of ∇.

Thus, from (1.2) and (1.3) (1.6), we obtain [13]

$$\begin{aligned}R(X,Y)Z=r(X,Y)Z&+((D_X\omega)(Z)-(1/2)g(X,Z)\omega(G_*\omega))Y-((D_Y\omega)(Z)-\\&-(1/2)g(Y,Z)\omega(G_*\omega))X+g(X,Z)(D_YG_*\omega-(1/2)\omega(G_*\omega)Y)-\\&-g(Y,Z)(D_XG_*\omega-(1/2)\omega(G_*\omega)X).\end{aligned} \tag{1.7}$$

If A is a tensor field of type (1.1) defined by

$$A(Y)=D_YG_*\omega-(1/2)\omega(G_*\omega)Y, \tag{1.8}$$

and a is the trace of A, then we have, from (1.7),

$$\text{Ric}(X,Z)=\text{ric}(X,Z)+(n-2)g(AX,Z)+ag(X,Z). \tag{1.9}$$

If R is the trace of R_0, where R_0 is tensor field of type (1.1) defined by

$$g(R_0X,Z)=\text{Ric}(X,Z), \tag{1.10}$$

then we have, from (1.9)

(1.11) $$R_0X = r_0X + (n-2)AX + aX.$$

and

(1.12) $$R = r + 2(n-1)a, \quad (R-r)/[2(n-1)].$$

Thus, from (1.9) and (1.12), we obtain

(1.13) $$(D_X\omega)(Z) - \frac{1}{2}g(X,Z)\omega(G_*\omega) = \frac{\mathrm{Ric}(X,Z) - \mathrm{ric}\,(X,Z)}{n-2} - \frac{R-r}{2(n-1)\,(n-2)}\,g(X,Z).$$

Definition (1.1). *If D satisfies.*

(1.14) $$D_XG_*\omega - (1/2)\omega(G_*\omega)X = fX,$$

where f is a scalar function, then D is called a S-concircular connection.

From (1.15), it results $(D_X\omega)\,(Z) - (1/2)g(X,Z)\omega(G_*\omega) = fg(X,Z)$, for any vector fields X, Z.

Proposition (1.1) *If the connection D is a S-concircular connection, then the Ricci tensor* Ric (X,Z) *is a symmetric tensor and the 1-form ω is closed* $(d\omega = 0)$.

Proof. From (1.4) it results: $(D_X\omega)\,(Z) - (D_Y\omega)\,(Z) = 0$. On the other hand, we have, from (1.1), $\omega(T(X,Y)) = 0$. Comparing (1.17) with (1.18), we find $d\omega = 0$. Thus, from (1.14) and (1.17), we obtain Ric $(X,Z) =$ Ric (Z,X).

Proposition (1.2) [9]. *The connection D is a S-concircular connection if and only if*

(1.15) $$T(X,Y)Z = t(X,Y)Z,$$

where $T(X,Y)Z$ is the D-concircular curvature tensor

$$T(X,Y)Z = R(X,Y)Z - R/[n(n,-1)]\,(g(X,Z)Y - g(Y,Z)X),$$

and $t(X,Y)Z$ is the concircular curvature tensor of K. Yano [14]:

$$t(X,Y)Z = r(X,Y)Z - r/[n(n-1)]\,(g(X,Z)Y - g(Y,Z)X).$$

If $R(X,Y)Z = 0$, $(D_XT)\,(Y,Z) = 0$,

then the manifold is a group-manifold [13] and the Riemannian manifold (L_n, g, ∇) is of constant curvature.

From (1.16), we obtain

$$(\nabla_X\omega)\,(Z) - \omega(X)\omega(Z) + (1/2)g(X,Z)\omega(G_*\omega) = fg(X,Z).$$

Then we have, from (1.14) (1.24) and (1.25),

$$(\nabla_X\omega)\,(Z) - \omega(X)\omega(Z) + (1/2)g(X,Z)\omega(G_*\omega) = -r/[2n(n-1)]g(X,Z).$$

Consequence (1.1). If the connection D is a semi-symmetric connection and the manifold (L_n, g, ∇) is a group manifold, then (L_n, g, D) is *a* S-concircular manifold and $f = -r/[2n(n-1)]$.

Consequence (1.2). *If D is a S-concircular connection, then* $f = (R-r)/[2n(n-1)]$. We have, from (1.21) Ric $(X,Z) - (R/n)g(X,Z) =$ ric $(X,Z) - (r/n)g(X,Z)$. Thus, from (1.14) and (1.29), results (1.28).

Proposition (1.2). *If D is a S-concircular connection and* $T(X,Y)Z = 0$, (L_n, g) *is of constant curvature.*

Proof. We have from (1.21) and $T(x,y)z = 0$, $t(x,y,z) = 0$, which shows that (L_n, g, ∇) is of constant curvature K, then $R = -r/[n/n-1)]$.

Consequence (1.3). We have, from $T(X,Y)Z = 0$,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

$$\omega(R(X,Y)Z) = R/[n(n-1)]\,(g(X,Z)\omega(Y) - g(Y,Z)\omega(X))$$

$$\omega(r(X,Y)Z) = r/[n(n-1)]\,(g(X,Z)\omega(Y) - g(Y,Z)\omega(X)).$$

Proposition (1.3). *If D is a S-concircular connection and* $T(X,Y)Z = 0$, *then the 1-form ω is a solution of equations*

$$(\nabla_X\omega)(Z) - \omega(X)\omega(Z) + (1/2)g(X,Z)\omega(G_*\omega) - (1/2)\,(K + ce^h/[n(n-1)]. g(X,Z) = 0$$

where, locally, $\omega = 1/2\,dh$, and h is a scalar function.

Proof. From the Ricci identity $(\nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega)\,Z = -\omega(r(X,Y)Z)$, and the previous equalities, it results $((dR)\,(X) - 2R\omega(X))Y - ((dR)\,(Y) - 2R\omega(Y))X = 0$.

Or $(n-1)\,((dR)(X) - 2R\omega(X)) = 0$. Then $(\nabla_X\omega)\,(Z) - (\nabla_Z\omega)\,(X) = 0$. Or $X\omega(Z) - Z\omega(X) - \omega([X,Z]) = 0$, from which there results that, locally, exists a scalar function h, so that: $\omega = (1/2)dh$, or as a consequence $R = ce^h$; $c =$ ct.

Since $\omega \neq 0$, locally, we have, $dh \neq 0$ and $R \neq$ ct.

§ 2. **The inverse problem.** If (L_n, g, ∇) has with constant curvature, then

(2.1) $$t(X,Y)Z = 0.$$

We shall consider, locally, the equations:

(2.2) $$(\nabla_X\omega)\,(Z) - \omega(X)\omega(Z) + (1/2)g(X,Z)\omega(G_*\omega) - (1/2)\,(k + ce^h/[n(n-1)]g(X,Z) = 0$$

(2.3) $$\omega = (1/2)\,dh,$$

h being unknown differentiable scalar function and (c) a constant ($c \geqslant 0$ if $k > 0$ and $c \leqslant 0$ if $k < 0$). Because

(2.4) $$(\nabla_X\omega)\,(Z) - (D_Z\omega)\,(X) = 0$$

from (2.2) (2.3) and (2.4) it results $(\nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega)\,(Z) = k(g(X,Z)\omega(Y) - g(Y,Z)\omega(X))$. Thus or $-\omega(r(X,Y)Z) = k(g(X,Z)\omega(Y) - g(Y,Z)\omega(X))$

(2.5) $$\omega(r(X,Y)Z) - r/[n(n-1)]\,(g(X,Z)\omega(Y) - g(Y,Z)\omega(X)) = 0.$$

From (2.1), it results $\omega(r(X,Y)Z) = r/[n(n-1)]\,(g(X,Z)\omega(Y) - g(Y,Z)\omega(X))$, that is the condition (2.5) is verified for every X, Y, Z. It results:

Proposition (2.1). *If the Riemannian manifold (L_n, g, ∇) is of constant curvature, then equations* (2.2) (2.3) *are completely integrable, and determine a 1-form ω.*

Consequence (2.1). $h \neq$ ct, locally, because in the contrary case we should have $\omega = 0$ and from (2.2) it would result $K = ce^h/[n(n-1]$. Because $c \geqslant 0$ if

$K>0$, from (2.9) results $K\leqslant 0$, contradiction. Similarly $c\leqslant 0$ if $K<0$, which contradicts $K\geqslant 0$ as it results from the above equality.

Let D be the connection (1.3), ω being the solution of the equations (2.2)(2.3)($c\neq 0$, $c>0$ if $K>0$ and $c<0$ if $K<0$). The connection is semi-symmetric S-concircular with $f=(ce^{h}-r)/[2n(n-1)]$. From (1.28) and (2.10) results $R=ce^{h}$ and $R\neq$ ct. If $K>0$ then $R>0$. If $K<0$ then $R<0$. Or.

Proposition (2.2). *If the Riemannian manifold* (L_n, g, ∇) *is of constant curvature, then there exists, in* L_n, *a semi-symmetric metric connection, whose* D-*concircular curvature tensor vanishes, and the scalare curvature* $R>0$ *if* $K>0$ *or* $R<0$ *id* $K<0$.

This proposition generalises the result of K. Yano [13] and the result from [11], obtained for $c=1$.

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CONFRONTO ED EQUIVALENZE TRA LE DIVERSE GENERALIZZAZIONI DELLA DERIVAZIONE TORSIONALE DI A. G. WALKER

DI

M. PAOLA PIU

Dedicato al Professore Ettore Picasso

Introduzione. Sia M una varietà differenziabile m-dimensionale, e sia J un campo di endomorfismi che definisce su M una struttura *quasi complessa* ($J^2=-I$).

Alla torsione della struttura quasi complessa A. G. Walker ha associato un operatore differenziale di grado 2 da lui chiamato *derivazione torsionale* [W1]. Con metodi e in contesti diversi, la derivazione di Walker è stata estesa a strutture definite da campi tensoriali di tipo (1,1) diverse e più generali della struttura quasi complessa da K. Yano [Y], da T. J. Willmore [WL], da Th. Hangan [HN], dallo stesso Walker [W2] e da C. J. Hsu [H2].

Questa Nota è costituita essenzialmente da tre parti. Nella prima parte dimostreremo innanzitutto che la derivazione trovata da Yano può essere ridescritta in forma invariante in termini di derivazioni di Lie. Quindi, tenendo conto delle formule che si trovano in [P], saremo in grado di mettere agevolmente a confronto i risultati di Yano, di Willmore, di Hangan e di Walker, e di riconoscere che la derivazione trovata da Hangan è diversa dalle altre.

Nella seconda parte si dimostra che, sulla base dei risultati ottenuti, è possibile far vedere che le funzioni h'^{i}_{jkl}, di cui Hangan fa uso per costruire una derivazione torsionale [HN], possono ricondursi, nel caso di una struttura quasi complessa, alle componenti h^{i}_{jkl} introdotte da Walker in [W1].

Infine, la derivazione di Hsu viene messa in forma invariante. Essa risulta contenere tutte le altre, ad eccezione di quella di Hangan, come casi particolari.

1. Sia M una varietà differenziabile di classe C^{∞} e sia $\mathfrak{X}(M)$ l'algebra di Lie dei campi di vettori C^{∞} su M. Se $J=(h^{i}_{j})$ è un campo tensoriale di tipo (1,1) su M, la sua *torsione* $H=(H^{i}_{jk})$ è la 2-forma vettoriale definita per ogni coppia di campi vettoriali $X, Y\in\mathfrak{X}(M)$ dall'espressione (Nijenhuis).

$$(1.1)\qquad 4H(X,Y)=J[JX,Y]+J[X,JY]-[JX,JY]-J^2[X,Y]=$$
$$=[J,J](X,Y).$$

Supponiamo che J definisca su M una struttura *quasi complessa* ($J^2=-I$) e *quasi prodotto* con due distribuzioni ($J^2=I$), non integrabile ($H\neq 0$). Su