We denote by $E^{2n+1}(-3)$ the Sasakian space form of constant $\varphi$-sectional curvature $c=-3$ endowed with the standard Sasakian structure in a Euclidean space (see [1], p. 99).

**Theorem 3.2.** *Let $M$ be an $(m+1)$-dimensional contact umbilical submanifold with respect to the $D$-parallel mean curvature vector $H$ in $E^{2n+1}(-3)$. If $\rho=(m^2+m+1)\,\|H\|^2$, then all sectional curvatures of $M$ are nonnegative.*

*Proof.* By the hypothesis on the scalar curvature of $M$ we have (3.9). Hence,
$h(X,\ Y)=g(h(X,\ Y),\ N_1)\,N_1=g(X,\ Y)\,H$, for any $X,\ Y\in\Gamma(D)$.
On the other hand, by Theorem 3.1 $M$ is an anti invariant submanifold of $E^{2n+1}(-3)$. Then by using (1.9), (1.10), (1.11) and (3.4) we obtain

$$K_M(X\wedge Y)=\|H\|^2>0 \text{ and } K_M(X\wedge\xi)=0,$$

for any $X,\ Y\in\Gamma(D)$. Thus all sectional curvatures of $M$ are nonnegative.

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# BOUNDARY-VALUE STOCHASTIC PROBLEMS (II)

BY

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**§1. Notations.** This paper represents the sequel of the paper *Boundary-value stochastic problem (I)* from the previous issue of this review. We recall here a few notations.

Let $D$ be a bounded and open subset of $R^N$ with sufficiently smoth boundary. We set $Q=\,]0,\ T[\times D$, $\Sigma=\,]0,\ T[\times\Gamma$, $V=H^1(D)$, $H=L^2(D)$, $W(D)=W(0,\ T;\ V)$, $\Theta=\Omega\times]0,\ T[\times D=\Omega\times Q$, $W_{ad}(\Theta)=W_{ad}(\Omega\times]0,\ T[;\ V)$, $L^2_{ad}(\Theta)=L^2_{ad}(\Omega\times]0,\ T[;\ H)$, $L^2_{ad}(\Omega\times\Sigma)=L^2_{ad}(\Omega\times]0,\ T[;L^2(\Gamma))$, $L^2_0(\Omega\times D)=L^2(\Omega,\ \mathcal{F}_0,\ P;\ H)$, where $(\Omega,\ \mathcal{F},\ P,\ (\mathcal{F}_t)_{t\geq 0})$ is a stochastic complete basis;
$L^2_{ad}(\Omega\times]0,\ T[;\ X)$ (with $X$ a Hilbert space) is the Hilbert subspace of the nonanticipative processes from $L^2(\Omega\times]0,\ T[;\ X)$;

$V\subset H\subset V^*$;

$W(0,\ T;\ V)=\{u\in L^2(0,\ T;\ V)\cap C([0,\ T];H):u(t)=u_0+\int_0^t u'(s)\,ds,\ u_0\in H,$
$u'\in L^2(0,\ T;\ V^*)\}$;

$W_{ad}(\Omega\times]0,\ T[;\ V)=\{(u\in L^2_{ad}(\Omega\times]0,\ T[;\ V)\cap L^2_{ad}(\Omega;\ C([0,\ T];\ H)):$
$u(\omega,t)=u_0(\omega)+\int_0^t u'(\omega,\ s)\,ds,\ u_0\in L^2(\Omega,\ \mathcal{F}_0,\ P;\ H),\ u'\in L^2_{ad}(\Omega\times]0,\ T[;\ V^*)\}$.

By $\mathcal{M}^2(0,\ T;\ H)$ we shall denote the space of continuous square $H$-valued martingales (it is a closed subspace of $L^2_{ad}(\Omega;\ C([0,\ T];\ H)))$.

**§2. Hypotheses.** We shall consider the boundary-value stochastic system.

$$
\begin{aligned}
&du+Au\,dt=f(\omega,\ t,\ x)\,dt+dM(\omega,\ t,\ x) \text{ in } \Theta=\Omega\times Q,\\
&\frac{\partial u}{\partial\nu}+\beta(u)\ni g(\omega,\ t,\ \sigma) \text{ in } \Omega\times\Sigma,\\
&u(\omega,\ 0,\ x)=u_0(\omega,\ x) \text{ in } \Omega\times D,
\end{aligned}
\tag{2.1}
$$

where $\partial/\partial\nu$ is the outward normal derivative associated with $A$.

We assume that:

$$
Ay=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y}{\partial x_i}\right)+a_0y,
\tag{H$_1$}
$$

where $a_{ij} \in C^1(\bar{D})$, $a_0 \in L^\infty(D)$, $a_{ij} = a_{ji}$, $a_0(x) \geq 0$ a.e. $x \in D$, and there exists $C > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j \geq C \,\|\xi\|^2 \text{ a.e. on } D, \text{ for all } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N ;$$

($H_2$) $\quad f \in L^2_{ad}(\Theta)$, $g \in L^2_{ad}(\Omega \times \Sigma)$, $u_0 \in L^2_0(\Omega \times D)$ ;

($H_3$) $M \in \mathcal{M}^2(0, T ; L^2(D))$, with $K$ covariance process and $\lambda$ Doleans measure, verifies the condition

$$\sup_{t \in [0,T]} \int_{\Omega \times [0,t]} \| A S(t-s)\, K^{1/2} \|_2^2 \, d\lambda < +\infty,$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm in $L^2(D)$ and $S(t)$ is the semigroup (strongly continuous and analytic generated by $A : H^2(D) \cap H^1_0(D) \subset L^2(D) \to \to L^2(D)$ from $(H_1)$.

($H_4$) $\beta$ is a graph maximal monotone in $\mathbf{R} \times \mathbf{R}$ such that $0 \in \beta(0)$, what is equivalent with $\beta = \partial\varphi$ (the subdifferential of $\varphi$), where $\varphi : \mathbf{R} \to ]-\infty, +\infty]$ is a proper convex lower-semicontinuous function and $\varphi(x) \geq \geq \varphi(0)$ for all $x \in \mathbf{R}$.

**Definition 2.1.** *A function $u \in L^2_{ad}(\Omega \times ]0, T[ ; H^1(D)) \cap L^2_{ad}(\Omega ; C([0, T] ; L^2(D)))$ is a strong solution to (2.1) if there exists $w \in L^2_{ad}(\Omega \times \Sigma)$ such that :*

(2.2) $$w(\omega, t, x) \in \beta(u(\omega, t, x)) \text{ a.e. } on\ \Omega \times \Sigma$$

(2.3) $$-\int_Q u \frac{\partial v}{\partial t}\, dt dx + \int_0^T a(u, v)\, dt + \int_\Sigma (w-g)\, v dt d\sigma = \int_D u_0 v(\omega, 0, x)\, dx + + \int_0^T \int_D v(\omega, t, x)\, dM(\omega, t, x)\, dx, \ \omega\text{-a.s. ; } for\ all\ v \in W_{ad}(\Theta),$$
$$v(\omega, T, x) = 0 \text{ a.e. } on\ \Omega \times D.$$

Here $a : H^1(D) \times H^1(D) \to \mathbf{R}$ is the bilinear functional

(2.4) $$a(y, z) = \sum_{i,j=1}^{N} \int_D a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} + \int_D a_0 yz dx \ ; \ y, z \in H^1(D)$$

§3. **The main result. Theorem 3.1.** *If $\varphi(u_0) \in L^1(\Omega \times D)$ then, under hypotheses $(H_1 - H_4)$, the system (2.1) has a unique strong solution.*

*Proof.* The uniqueness of the solution is obtained using Proposition 1.2 from [5]. To prove the existence we shall divide System 2.1 in three systems

(3.1) $$\begin{aligned} & dy_1 + A y_1 dt = f(t)\, dt \text{ in } \Theta, \\ & y_1(\omega, t, x) = 0 \text{ on } \Omega \times \Sigma, \ y_1(\omega, 0, x) = 0 \text{ on } \Omega \times D ; \end{aligned}$$

(3.2) $$\begin{aligned} & dy_2(t) + A y_2 dt = dM(t) \text{ in } \Theta, \\ & y_2(\omega, t, x) = 0 \text{ on } \Omega \times \Sigma, \ y_2(\omega, 0, x) = 0 \text{ on } \Omega \times D ; \end{aligned}$$





(3.3) $$\begin{aligned} & dz(t) + Az dt = 0 \text{ in } \Theta, \\ & \frac{\partial z}{\partial \nu} + \beta(z) \ni h \text{ in } \Omega \times \Sigma, \ z(\omega, 0, x) = u_0(\omega, x) \text{ in } \Omega \times D. \end{aligned}$$

and we will show that, for $h = g - \partial(y_1 + y_2)/\partial\nu$, $u = (y_1 + y_2) + z$ is a strong solution of System (2.1).

*Remark 3.1.* We recall from [4], Chap. 4, that the mapping $y \to \partial y/\partial\nu$ is is continuous from $H^{1,2}(Q)$ to $L^2(\Sigma)$ and the system :

$$\begin{cases} y' + Ay = f \text{ in } Q = ]0, T[ \times D, \\ y = 0 \text{ on } \Sigma = ]0, T[ \times \Gamma, \ y(0, x) = 0 \text{ on } D', \end{cases}$$

where $f \in L^2(Q)$ and $A$ verifies (H1) has a unique solution $y \in H^{1,2}(Q)$ and $\|y\|_{H^{1,2}(Q)} \leq C \|f\|_{L^2(Q)}$. By this Remark we infer easily the following Lemma

**Lemma 3.1.** *If hypotheses (H1) are verified and $f \in L^2_{ad}(\Theta)$, then System 3.1 has a unique solution $y_1 \in L^2_{ad}(\Omega, H^{1,2}(Q))$ in the following sense :*

(3.4) $$\begin{aligned} & y_1(t) + \int_0^t A y_1 ds = \int_0^t f(s)\, ds \ in\ L^2(\Omega ; C([0, T] ; L^2(D))), \\ & y_1(\omega, t, x) = 0 \text{ a.e. } on\ \Omega \times ]0, T[ \times \Gamma \end{aligned}$$

*Moreover $\partial y_1/\partial\nu \in L^2_{ad}(\Omega \times \Sigma)$ and*

(3.5) $$-\int_Q y_1 \frac{\partial v}{\partial t}\, dt dx + \int_0^T a(y_1, v)\, dt - \int_\Sigma \frac{\partial y_1}{\partial \nu} v|_\Gamma\, dt d\sigma = \int_Q f v dt dx,$$
$$\omega\text{-a.s., } for\ all\ v \in W_{ad}(\Theta),\ v(\omega, T, x) = \text{a.e. } on\ \Omega \times D.$$

Now we study system (3.2). The linear operator $A : D(A) \subset L^2(D) \to \to L^2(D) = H$ is closed and densely defined on $L^2(D)$, $D(A) = H^2(D) \cap H^1_0(D)$. $A$ generates an infinitesimal strongly continuous (even analytic) semigroup $S(t)$. The graph norm is equivalent on $D(A)$ with the $H^2(D)$-norm. By Proposition 1.1 and Proposition 1.2 from [5] and by the continuity of the mapping $y \to \partial y/\partial\nu$ from $H^2(D)$ to $L^2(\Gamma)$ we have :

**Lemma 3.2.** *If hypotheses (H1) and (H3) hold, then System (3.2) has a unique solution $y_2 \in L^2_{ad}(\Omega \times ]0, T[ ; D(A))\ L^2_{ad}(\Omega ; C([0, T] ; L^2(D)))$ in the following sense*

(3.6) $$\begin{cases} y_2(t) + \int_0^t A y_2(s)\, ds = M(t) \ in\ L^2(\Omega ; C([0, T] ; L^2(D))), \\ y_2(\omega, t, x) = 0 \text{ a.e. } on\ \Omega \times ]0, T[ \times \Gamma \end{cases}$$

*Moreover $\partial y_2/\partial\nu \in L^2_{ad}(\Omega \times \Sigma)$ and*

(3.7) $$-\int_Q y_2 \frac{\partial v}{\partial t}\, dt dx + \int_0^T a(y_2, v)\, dt - \int_\Sigma \frac{\partial y_2}{\partial \nu} v dt d\sigma = \int_0^T (v(t), dM(t))_{L^2(D)}$$
$$\omega\text{-a.s., } for\ all\ v \in W_{ad}(\Theta),\ v(\omega, T, x) = 0.$$

Finally we consider System (3.3) with $h=g-\partial(y_1+y_2)/\partial\nu\in L^2_{ad}(\Omega\times\Sigma)$. For $\varepsilon>0$ and $r\in\mathbf{R}$ we denote

a) $\beta_\varepsilon(r)=\varepsilon^{-1}(r-(1+\varepsilon\beta)^{-1}r)$,

(3.8) b) $\beta^\varepsilon(r)=\int_{-\infty}^{+\infty}\beta_\varepsilon(r-\varepsilon\theta)\,\rho(\theta)\,d\theta$, („mollifier" approximation)

c) $\varphi^\varepsilon(r)=\int_0^r\beta^\varepsilon(r)\,dr$,

where $\rho\in C_0^\infty(R)$, $\rho(r)=\rho(-r)\geqslant 0$ for all $r\in R$, $\rho(r)=0$, for $|r|\geqslant 1$ and

$$\int_{-\infty}^{+\infty}\rho(r)\,dr=1\ ;\ \beta=\partial\varphi\,.$$

We remark the following properties :

a) $\beta_\varepsilon$ and $\beta^\varepsilon$ are Lipschitzian (with Lipschitz-constant $\varepsilon^{-1}$) and monotonically increasing functions ; $\beta^\varepsilon$ are infinitely differentiable functions,

b) $\beta_\varepsilon(0)=0$ ;

(3.9) c) $|\beta^\varepsilon(r)-\beta_\varepsilon(r)|\leqslant 1$ ;

d) $-|r|\leqslant\varphi^\varepsilon(r)\leqslant|r|+\varphi(r)-\varphi(0)$ ;

e) if $(S, \mathcal{S}, \mu)$ is a space with measure, $y_\varepsilon\in L^2(S, \mathcal{S}, \mu;\ \mathbf{R})=L^2(S)$ such that $\lim\limits_{\varepsilon\to 0} y_\varepsilon=y$ in $L^2(S)$ and $\beta^\varepsilon(y_\varepsilon)$, $\varepsilon\in]0,1]$, is bounded in $L^2(S)$, then $y(s)\in D(\beta)$ $\mu$-a.e. and all $w$, weak limits in $L^2(S)$ of $\beta^\varepsilon(y)$ when $\varepsilon\to 0$, have the property : $w(s)\in\beta(y(s))$ $\mu$-a.e.

The properties (b), (c), (d) are imediately. Here we shall recall from [1] the proof for (e).

We set $z_\varepsilon(s,\theta)=\beta_\varepsilon(y_\varepsilon(s)-\varepsilon\theta)$, $|\varepsilon|\leqslant 1$. By (3.9—a ; c) it follows : $|z_\varepsilon(s,\theta)|\leqslant|\beta_\varepsilon(y_\varepsilon-\varepsilon\theta)-\beta_\varepsilon(y_\varepsilon)|+|\beta_\varepsilon(y_\varepsilon)-\beta^\varepsilon(y_\varepsilon)|+|\beta^\varepsilon(y_\varepsilon)|\leqslant 2+|\beta^\varepsilon(y_\varepsilon)|$.

Hence $z_\varepsilon$ is bounded in $L^2(S\times]-1,1[)$ and for on some subsequence $\varepsilon_n\to 0$

(3.10) $\beta^{\varepsilon_n}(y_{\varepsilon_n})\to w$ (*weakly*) in $L^2(S\times]-1,1[)$ and $z_{\varepsilon_n}\to z$ (*weakly*) in $L^2(S\times]-1,1[)$.

On the other hand, if we denote $J_\varepsilon r=(1+\varepsilon\beta)^{-1}(r)$, we have

(3.11) $$r-J_\varepsilon r=\varepsilon\beta_\varepsilon(r),\ \beta_\varepsilon(r)\in\beta(J_\varepsilon r).$$

Therefore $J_\varepsilon(y_\varepsilon-\varepsilon\theta)$ is strongly convergent, for $\varepsilon\to 0$, to $y$ in $L^2(S\times]-1,1[)$, and

(3.12) $$y(s)\in D(\beta)\ \mu\text{-a.e. and } z(s,\theta)\in\beta(y(s))\ \ \mu\times d\theta\text{-a.e.},$$

since $\beta$ is demiclosed. Finally by (3.8—b), (3.10), (3.12) it follows (3.9—e).

Taking into account the monotonicity of $\beta$ and (3.11) we also have :



(3.13) a) $(\beta_\varepsilon(r)-\beta_\delta(s))\,(r-s)\geqslant-\frac{1}{4}(|\beta_\varepsilon(r)|^2+|\beta_\delta(s)|^2)\geqslant(\varepsilon+\delta)$

b) $(\beta^\varepsilon(r)-\beta^\delta(s))\,(r-s)\geqslant-8\,(1+|\beta^\varepsilon(r)|^2+|\beta^\delta(s)|^2)\,(\varepsilon+\delta)$.

For each $\varepsilon>0$ consider the approximating system

$$\frac{dz}{dt}+Az=0,\ \text{in }\Theta,$$

(3.14) $$\frac{\partial z}{\partial\nu}+\beta^\varepsilon(z)=h,\ \text{in }\Omega\times\Sigma,\ z(\omega,0,x)=u_0(\omega,x),\ \text{in }\Omega\times D.$$

Let $A_\varepsilon: H^1(D)\to(H^1(D))^*$ be the operator defined by

(3.15) $$<A_\varepsilon z,\ y>=a(z,y)+\int_\Gamma\beta^\varepsilon(z)\,y\,d\sigma,\ z,\ y\in H^1(D),$$

and $\tilde h\in L^2_{ad}(\Omega\times]0,T[\,;(H^1(D))^*)$ be given by

(3.16) $$<\tilde h(\omega,t),\ y>=\int_\Gamma h(\omega,t,\sigma)\,y(\sigma)\,d\sigma,\ y\in H^1(D).$$

System (3.14) can be written (in the sense of Definition 2.1 with $M=0$, $f=0$ and $g$ replaced by $h$) as

(3.17) $$\frac{dz}{dt}(\omega,t)+A_\varepsilon z(\omega,t)=\tilde h(\omega,t),\quad z(\omega,0)=u_0(\omega),$$

which has a unique solution $z_\varepsilon\in W_{ad}(\Theta)$ because $A_\varepsilon$ is a continuous, monotone, coercive and sublinear operator from $H^1(D)$ to $(H^1(D))^*$.

**Lemma 3.3.** *If hypotheses (H1) hold,* $u_0\in L^2(\Omega,\mathcal{F}_0,P;\ L^2(D))$, $\varphi(u_0)\in L^1(\Omega\times D)$ *and* $h\in L_{ad}(\Omega\times\Sigma)$ *then System* (3.3) *has a unique solution* $z\in W_{ad}(\Theta)$ *in the sense of Definition* 2.1, *i.e.*

(3.18) *there exists* $w\in L^2_{ad}(\Omega\times\Sigma)$ *such that* $w(\omega,t,\sigma)\in\beta(z(\omega,t,\sigma))$ a.e. *on* $\Omega\times\Sigma$

*and*

(3.19) $$-\int_Q\frac{\partial v}{\partial t}z\,dt\,dx+\int_0^T a(z,v)\,dt+\int_\Sigma(w-h)\,v\,dt\,d\sigma=\int_D u_0 v(\omega,0,x)\,dx$$

$\omega$-a.s. ; *for all* $v\in W_{ad}(\Theta)$, $v(\omega,T,x)=0$ a.e. *on* $\Omega\times D$.

*Moreover for* $\varepsilon\to 0$

(3.20) a) $z_\varepsilon\to z$ *strongly in* $L^2_{ad}(\Omega;\ C([0,T];\ L^2(D)))\cap L^2_{ad}(\Omega\times]0,T[\,;$ $H^1(D)$ *and weakly in* $L^2(\Omega;\ W(Q))$

b) $\omega$-a.s. $z_\varepsilon(\omega)\to z(\omega)$ *strongly in* $C([0,T];\ L^2(D))\cap L^2(]0,T[\,;$ $H^1(D))$

*and there exists* $C>0$ *a constant independent of* $\omega$, $h$, $u_0$ *such that*

$$\| z \|^2_{W(Q)} + \| w \|_{L^2(\Sigma)} \leqslant C(\| h \|^2_{L^2(\Sigma)} + \| u_0 \|_{L^2(D)} + \tag{3.21}$$
$$+ \int_D \varphi(u_0(x))\, dx + 1),\ \omega\text{-a.s.}$$

*Proof.* Here we shall use the idea from [1] (Proposition 1). From (3.17), for $z = z_\varepsilon$, it follows by the energy equality that

$$\| z_\varepsilon(t) \|^2_{L^2(D)} + 2 \int_0^t < A_\varepsilon z_\varepsilon,\ z_\varepsilon > ds = \| u_0 \|^2_{L^2(D)} + \int_0^t (h(s),\ z_\varepsilon(s)_{L^2(\Gamma)})\ ds,$$

for all $t \in [0,\ T]$ ; $\omega$-a.s., where $<\ldots>$ is the duality between $(H^1(D))^*$ and $H^1(D)$. But

$$< A_\varepsilon z_\varepsilon,\ z_\varepsilon > \geqslant C \| grad\ z_\varepsilon \|^2_{L^2(D)} - \| z_\varepsilon \|_{L^2(\Gamma)},$$

$$\text{where } \| \text{grad } z_\varepsilon \|^2_{L^2(D)} = \int_D \sum_{i=1}^N | \partial z / \partial x_i |^2\, dx.$$

Hence by the continuity of the mapping $z \to z |_\Gamma$ from $H^1(D)$ to $L^2(\Gamma)$ we have

$$\sup_{s \in [0,\ T]} \| z_\varepsilon(s) \|^2_{L^2(D)} + \int_0^T \| \text{grad}\, z_\varepsilon(s) \|^2_{L^2(D)} \leqslant C_1(\| u_0 \|^2_{L^2(D)} + \| h \|^2_{L^2(\Sigma)} + 1). \tag{3.22}$$

(we shall denote by $C_i$ constants independent of $\omega$, $\varepsilon$, $u_0$, $h$).

Next we take the inner product of (3.17) with $\beta^\varepsilon(z_\varepsilon)$. Inasmuch as

$$a(\psi,\ \beta^\varepsilon(\psi)) \leqslant \beta\ (0) \int_D a_0(x)\, \psi(x)\, dx \leqslant -C_2(\| \psi \|^2_{L^2(D)} + 1)$$

for all $\psi \in H^1(D)$, we have :

$$\int_D \varphi(z_\varepsilon(\omega,\ T,\ x))\, dx + \int_\Sigma (\beta^\varepsilon(z_\varepsilon) - h)\, \beta^\varepsilon(z_\varepsilon)\, d\sigma dt \leqslant \tag{3.23}$$
$$\leqslant \int_D \varphi^\varepsilon(u_0(\omega,\ x))\, dx + C_2(\| z_\varepsilon \|^2_{L^2(Q)} + 1),$$

$\omega$-a.s. Along with (3.9-d) and (3.22), (3.23) yields

$$\| \beta^\varepsilon(z_\varepsilon) \|^2_{L^2(\Sigma)} \leqslant C_3(\| u_0 \|^2_{L^2(D)} + \| h \|^2_{L^2(\Sigma)} + \int_D \varphi(u_0)\ dx + 1),\ \omega\text{-a.s.} \tag{3.24}$$

It is easy to show that

$$\| A_\varepsilon z_\varepsilon \|^2_{(H^1(D))^*} \leqslant C_4\ (\| z_\varepsilon \|^2_{H^1(D)} + \| \beta^\varepsilon(z_\varepsilon) \|^2_{L^2(\Gamma)}). \tag{3.25}$$

Inequalities (3.22), (3.24), (3.25) and Eq. (3.17) imply that

$$\| z_\varepsilon \|^2_{W(Q)} + \| \beta^\varepsilon(z_\varepsilon) \|^2_{L^2(\Sigma)} \leqslant C_5\ (1 + \| h \|^2_{L^2(\Sigma)} + \tag{3.26}$$
$$+ \| u_0 \|^2_{L^2(D)} + \int_D \varphi(u_0)\ dx),\ \omega\text{-a.s.}$$

Now from (3.17) for $\varepsilon$, $\delta > 0$ we have

$$\| z_\varepsilon(t) - z\delta(t) \|^2_{L^2(D)} + \| \text{grad}(z_\varepsilon - z\delta) \|^2_{L^2(Q)} +$$
$$+ \int_\Sigma (\beta^\varepsilon(z_\varepsilon) - \beta^\delta(z_\delta))\ (z_\varepsilon - z_\delta)\ d\sigma dt \leqslant 0,$$



where from by (3.13-b) and (3.26) we get

$$\| z_\varepsilon - z_\delta \|^2_{C([0,T];\, L^2(D))} + \| \text{grad}\ (z_\varepsilon - z_\delta) \|^2_{L^2(Q)} \leqslant \bar{K}(\omega)\ (\varepsilon + \delta), \tag{3.27}$$

$\omega$-a.s., where $\bar{K}(\cdot) \in L^1\Omega()$ and is independent of $\varepsilon$ and $\lambda$. Hence it exists $\lim_{\varepsilon \to 0} z_\varepsilon = z$ in $L^2_{ad}\ (\Omega\ ; C([0,\ T]\ ;\ L^2(D))) \cap L^2_{ad}\ (\Omega \times ]\,0,\ T[\ ;\ H^1(D))$. Moreover, for almost all $\omega$, $z_\varepsilon(\omega) \to z(\omega)$ in $C([0,\ T]\ ;\ L^2(D)) \cap L^2(0,\ T\ ;\ H^1(D))$ and from (3.26) $z_\varepsilon \to z$ weakly in $L^2(\Omega\ ;\ W(Q))$.
By (3.24) there exist $w \in L^2(\Omega \times \Sigma)$ and $\varepsilon_n \to 0$ such that $\beta^{\varepsilon_n}(z_{\varepsilon_n}) \to w$ (weakly) in $L^2(\Omega \times \Sigma)$ and $\beta^{\varepsilon_n}(z(\omega)) \to w(\omega)$ (weakly) in $L^2(\Sigma)$, $\omega$-a.s. Then by (3.9-e) we have $w(\omega,\ t,\ \sigma) \in \beta(z(\omega,\ t,\ \sigma))$ a.e. on $\Omega \times \Sigma$ and by multiplying in (3.17) by $v \in W_{ad}(\Theta)$, $v(\omega,\ T,\ x) = 0$ a.e. on $\Omega \times D$, we get to limit (3.19). From (3.26) it results (3.21). The uniqueness of $z$ can be obtained easily from (3.19). This completes the proof of Lemma 3.3.

*Proof. of Theorem* 3.1. (continuation). If we add member by member the relations (3.5), (3.7) and (3.19) with $h = g - \partial/\partial\nu\ (y_1 + y_2)$ the proof is finished.

*Remark* 3.1. The solution for System (2.1) has the form

$$u(t) = z(t) + \int_0^t S(t-s)\, f(s)\, ds + \int_0^t S(t-s)\ dM(s) \tag{3.28}$$

where $z$ is solution for (3.3) and $S(t)$ is the semigroup generated by the operator $A$ in $L^2(D)$.

**§4. Two elementary examples. (Noisy heat equations.** We shall consider stochastic versions for the deterministic problems given by V. Barbu in [1] (see also [2] and [3]).

Let $(\Omega,\ \mathcal{F},\ P,\ (\mathcal{F}_t)_{t \geqslant 0})$ be a fixed probabilized stochastic basis.

*Example 1.* Let $D = \{(x,\ y) \in R^2 : x^2 + y^2 < 1\}$ and $\Gamma$ be the boundary of $D$, $Q = ]0,\ T[ \times D$, $\Sigma = ]0,\ T[ \times \Gamma$ ; $\Theta = \Omega \times Q$. We shall consider the following system

$$u(\omega,\ t,\ x,\ y) - \int_0^t [u_{xx}(\omega,\ s,\ x,\ y) + u_{yy}(\omega,\ s,\ x,\ y)]\ ds = u_0(\omega,\ x,\ y) +$$
$$+ \int_0^t f(\omega,\ s,\ x,\ y)\ ds + W(\omega,\ t,\ x,\ y) \ \text{ a.e. on } \Omega \times ]\,0,\ T[ \times D \tag{4.1}$$
$$\frac{\partial u}{\partial n}(\omega,\ t,\ x,\ y) + \beta(u(\omega,\ t,\ x,\ y)) \ni g(\omega,\ t,\ x,\ y) \text{ a.e. on } \Omega \times ]\,0,\ T[ \times \Gamma,$$

where $u_{xx}$, $u_{yy}$ denote the second derivatives of $u$ with respect to $x$ and $y$, respectively ; $\partial u / \partial n$ is the outward normal derivative of $u$ on $\Gamma$.

In this case the operator $A$ will be $Az = -z_{xx} - z_{yy}$ for

$$z \in D(A) \subset L^2(D)\ ;\ D(A) = H^2(D) \cap H^1_0\ (D).$$

We shall assume that :

A. $W(t)$ is a $L^2(D)$-valued Wiener process.

$$W(\omega,\ t,\ x,\ y) = \sum_{n=1}^\infty \beta_n(\omega,\ t)\ e_n(x,\ y).$$

where $e_n(x, y)=\sqrt{2/\pi}\sin n\pi(x^2+y^2)$, $e_n \in D(A)$ ; $\beta_n$ are mutually independent real Wiener processes with incremental covariance $\alpha_n$, $\sum_{n=1}^{\infty}\alpha_n<+\infty$.

B. $\beta$ is a graph maximal monotone in $\mathbf{R}\times R$ such that $0\in\beta(0)$.

C. $f\in L^2_{ad}(\Omega\times]0, T[ ; H)$, $g\in L^2_{ad}(\Omega\times\Sigma)$ $u_0\in L^2(\Omega, \mathcal{F}_0, P ; H)$, where $H=L^2(D)$.

Since the covariance operator, $K$, of $W$ is :

$$Kz=\sum_{n=1}^{\infty}\alpha_n(z, e_n)\,e_n \tag{4.2}$$

and the Doleans measure is $\lambda=P\times dt$, then the condition

$$\sup_{t\in[0,T]}\int_{\Omega\times[0,t]}\|AS(t-s)\,K^{1/2}\|_2^2\,d\lambda<+\infty$$

holds if, for example :

$$\sum_{n=1}^{\infty}\alpha_n n^4<+\infty. \tag{4.3}$$

We remark that the hypotheses of Theorem 3.1 are verified if the assumptions (A), (B), (C), (4.3) hold and $\varphi(u_0)\in L^1(\Omega\times D)$. Then system (4.1) has a unique solution $u\in L^2_{ad}(\Omega\times]0, T[ ; H^1(D))\cap L^2_{ad}(\Omega ; C(0, T ; L^2(D)))$ in the sense of Definition 2.1.

*Example* 2. Let us consider the problem : there exists $u\in L^2_{ad}(\Omega\times]0, T[ ; H^1(0,1))\cap L^2_{ad}(\Omega ; C(0, T ; L^2(0,1)))$ such that :

$$\begin{aligned}
&\text{a) } u(\omega, t, 0)\geqslant 0,\ u(\omega, t, 1)\geqslant 0 \text{ a.e. on } \Omega\times]0, T[\\
&\text{b) } u_x(\omega, t, 0)+g_0(\omega, t)=0, \text{ if } u(\omega, t, 0)>0,\\
&\quad u_x(\omega, t, 0)+g_0(\omega, t)\leqslant 0, \text{ if } u(\omega, t, 0)=0,\\
&\quad u_x(\omega, t, 1)+g_1(\omega, t)=0, \text{ if } u(\omega, t, 1)>0,\\
&\quad u_x(\omega, t, 1)+g_1(\omega, t)\geqslant 0, \text{ if } u(\omega, t, 1)=0,\\
&\quad \text{a.e. on } \Omega\times]0, T[\\
&\text{c) } u(\omega, t, x)-\int_0^t u_{xx}(\omega, s, x)\,ds=u_0(\omega, x)+\int_0^t f(\omega, s, x)\,ds+\\
&\qquad +W(\omega, t, x), \text{ a.e. on } \Omega\times]0, T[\times]0, 1[,
\end{aligned} \tag{4.4}$$

where $u_x$, $u_{xx}$ denote the first and second derivatives of $u$ with respect to $x$, respectively, and

A. $f\in L^2_{ad}(\Omega\times]0, T[ ; L^2(0,1))$, $g_0, g_1\in L_{ad}(\Omega\times]0, T[)$ $u_0\in L^2(\Omega, \mathcal{F}_0, P ; L^2(0, 1))$ are fixed ;

B. $W$ is an $L^2(0,1)$-valued Wiener process

$$W(\omega, t)=\sum_{n=1}^{\infty}\beta_n(\omega, t)\,e_n(x),$$



where $e_n(x)=\sqrt{2}\sin n\pi x$ ; $\beta_n$ are mutually independent real Wiener processes with incremental covariance $\alpha_n$, $\sum_{n=1}^{\infty}\alpha_n<+\infty$.

In this case the operator $A$ will be $Az=-z_{xx}$ for $z\in D(A)\subset L^2(0,1)$, $D(A)=H^2(0,1)\cap H_0^1(0,1)=\{z\in L^2(0,1) : z_x,\ z_{xx}\in L^2(0,1),\ z(0)=z(1)=0\}$, and the semigroup $S(t)$ generated by $A$ is

$$S(t)=\sum_{n=1}^{\infty}e^{-n^2\pi^2 t}(z, e_n)\,e_n$$

where $(.,.)$ denotes the inner product in $L^2(0,1)$.

The covariance operator, $K$, of $W$ is the following

$$Kz=\sum_{n=1}^{\infty}\alpha_n(z, e_n)\,e_n$$

and the Doleans measure is $\lambda=P\times dt$.

Since

$$\|AS(t-s)\,K^{1/2}\|_2^2=\sum_{n=1}^{\infty}\alpha_n\|AS(t-s)\,e_n\|^2=\sum_{n=1}^{\infty}\alpha_n\|Ae^{-n^2\pi^2(t-s)}e_n\|^2=$$

$$=\sum_{\pi=1}^{\infty}\alpha_n\pi^4 n^4 e^{-2n^2\pi^2(t-s)},$$

where $\|\cdot\|$ is the norm in $L^2(0,1)$ and $\|\cdot\|_2$ is the Hilbert-Schmidt norm, then the condition from (H3) becomes

$$\sum_{n=1}^{\infty}\alpha_n n^2<+\infty. \tag{4.5}$$

If we denote by $I_{[0,+\infty[}$ the indicator convex function of $[0, +\infty[$ then in the problem (4.4) the conditions (a) and (b) may be written in the following form :

$$\begin{aligned}
&-u_x(\omega, t, 0)+\partial I_{[0,+\infty[}(u(\omega, t, 0))\ni g_0(\omega, t),\\
&u_x(\omega, t, 1)+\partial I_{[0,+\infty[}(u(\omega, t, 1))\ni g_1(\omega, t),
\end{aligned} \tag{4.6}$$

a.e. on $\Omega\times]0, T[$.

We remark that the hypotheses of Theorem 3.1 are verified if the assumptions (A), (B), (4.5) hold and $u_0(\omega, x)\geqslant 0$ a.e. on $\Omega\times]0,1[$. Then in the same way as Theorem 3.1 (one considers $<A_\varepsilon z, y>=\int_0^1 z_x y_x dx+$ $+\beta^\varepsilon(z(0))\,y(0)+\beta^\varepsilon(z(1))\,y(1)$, $<\tilde{h}(\omega, t), y>=h_0(\omega, t)\,y(0)+h_1(\omega, t)\,y(1)$, $h_0(\omega, t)=g_0(t)+\frac{\partial}{\partial x}(y_1+y_2)\,(t, 0)$, $h_1(\omega, t)=g_1(t)-\frac{\partial}{\partial x}(y_1+y_2)\,(t, 1))$ we can prove that :

Problem (4.4) has an unique solution $u$ in the sense of Definition 2.1, i.e. there exist $w_0, w_1\in L^2_{ad}(\Omega\times]0, T[$ such that

$$
\begin{aligned}
&w_0(\omega, t) \leqslant 0, \text{ if } u(\omega, t, 0)=0 \qquad w_1(\omega, t) \leqslant 0, \text{ if } u(\omega, t, 1)=0\\
&w_0(\omega, t) = 0, \text{ if } u(\omega, t, 0)>0 \qquad w_1(\omega, t) = 0, \text{ if } u(\omega, t, 1)>0
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&\int_0^T\int_0^1 uv_t\,dx\,dt+\int_0^T\int_0^1 u_x v_x\,dx\,dt+\int_0^T (w_0-g_0)\,v(\omega, t, 0)\,dt+\\
&+\int_0^T (w_1-g_1)\,v(\omega, t, 1)\,dt=\int_0^1 u_0 v(\omega, 0, x)\,dx+\int_0^T\int_0^1 fv\,dx\,dt+\\
&+\int_0^T\int_0^1 v(\omega, t, x)\,dW(\omega, t, x)\,dx, \quad \omega\text{-a.s.};
\end{aligned}
\tag{4.8}
$$

for all $v \in W_{ad}(\Omega \times ]0, T[\times ]0, 1)$ $v(\omega, T, x)=0$ a.e. on $\Omega \times ]0, 1[$.

From Proposition 1.2 from [5] and (2.3) the equality (4.8) is equivalent) with

$$
\begin{aligned}
&\int_0^T\int_0^1 [(u-W)\,v_t+fv-(w_0-g)\,v(\omega, t, 0)-(w_1-g_1)\,v(\omega, t, 1)-\\
&-u_x v_x]\,dx\,dt+\int_0^1 u_0 v(\omega, 0, x)\,dx=0, \quad \omega\text{-a.s.}
\end{aligned}
\tag{4.9}
$$

for all $v \in L^2_{ad}(\Omega \times ]0, T[; H^1(0,1))$ such that $v_t \in L^2_{ad}(\Omega \times ]0, T[; H^1(0,1)$ and $v(\omega, T, x)=0$ a.e. on $\Omega \times ]0, 1[$.

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## *RECENZII*

WACLAW SIERPINSKI: *Oeuvres choisies. Tome I: Bibliographie. Théorie des nom-*
*bres et analyse mathématique*, Publié par les soins de S. Hartman et A. Schinzel, Académie
Polonaise des Sciences, Institut Mathématique, PWN-Éditions Scientifiques de Pologne,
Warszawa, 1974, 300 p., 140 Zl; *Tome II: Théorie des ensembles et ses applications,*
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Académie Polonaise des Sciences. Institut Mathématique, PWN-Éditions Scientifiques de
Pologne, Warszawa, 1976, 688 p., 250 Zl.

W. Sierpinski s-a născut la 14 martie 1882 la Varșovia. Studiile liceale și facultatea
le-a făcut la Varșovia. În anul 1904 își dă doctoratul în matematici iar în anul 1906
devine doctor în filozofie. Pînă în anul 1910 a fost profesor de matematică în învă-
țămîntul liceal, din 1910 a fost profesor la universitățile din Lwów, Cracovia și Varșovia.
A fost membru al Academiei Poloneze de Științe. De asemenea a fost membru a douăsprezece
academii de științe din alte țări, printre care și al Academiei R.S.R. A murit la 21 octom-
brie 1969 la Varșovia. O prezentare a vieții sale este făcută de către K. Kuratowski în vo-
lumul I. De asemenea, volumul I conține lista publicațiilor lui W. Sierpinski, compusă
din 720 lucrări și comunicări științifice, 106 conferințe și articole de popularizare și istorie,
50 de cărți și 12 cursuri universitare. Dintre cele 720 de lucrări și comunicări științifice,
colectivul de redacție a ales lucrările cele mai semnificative (prin actualitatea lor, importanța
teoretică, frumusețea rezultatelor și metodelor folosite) și le-a grupat în trei volume;
primul conținînd lucrări de teoria numerelor și analiză matematică, iar celelalte două vo-
lume fiind dedicate lucrărilor de teoria mulțimilor și aplicații (al doilea volum conține
lucrări apărute în perioada 1908—1929 iar volumul al treilea conține lucrări apărute între
anii 1930 și 1966). Prezentarea lucrărilor de teoria numerelor este făcută de către A. Schinzel
în volumul I, prezentarea lucrărilor de analiză matematică este făcută de către S. Hartman
în volumul I, prezentarea lucrărilor de Teoria mulțimilor și aplicații este făcută în volumul
al doilea de către A. Mostowski (pentru lucrările de teoria generală a mulțimilor), S. Hartman
și E. Marczewski (pentru lucrările de mulțimi analitice și proiective), E. Marczewski (pen-
tru lucrările de topologie generală), S. Hartman (pentru lucrările de măsură și categorii, con-
gruența mulțimilor) și de S. Hartman (pentru lucrările de funcții de o variabilă reală).
Cele trei volume conțin 11 lucrări de teoria numerelor, 10 lucrări de analiză matematică,
și peste 150 de lucrări de teoria mulțimilor și aplicații.

*V. Popescu*

A. OSTROWSKI: *Collected mathematical papers*. Vol. 3. Birkhäuser Verlag, Basel,
Boston, Stuttgart, 1984; 532 p., SFr 68,00

Acesta este cel de-al treilea volum dintre cele șase volume în care se vor publica
articolele profesorului A. Ostrowski. Conține articole consacrate unor subiecte de teoria
numerelor (17 articole), geometrie (6 articole), topologie (8 articole), convergență (6 articole).
Toate aceste articole apar acum pentru prima dată în volum.

*Mirela Ștefănescu*