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# ON CONTACT UMBILICAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

BY

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The geometry of anti-invariant submanifolds in Sasakian manifolds is an interesting topic in differential geometry. K. Yano and M. Kon proved that any submanifold normal to the structure vector field in a Sasakian manifold is an anti-invariant submanifold (see [3], p. 40). The purpose of the present paper is to introduce and study a class of anti-invariant submanifolds which are tangent to the structure vector field.

In an introductory paragraph we give definitions and fundamental formulas for submanifolds. In the next paragraphs we study contact umbilical submanifolds with respect to a normal section in Sasakian space forms. The main results are given in Theorems 2.1, 3.1 and 3.2. N. Sato studied in [2] a class of umbilical totally real submanifolds in complex space forms and obtained interesting results.

§1. **Preliminaries.** Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with structure tensor fields $(\varphi, \xi, \eta, g)$, where $\varphi$ is a tensor field of type (1,1), $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $\tilde{M}$. These tensor fields satisfy (see [1], p. 19).

$$\varphi^2 X = -X + \eta(X)\,\xi,\quad \varphi\xi = 0,\quad \eta(\xi) = 1,\quad \eta(\varphi X) = 0 \quad \text{and} \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\cdot\eta(Y),\quad \eta(X) = g(X, \xi), \tag{1.2}$$

for any $X$, $Y$ tangent to $\tilde{M}$. We denote by $\tilde{\nabla}$ the covariant differentiation with respect to the metric $g$ on $\tilde{M}$. Then it is well known (see [4], p. 20) that $\tilde{M}$ is a Sasakian manifold if and only if we have

$$(\tilde{\nabla}_X \varphi)\, Y = -g(X, Y)\,\xi + \eta(Y)\, X, \tag{1.3}$$

for any $X$, $Y$ tangent to $\tilde{M}$. From (1.3) it follows

$$\tilde{\nabla}_X \xi = \varphi X. \tag{1.4}$$

Now, let $M$ be an $(m+1)$-dimensional Riemannian manifold isometrically immersed in a Sasakian manifold $\tilde{M}$. We say that $M$ is an *anti-invariant submanifold* of $\tilde{M}$ if we have

$$g(X, \varphi Y) = 0, \tag{1.5}$$

for any $X$, $Y$ tangent to $M$, or equivalently $\varphi(T_xM)\subset T_xM^{\perp}$ for each $x\in M$. In the present paper we find a new class of anti-invariant submanifolds tangent to the structure vector field and study their properties.

We need some basic formulas from geometry of submanifolds in Riemannian manifolds. Suppose $M$ is a submanifold of $\tilde{M}$ and denote by $\nabla$ and $\nabla^{\perp}$ the Levi-Civita connection on $M$ and respectively the Riemannian connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X,\ Y) \text{ and respectively} \tag{1.6}$$

$$\tilde{\nabla}_X N=-A_N X+\nabla^{\perp}_X N, \tag{1.7}$$

for any $X$, $Y$ tangent to $M$ and normal section $N$, where $h$ is the second fundamental form of $M$ and $A_N$ is the fundamental tensor of Weingarten with respect to $N$. Moreover we have

$$g(h(X,\ Y),\ N)=g(A_N X,\ Y), \tag{1.8}$$

for any $X$, $Y$ tangent to $M$ and normal section $N$. If $M$ is an anti-invariant submanifold of a Sasakian manifold $\tilde{M}$, then by using (1.4) and (1.6) we obtain

$$h(X,\ \xi)=\varphi X,\quad h(\xi,\ \xi)=0, \tag{1.9}$$

for any $X$ tangent to $M$.

We denote by $\tilde{R}$ and $R$ the curvature tensors of Levi-Civita connections $\tilde{\nabla}$ and respectively $\nabla$. Then we have the equation of Gauss given by

$$\begin{aligned} g(\tilde{R}(X,\ Y)Z,\ W)=&\ g(R(X,\ Y)Z,\ W)+g(h(X,\ Z),\ h(Y,\ W))\\ &-g(h(Y,\ Z),\ h(X,\ W)), \end{aligned} \tag{1.10}$$

for any $X$, $Y$, $Z$, $W$ tangent to $M$.

Now, let $\tilde{M}(c)$ be a Sasakian space form of constant $\varphi$-sectional curvature $c$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by (see [1], p. 97)

$$\begin{aligned} \tilde{R}(X,\ Y)Z=&\ [(c+3)/4]\{g(Y,\ Z)X-g(X,\ Z)Y\}+\\ &+[(c-1)/4]\{\eta(X)\cdot\eta(Z)Y-\eta(Y)\cdot\eta(Z)X+g(X,Z)\cdot\eta(Y)\xi-\\ &-g(Y,\ Z)\eta(X)\xi+g(Z,\ \varphi Y)\varphi X-g(Z,\ \varphi X)\varphi Y+2g(X,\ \varphi Y)\varphi Z\}, \end{aligned} \tag{1.11}$$

for any $X$, $Y$, $Z$ tangent to $\tilde{M}(c)$.

## §2. Contact umbilical submanifolds with respect to a normal section.

Let $M$ be an $(m+1)$-dimensional submanifold tangent to the structure vector field $\xi$ in a $(2n+1)$-dimensional Sasakian manifold $M$ and $N$ be a normal section to $M$. Denote by $\{\xi\}$ the distribution spanned by $\xi$ on $M$ and by $D$ the complementary orthogonal distribution to $\{\xi\}$ in $TM$. Thus we have

$$TM=D\oplus\{\xi\}. \tag{2.1}$$



The module of all differentiable sections of $D$ is denoted by $\Gamma(D)$.

We say that $M$ is *contact umbilical with respect to the normal section* $N$ if the fundamental tensor of Weingarten with respect to $N$ satisfies

$$A_N X=\|N\|^2 X+g(A_N\xi,\ X)\,\xi, \tag{2.2}$$

for any $X\in\Gamma(D)$. We recall that $M$ is a *contact totally umbilical submanifold* if its second fundamental form $h$ satisfies (see [3], p. 157)

$$h(X,\ Y)=g(\varphi X,\ \varphi Y)\,H+\eta(X)\,h(Y,\ \xi)+\eta(Y)\,h(X,\ \xi), \tag{2.3}$$

where $H$ is the mean curvature vector of $M$ given by

$$H=1/(m+1)\ \mathrm{trace}\ (h). \tag{2.4}$$

**Proposition 2.1.** *Let $M$ be a contact totally umbilical submanifold of a Sasakian manifold $\tilde{M}$. Then $M$ is a contact umbilical submanifold with respect to the mean curvature vector $H$.*

*Proof.* For any $X\in\Gamma(D)$ and $Y$ tangent to $M$, by using (1.2), (1.8) and (2.3) we obtain

$$\begin{aligned} g(A_H X,\ Y)&=g(h(X,\ Y),\ H)=g(X,\ Y)\,\|H\|^2+\eta(Y)\,g(A_H\xi,\ X)=\\ &=g(\|H\|^2 X+g(A_H\xi,\ X)\,\xi,\ Y), \end{aligned}$$

since $\eta(X)=0$. Thus we have the assertion of the proposition.

A normal section $N$ is said to be $D$ parallel if we have

$$\nabla^{\perp}_X N=0,\ \text{for any } X\in\Gamma(D). \tag{2.5}$$

**Proposition 2.2.** *Let $M$ be a contact umbilical submanifold with respect to a $D$ parallel normal section $N$ in a Sasakian manifold $\tilde{M}$. Then we have*

$$\tilde{\nabla}_X N=-A_N X=-\|N\|^2 X-g(A_N\xi,\ X)\,\xi, \tag{2.6}$$

*for any $X\in\Gamma(D)$.*

The proof follows by using (1.7), (2.2) and (2.5).

**Proposition 2.3.** *Let $M$ be a contact umbilical submanifold with respect to a $D$-parallel normal section $N$ in a Sasakian manifold $\tilde{M}$. Then we have*

$$\begin{aligned} \tilde{R}(X,\ Y)N=&\ \{Y(g(A_N\xi,\ X))-X(g(A_N\xi,\ Y))+g(A_N\xi,\ [X,\ Y])\}\\ &\xi-g(A_N\xi,\ Y)\,\varphi X+g(A_N\xi,\ X)\,\varphi Y, \end{aligned} \tag{2.7}$$

*for any $X$, $Y\in\Gamma(D)$.*

*Proof.* Differentiating covariantly (2.6) with respect to $Y\in\Gamma(D)$ and taking account of (1.4) we obtain

$$\tilde{\nabla}_Y\tilde{\nabla}_X N=-\|N\|^2\tilde{\nabla}_Y X-Y(g(A_N\xi,\ X))\,\xi-g(A_N\xi,\ X)\,\varphi Y, \tag{2.8}$$

since we have $Y(\|N\|^2)=Y(g(N,\ N))=2g(\nabla^{\perp}_Y N,\ N)=0$. Thus (2.7) follows by a direct computation using (2.6), (2.8) and taking into account that $\tilde{\nabla}$ is a less torsion connection on $\tilde{M}$ and

$$\tilde{R}(X, Y)N = \tilde{\nabla}_X \tilde{\nabla}_Y N - \tilde{\nabla}_Y \tilde{\nabla}_X N - \tilde{\nabla}_{[X, Y]} N.$$

**Theorem 2.1.** *Let $M$ be a contact umbilical submanifold with respect to a nonzero $D$-parallel normal section $N$ in a Sasakian space form $\tilde{M}(c)$ with $c \neq 1$. Then $M$ is an anti-invariant submanifold of $\tilde{M}(c)$.*

*Proof.* First we note that it is sufficient to prove (1.5) for any $X, Y \in \Gamma(D)$. Then by using (2.7) we obtain

$$g(\tilde{R}(X, Y)N, \varphi N) = 0, \tag{2.9}$$

since we have

$$g(\xi, \varphi N) = -g(\varphi\xi, N) = 0 \text{ and } g(\varphi X, \varphi N) = g(X, N) - \eta(X)\cdot\eta(N) = 0.$$

On the other hand, by using (1.11) we get

$$g(\tilde{R}(X, Y)N, \varphi N) = [(c-1)/2]\, g(X, \varphi Y)\, \| N \|^2. \tag{2.10}$$

Thus (1.5) follows from (2.9) and (2.10) taking account that $c \neq 1$ and $\|N\| \neq 0$.

Combining Proposition 2.1 and Theorem 2.1 we obtain

**Corollary 2.1.** *Any contact totally umbilical submanifold with nonzero $D$-parallel mean curvature vector in $\tilde{M}(c)$ with $c \neq 1$ is an anti-invariant submanifold.*

**§3. Contact umbilical submanifolds with respect to the mean curvature vector.** Throughout this section we suppose $M$ is an $(m+1)$-dimensional submanifold with nonzero mean curvature vector $H$ in a $(2n+1)$-dimensional Sasakian space form $\tilde{M}(c)$. We denote by $r$ the codimension of $M$ in $\tilde{M}(c)$. Also we denote by $P$ the projection morphism to the distribution $D$. Thus each vector field $X$ tangent to $M$ can be written as follows

$$X = PX + \eta(X)\,\xi. \tag{3.1}$$

The Ricci tensor $S$ and the scalar curvature $\rho$ of $M$ are given by

$$S(X \cdot Y) = \sum_{i=1}^{m+1} g(R(E_i, X)Y, E_i) \text{ and respectively} \tag{3.2}$$

$$\rho = \sum_{i=1}^{m+1} S(E_i, E_i), \tag{3.3}$$

for any $X, Y$ tangent to $M$, where $\{E_i\}$ is an orthonormal field of frames on $M$ and $E_{m+1} = \xi$. Also the sectional curvature $K_M$ of $M$ determined by orthonormal vectors $\{X, Y\}$ is given by

$$K_M(X \wedge Y) = g(R(X, Y)Y, X) \tag{3.4}$$

**Theorem 3.1.** *Let $M$ be a contact umbilical submanifold with respect to the $D$-parallel mean curvature vector $H$ in a Sasakian space form $\tilde{M}(c)$, $c \neq 1$. Then we have*

(i) *$M$ is an anti-invariant submanifold of $\tilde{M}(c)$,*

(ii) *the scalar curvature of $M$ satisfies*



$$\rho \leq [m(m-1)(3+c)/4] + (m^2+m+1)\, \| H \|^2. \tag{3.5}$$

(iii) *if the equality sign of (3.5) holds and if $H$ is orthogonal to $\varphi(TM)$ then $M$ is totally contact umbilical.*

*Proof.* The assertion (i) follows from Theorem 2.1. In order to obtain (3.5) we note that by a direct computation using (1.9), (1.10) and (3.2) we obtain

$$S(X, Y) = [(m-1)(c+3)/4]\, g(X, Y) + (1 - [(c-1)(m-1)/4]\, \eta(X)\cdot$$

$$\cdot\, \eta(Y) + (m+1)\, g(h(X, Y), H) - \sum_{i=1}^{m} g(h(E_i, X), h(E_i, Y)), \tag{3.6}$$

for any $X, Y$ tangent to $M$. Then by using (3.3) and (3.6) we get

$$\rho = [m(m-1)(c+3)/4] + (m+1)^2 \| H \|^2 - \sum_{i,j=1}^{m} g(h(E_i, E_j), h(E_i, E_j)). \tag{3.7}$$

Now we choose an orthonormal field of frames $\{N_\alpha\}$ $\alpha = 1, \ldots, r$ normal to $M$, in such a way that $N_1 = 1/\| H \| \cdot H$. Then by using (1.8) and (2.2) we obtain

$$g(h(E_i, E_j), N_1) = \| H \|\, g(E_i, E_j),\; i, j = 1, \ldots, m.$$

Thus (3.7) becomes

$$\rho = [m(m-1)(c+3)/4] + (m^2+m+1)\, \| H \|^2 - \sum_{i,j=1}^{m} \sum_{\alpha=2}^{r} g(h(E_i, E_j), N_\alpha)^2, \tag{3.8}$$

and the assertion (ii) follows from (3.8). Next suppose the equality sign of (3.5) holds. Then by (3.8) we obtain

$$g(h(PX, PY), N_\alpha) = 0,\; \alpha = 2, \ldots, r. \tag{3.9}$$

By using (1.8), (2.2) and (3.1) we get

$$g(h(X, Y), N_1) = g(PX, PY)\, \| H \|. \tag{3.10}$$

for any $X, Y$ tangent to $M$, since $H$ is orthogonal to $\varphi(TM)$. On the other hand, by using (1.9) and (3.9) we obtain

$$g(h(X, Y), N_\alpha) = \eta(X)\, g(\varphi PY, N_\alpha) + \eta(Y)\, g(\varphi PX, N_\alpha), \tag{3.11}$$

for all $X, Y$ tangent to $M$ and $\alpha = 2, \ldots, r$. We choose $\{N_\alpha\}$ $\alpha = 2, \ldots, r$ such that $m$ vector fields from them be $\{\varphi E_1, \ldots, \varphi E_m\}$. This is possible since we supposed $H$ be orthogonal to $\varphi(TM)$. Finally, by (1.9), (3.1), (3.10) and (3.11) we have

$$h(X, Y) = \sum_{\alpha=1}^{r} g(h(X, Y), N_\alpha)\, N_\alpha = g(PX, PY)\, H + \eta(X)\, \varphi Y + \eta(Y)\, \varphi X =$$

$$= g(X, Y)\, H - \eta(X)\cdot\eta(Y)\, H + \eta(X)\, h(Y, \xi) + \eta(Y)\, h(X, \xi),$$

that is $M$ is contact totally umbilical. The proof is complete.

We denote by $E^{2n+1}(-3)$ the Sasakian space form of constant $\varphi$-sectional curvature $c=-3$ endowed with the standard Sasakian structure in a Euclidean space (see [1], p. 99).

**Theorem 3.2.** *Let $M$ be an $(m+1)$-dimensional contact umbilical submanifold with respect to the D-parallel mean curvature vector $H$ in $E^{2n+1}(-3)$. If $\rho=(m^2+m+1)\,\|H\|^2$, then all sectional curvatures of $M$ are nonnegative.*

*Proof.* By the hypothesis on the scalar curvature of $M$ we have (3.9). Hence.

$h(X, Y)=g(h(X, Y), N_1)\,N_1=g(X, Y)\,H$, for any $X, Y \in \Gamma(D)$.

On the other hand, by Theorem 3.1 $M$ is an anti-invariant submanifold of $E^{2n+1}(-3)$. Then by using (1.9), (1.10), (1.11) and (3.4) we obtain

$$K_M(X\wedge Y)=\|H\|^2>0 \text{ and } K_M(X\wedge \xi)=0,$$

for any $X, Y \in \Gamma(D)$. Thus all sectional curvatures of $M$ are nonnegative.

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# BOUNDARY-VALUE STOCHASTIC PROBLEMS (II)

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**§1. Notations.** This paper represents the sequel of the paper *Boundary-value stochastic problem (I)* from the previous issue of this review. We recall here a few notations.

Let $D$ be a bounded and open subset of $R^N$ with sufficiently smoth boundary. We set $Q=\,]0, T[\times D$, $\Sigma=\,]0, T[\times\Gamma$, $V=H^1(D)$, $H=L^2(D)$, $W(D)=W(0, T; V)$, $\Theta=\Omega\times]0, T[\times D=\Omega\times Q$, $W_{ad}(\Theta)=W_{ad}(\Omega\times]0, T[; V)$, $L^2_{ad}(\Theta)=L^2_{ad}(\Omega\times]0, T[; H)$, $L^2_{ad}(\Omega\times\Sigma)=L^2_{ad}(\Omega\times]0, T[; L^2(\Gamma))$, $L^2_0(\Omega\times D)=L^2(\Omega, \mathcal{F}_0, P; H)$, where $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ is a stochastic complete basis;

$L^2_{ad}(\Omega\times]0, T[; X)$ (with $X$ a Hilbert space) is the Hilbert subspace of the nonanticipative processes from $L^2(\Omega\times]0, T[; X)$;

$V\subset H\subset V^*$;

$W(0, T; V)=\{u\in L^2(0, T; V)\cap C([0, T]; H): u(t)=u_0+\int_0^t u'(s)\,ds,\ u_0\in H,$

$u'\in L^2(0, T; V^*)\}$;

$W_{ad}(\Omega\times]0, T[; V)=\{(u\in L^2_{ad}(\Omega\times]0, T[; V)\cap L^2_{ad}(\Omega; C([0, T]; H)):$

$u(\omega, t)=u_0(\omega)+\int_0^t u'(\omega, s)\,ds,\ u_0\in L^2(\Omega, \mathcal{F}_0, P; H),\ u'\in L^2_{ad}(\Omega\times]0, T[; V^*)\}$.

By $\mathcal{M}^2(0, T; H)$ we shall denote the space of continuous square $H$-valued martingales (it is a closed subspace of $L^2_{ad}(\Omega; C([0, T]; H))$).

**§2. Hypotheses.** We shall consider the boundary-value stochastic system.

$$\text{(2.1)}\quad \begin{aligned} &du+Au\,dt=f(\omega, t, x)\,dt+dM(\omega, t, x) \text{ in } \Theta=\Omega\times Q,\\ &\frac{\partial u}{\partial \nu}+\beta(u)\ni g(\omega, t, \sigma) \text{ in } \Omega\times\Sigma,\\ &u(\omega, 0, x)=u_0(\omega, x) \text{ in } \Omega\times D,\end{aligned}$$

where $\partial/\partial\nu$ is the outward normal derivative associated with $A$.

We assume that:

$$\text{(H}_1\text{)}\quad Ay=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\,\frac{\partial y}{\partial x_i}\right)+a_0 y,$$