Nous allons montrer la fonctorialité de Ψ. Soit $\alpha : X_1 \longrightarrow X$ un monomorphisme ; il y a un seul morphisme $\bar{\alpha} : X_1/G \longrightarrow X/G$ qui rend commutatif le diagramme suivant :

$$\begin{array}{ccc} X & \xrightarrow{\ \mu\ } & X/G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\ X_1 & \xrightarrow{\ \mu_1\ } & X_1/G \end{array}$$

Il faut vérifier la commutativité du diagramme suivant :

$$\begin{array}{ccc} {}_G\mathcal{E}(X, T(Y)) & \xrightarrow{\ \Psi(X_1, Y)\ } & \mathcal{E}(X/G, Y) \\ \downarrow{\scriptstyle \bar{h}_y(\alpha)} & & \downarrow{\scriptstyle h_y(\bar{\alpha})} \\ {}_G\mathcal{E}(X_1, T(Y)) & \xrightarrow{\ \Psi(X, Y_1)\ } & \mathcal{E}(X_1/G, Y_1) \end{array}$$

Soit $F : X \longrightarrow T(Y)(=Y)$ un monomorphisme : les relations suivantes sont vraies :

$$h_y(\bar{\alpha})\Psi(X, Y)(F) = h_y(\bar{\alpha})(\bar{F}) = \bar{F}\bar{\alpha}$$

$$\Psi(X_1, Y)\ h_y(\alpha)(F) = \Psi(X_1, Y)(F\alpha) = \overline{F\alpha} = \bar{F}\bar{\alpha}.$$

Parce que le morphisme $\overline{F\alpha} : X_1/G \longrightarrow Y$ est l'unique morphisme tel que $F\alpha = \overline{F\alpha}\mu_1$ et le morphisme $\bar{F}\bar{\alpha}$ vérifie l'égalité $\overline{F\alpha}\mu_1 = \bar{F}\mu\alpha = F\alpha$, il résulte l'égalité $\overline{F\alpha} = \bar{F}\bar{\alpha}$ ce qui montre la commutativité du diagramme de plus haut.

Soit $\beta : Y \longrightarrow Y_1$; alors le diagramme suivant est commutatif :

$$\begin{array}{ccc} {}_G\mathcal{E}(X, T(Y)) & \xrightarrow{\ \Psi(X, Y)\ } & \mathcal{E}(X/G, Y) \\ \downarrow{\scriptstyle \bar{h}_X(\beta)} & & \downarrow{\scriptstyle \bar{h}_{X/G}(\beta)} \\ {}_G\mathcal{E}(X, T(Y_1)) & \xrightarrow{\ \Psi(X, Y_1)\ } & \mathcal{E}(X/G, Y_1) \end{array}$$

$$\bar{h}_{X/G}(\beta)\Psi(X, Y)(F) = h_{X/G}(\beta)(\bar{F}) = \beta\bar{F}$$

$$\Psi(X, Y_1)\bar{h}_Y(\beta)(F) = \Psi(X, Y_1)(\beta F) = \overline{\beta F} = \beta\bar{F}.$$

Parce que le morphisme $\overline{\beta F}$ est l'unique morphisme tel que $\beta F = \overline{\beta F}\mu$ et le morphisme $\beta\bar{F}$, vérifie l'égalité $\beta F = \beta\bar{F}\mu$ il résulte l'égalité $\overline{\beta F} = \beta\bar{F}$ ce qui montre la commutativité du diagramme de plus haut et donc il résulte que Ψ est un isomorphisme fonctoriel.

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Reçu le 12.XII.1981

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Tomul XXX—2, s. I a Matematică

ON THE CONSTRUCTION OF BOL LOOPS. II

BY

A. R. T. SOLARIN and B. L. SHARMA

1. **Introduction.** A loop $L(.)$ is a right Bol loop if and only if for all $x, y, z, \in L$,

$$(1) \qquad (xy.z)y = x(yz.y).$$

Bol loops had been discussed by Bol [1] and Robinson [5]. Burn [3] has proved that for any prime p, a Bol loop of order $2p$ or p^2 is necessarily a group. In an earlier paper [7], we disproved a conjecture that Bol loops of order $2p^2$, where p is an odd prime are necessarily groups by constructing Bol loops of order $2p^2$ which are non-associative. In continuation of the previous study, we consider existence of non-associative Boll loops of order p^2q where p and q are odd prime integers. We shall establish their existence for two generators. In particular we settle some of the open problems stated by Harald Niederreiter and Karl H. Robinson [4] that Bol loops of orders 18, 35, 50, 70, 77, 85 and 98 can not be determined. Here we settle 18, 50 and 98.

2. In our previous paper [7] we proved the following theorem.

Theorem 1. *Let H be a non abelian subgroup of a group G and let $A = H \times G$. For (h_1, g_1) and (h_2, g_2) in A, define*

$$(2) \qquad (h_1, g_1)\circ(h_2, g_2) = (h_1h_2,\ h_2g_1h_2^{-1}g_2)\ ;$$

then $A(0)$ is a Bol loop.

In the next theorem we shall prove that any Bol loop obtained by the above theorem contains a sub-loop which is also a Bol loop. For example if G is a non abelian group of order pq, where q divides $p-1$, then $A = G \times G$ contains a Bol loop of order p^2q.

Theorem 2. *Let G be a nonabelian group of order pq, and let $A = G \times G$. For (h_1, g_1) and (h_2, g_2) in A, define*

$$(2) \qquad (h_1, g_1)\circ(h_2, q_2) = (h_1h_2,\ h_2g_1h_2^{-1}g_2),$$

then $A(\circ)$ is a Bol loop of order p^2q^2 and A contains a subloop of order p^2q which is also a Bol loop.

Proof. By theorem 1, $A(\circ)$ is a Bol loop and $o(A) = p^2q^2$.

Since G is a non-abelian group of order pq then it has two generators x and y, such that $x^p=y^q=e$, where e is the identity element in G. Let $A_1=$ $=<(y, yx)>$, then A_1 is a subloop of A and

$$o(A_1)=pq. \tag{3}$$

Also let $A_2=<(x, x^{-1})>$, then A_2 is a subloop of A and

$$o(A_2)=p. \tag{4}$$

Now consider $A_3=A_1\times A_2$, A_3 is a subloop of A and we only need to show that $A_3(\circ)$ is a Bol loop. In establishing it, is sufficient to show that $A_3(0)$ is non-associative; since it is a subloop of a Bol loop its elements must satisfy the Bol identity.

If $a=(y, yx)$, $b=(y, yx)$, $c=(x, x^{-1})$, then a, b, $c\in A_3$; and

(5) $(a\circ b)\circ c=(y^2x, y^3x^{k^2})$, (6) $a\circ(b\circ c)=(y^2x, y^2x^{2k-1})$,

where $xy=yx^k$; $h^q\equiv 1(\mathrm{mod}\ p)$ and $q/p-1$.

Now from (5) and (6) $(a\circ b)\circ c=a\circ(b\circ c)$ whenever $k^3=2k-1\rightarrow k=1$, but $k\neq 1$, for non-abelian G. Therefore

$$(a\circ b)\circ c\neq a\circ(b\circ c). \tag{7}$$

Therefore A_3 is non-associative, hence it a Bol loop and

$$o(A_3)=p^2q. \tag{8}$$

The theorem below is essentially a special case of the theorem 2 for the dihedral groups which is of importance for our study.

Theorem 3. *Let D_p be a dihedral group of order $2p$. $A=Dp\times Dp$, $A(0)$ as in theorem 1; then A contains a subloop of order $2p^2$ which is a Bol loop.*

Proof. Since Dp is a nonabelian group of order $2p$, with two generators x and y such that $x^p=y^2=e$, where e is the identity element in Dp, then by theorem 2, Dp is a Bol loop of order $4p^2$ and it contains a subloop of order $2p^2$ which is a Bol loop.

Remark. In table III we display a Bol loop of order 18 ($p=3$) which is a subloop of $A=D_3\times D_3$.

3. In this section we construct Bol loops of order $4k$ for all positive integer $k\geqslant 2$.

Theorem 4. *Let $A(\circ)=D_k\times C_2$ where $D_k(.)$ is the dihedral group of order $2k$ with generators x and y, such that $x^p=y^2=e$. The binary operation $(\circ)$ is defined as follows:*

$$
(9)\quad
\begin{aligned}
&\text{(i)} && (y^{\beta_1}x^{\gamma_1}, e)\circ(y^{\beta_2}x^{\gamma_2}, e)=(y^{\beta_1}x^{\gamma_1}\cdot y^{\beta_2}x^{\gamma_2}, e),\\
&\text{(ii)} && (y^{\beta_1}x^{\gamma_1}, z)\circ(y^{\beta_2}x^{\gamma_2}, e)=(y^{\beta_1}x^{\gamma_1}\cdot y^{\beta_2}x^{\gamma_2}, z),\\
&\text{(iii)} && (y^{\beta_1}x^{\gamma_1}, e)\circ(y^{\beta_2}x^{\gamma_2}, z)=(y^{\beta_1+\beta_2}x^{\gamma_2-\gamma_1}, z),\\
&\text{(iv)} && (y^{\beta_1}x^{\gamma_1}, z)\circ(y^{\beta_2}x^{\gamma_2}, z)=(y^{\beta_1+\beta_2}x^{\gamma_1+\gamma_2}, e),\\
&\text{(v)} && y^{\beta_1}x^{\gamma_1}\cdot y^{\beta_2}x^{\gamma_2}=y^{\beta_1+\beta_2}x^{\gamma_2-\gamma_1},
\end{aligned}
$$

then $A(0)$ is a right Bol loop of order $4k$, where k is any integer greater than two.

Proof. First we show that $A(0)$ is non associative.
Let $A=(y^{\beta_1}x^{\gamma_1}, e)$; $B=(y^{\beta_2}x^{\gamma_2}, z)$; $c=(y^{\beta_3}x^{\gamma_3}, e)$

$$(A\circ B)\circ C=(y^{\beta_1+\beta_2+\beta_3}x^{\gamma_3-\gamma_1-\gamma_2}, z), \tag{10}$$

$$A\circ(B\circ C)=(y^{\beta_1+\beta_2+\beta_3}x^{\gamma_1+\gamma_2-\gamma_3}, z). \tag{11}$$

From (10) and (11) it is clear that $(A\circ B)\circ C\neq A\circ(B\circ C)$, therefore $A(.)$ is non associative.
Next we show that $A(0)$ satisfies (1).

(a) $A=(y^{\beta_1}x^{\gamma_1}, e)$; $B=(y^{\beta_2}x^{\gamma_2}, e)$; $C=(y^{\beta_3}x^{\gamma_3}, e)$.
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{2\gamma_2-\gamma_1-\gamma_3}, e)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{2\gamma_2-\gamma_1-\gamma_3}, e)$;

(b) $A=(y^{\beta_1}x^{\gamma_1}, e)$; $B=(y^{\beta_2}x^{\gamma_2}, e)$; $C=(y^{\beta_3}x^{\gamma_3}, z)$,
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_1-\gamma_3}, z)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_1-\gamma_3}, z)$;

(c) $A=(y^{\beta_1}x^{\gamma_1}, e)$; $B=(y^{\beta_2}x^{\gamma_2}, z)$; $C=(y^{\beta_3}x^{\gamma_3}, e)$,
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_3-\gamma_1}, e)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_3-\gamma_1}, e)$,

(d) $A=(y^{\beta_1}x^{\gamma_1}, e)$; $B=(y^{\beta_2}x^{\gamma_2}, z)$; $C=(y^{\beta_3}x^{\gamma_3}, z)$,
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_1+2\gamma_2+\gamma_3}, z)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_1+2\gamma_2+\gamma_3}, z)$;

(e) $A=(y^{\beta_1}x^{\gamma_1}, z)$; $B=(y^{\beta_2}x^{\gamma_2}, e)$; $C=(y^{\beta_3}x^{\gamma_3}, e)$,
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{2\gamma_2-\gamma_1-\gamma_3}, z)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{2\gamma_2-\gamma_1-\gamma_3}, z)$;

(f) $A=(y^{\beta_1}x^{\gamma_1}, z)$; $B=(y^{\beta_2}x^{\gamma_2}, e)$; $C(y^{\beta_3}x^{\gamma_3}, z)$
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_1-\gamma_3}, e)$, $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_1+\gamma_3}, e)$

(g) $A=(y^{\beta_1}x^{\gamma_1}, z)$; $B=(y^{\beta_2}x^{\gamma_2}, z)$; $C=(y^{\beta_3}x^{\gamma_3}, e)$
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_3-\gamma_1}, z)$ $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_3-\gamma_1}, z)$

(h) $A=(y^{\beta_1}x^{\gamma_1}, z)$; $B=(y^{\beta_2}x^{\gamma_2}, z)$; $C=(y^{\beta_3}x^{\gamma_3}, z)$
$(AB\circ C)B=(y^{\beta_1+\beta_3}x^{\gamma_1+2\gamma_2+\gamma_3}, e)$ $A(BC\circ B)=(y^{\beta_1+\beta_3}x^{\gamma_1+2\gamma_2+\gamma_3}, e)$.

Since $(AB\circ C)B=A(BC\circ B)$ in all cases, therefore A satisfies (1). Thus $A(\circ)$ is a Bol loop.

Theorem 4 above enables us to construct Bol loop of order 12. (see table I).

To conclude this section, we shall give some theorems on the construction of Bol loops of order $4n$, where n is an even positive integer, which are of two generators. It should be noted that the construction in theorem 4 is of three generators.

Theorem 5. *Let $G(\circ)=C_{4n}\times C_2$ and the binary operation is defined as follows:*

(i) $(x^p, e)\circ(x^q, e)=(x^{p+q}, e)$; (ii) $(x^p, e)\circ(x^q, a)=(x^{p+q}, a)$;

(iii) $(x^p, a)\circ(x^q, e)=(x^{p+q}, a)$; (iv) $(x^p, a)\circ(x^q, a)=(x^{p-(2n-1)q}, e)$,

then $G(\circ)$ is a Bol loop.

Proof. First we show that it satisfies (1).

(a) $A=(x^p, e)$, $B=(x^q, e)$, $C=(x^r, e)$, $(AB.C)B=(x^{p+q+r+q}, e)$, $A(BC.B)=(x^{p+q+r+q}, e)$;

(b) $A=(x^p, e)$, $B=(x^q, e)$, $C=(x^r, a)$, then $(AB.C)B=(x^{p+q+r+q}, a)$ $A(BC.B)=(x^{p+q+r+q}, a)$;

(c) $A=(x^p, e)$, $B=(x^p, a)$, $C=(x^r, e)$, then $(AB.C)B=(x^{p+r+q-(2n-1)q}, e)$,

$$A(BC.B)=(x^{p+r+q-(2n-1)q}, e);$$

(d) $A=(x^p, e)$ $B=(x^q, a)$, $C=(x^r, a)$, then $(AB.C)B=(x^{p+q-(2n-1)r+q}, a)$,

$$A(BC, B)=(x^{p+q-(2n-1)r+q}, a);$$

(e) $A=(x^p, a)$, $B=(x^q, e)$, $C=(x^r, a)$, then $(AB.C)B=(x^{p+q+r+q}, a)$,

$$A(BC.B)=(x^{p+q+r+q}, a);$$

(f) $A=(x^p, a)$, $B=(x^q, e)$, $C=(x^r, a)$, then $(AB.C)B=(x^{p+q-(2n-1)r+q}, e)$,

$$A(BC.B)=(x^{p-(2n-1)q-(2n-1)r}, e);$$

(g) $A=(x^p, a)$, $B=(x^q, a)$, $C=(x^r, e)$, then $((AB.C)B=(x^{p-(2n-1)q+r+q}, a)$,

$$A(BC, B)=(x^{p+r-(2n-1)q+q}, a),$$

(h) $A=(x^p, a)$, $B=(x^q, a)$, $C=(x^r, a)$, then $(AB.C)B=(x^{p+r-(2n-1)q-(2n-1)q}, e)$,

$$A(BC.B)=(x^{p-(2n-1)q-(2n-1)q+r}, e);$$

Since $(AB.C)B=A(BC.B)$ in all cases, therefore $(xy.z)y=x(yz.y)$ in G. Thus $G(\circ)$ is a Bol loop.

(B) (e, e) behaves as two sided identity:

(C) Inverses:

If $A=(x^p, e)$, then $A^{-1}=(x^{-1}, e)$; If $A=(x^p, a)$ then $A^{-1}=(x^{(2n-1)p}, a)$;

(D) Non-associativity:

$$(x^p, a)(x^q, a)\circ(x^r, e)=(x^{p-(2n-1)q+r}, e),\ (x^p, a)\circ(x^q, a)(x^q, e)=$$
$$(x^{p-1(2n-1)q-(2n-1)}, e);$$

thus it is non-associative.

(E) $$(x^{(2n-1)p}, a)\circ(x^p, a)(x^q, e)=(x^{(2n-1)p(2n-1)q-(2n-1)p}, e).$$

It does not satisfy the left inverse property. Hence $G(0)$ is a Bol loop.

By this construction loops of order $8n$, where $n=1, 2, 3, \ldots$ can be constructed. (see table II for Bol loop of order 8).

Theorem 6. *Let $G(\circ)=C_{4n}\times C_2$ and the binary operation is defined as follows:*

(i) $(x^p, e)\circ(x^q, e)=(x^{p+q}, e)$, (ii) $(x^p, e)\circ(x^q, a)=(x^{p+q}, a)$,

(iii) $(x^p, a)\circ(x^q, e)=(x^{p-(2n-1)q}, a)$, (iv) $(x^p, a)\circ(x^q, a)=(x^{p+q+2n}, e)$;

then $G(\circ)$ is a Bol loop.

Theorem 7. *Let $G=C_{4n}\times C_2$ and the binary operation is defined as follows:*

(i) $(x^p, e)\circ(x^q, e)=(x^{p+q}, e)$, (ii) $(x^p, e)\circ(x^q, a)=(x^{p+q}, a)$,

(iii) $(x^p, a)\circ(x^q, e)=(x^{p+q}, a)$, (iv) $(x^p, a)\circ(x^q, a)=(x^{p-(2n-1)q+2n}, e)$;

then $G(\circ)$ is a Bol loop.

Theorem 8. *Let $G=C_{4n}\times C_2$ and the binary operation is defined as follows:*

(i) $(x^p, e)\circ(x^q, e)=(x^{p+q}, e)$, (ii) $(x^p, e)\circ(x^q, a)=(x^{p+q}, a)$,

(iii) $(x^p, a)\circ(x^q, e)=(x^{p-(2n-1)q}, a)$, (iv) $(x^p, a)\circ(x^q, a)=(x^{p+q}, e))$;

then $G(\circ)$ is a Bol loop.

Theorems 4—8 give constructions of Bol loops of order $4k$, where K is an integer greater than one.

Final remark. We are now in a better position to determine for most orders whether a non-associative Bol loop of that order exist or not. It is still an open problem to determine the status of 35, 65, 70, 77 and 85.

TABLE I

(Bol loop of order 12; $D_3\times C_2$)

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	8	9	7	12	10	11
3	1	2	6	4	5	9	7	8	11	12	10
4	6	5	1	3	2	11	10	12	8	7	9
5	4	6	2	1	3	12	11	10	7	9	8
6	5	4	3	2	1	10	12	11	9	8	7
7	8	9	12	10	11	2	3	1	5	6	4
8	9	7	10	11	12	3	1	2	4	5	6
9	7	8	11	12	10	1	2	3	6	4	5
10	12	11	8	7	9	5	4	6	1	3	2
11	10	12	9	8	7	6	5	4	3	2	1
12	11	10	7	9	8	4	6	5	2	1	3

TABLE II

(Bol loop of order 8; $C_4\ X\ C_2$)

1	2	3	4	5	6	7	8
2	3	4	1	6	7	8	5
3	4	1	2	7	8	5	6
4	1	2	3	8	5	6	7
5	8	7	6	3	4	1	2
6	5	8	7	4	1	2	3
7	6	5	8	1	2	3	4
8	7	6	5	2	3	4	1

TABLE III

(Bol loop of order 18; $C_3\times C_3\times C_2$)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	3	1	5	6	4	8	9	7	11	12	10	14	15	13	17	18	16
2	1	2	6	4	5	9	7	8	12	10	11	15	13	14	18	16	17
4	5	6	7	8	9	1	2	3	18	16	17	12	10	11	15	13	14
5	6	4	8	9	7	2	3	1	16	17	18	10	11	12	13	14	15
6	4	5	9	7	8	3	1	2	17	18	16	11	12	10	14	15	13
7	8	9	1	2	3	4	5	6	14	15	13	17	18	16	11	12	10
8	9	7	2	3	1	5	6	4	15	13	14	18	16	17	12	10	11
9	7	8	3	1	2	6	4	5	13	14	15	16	17	18	10	11	12
10	11	12	13	14	15	16	17	18	1	2	3	5	6	4	9	7	8
11	12	10	14	15	13	17	18	16	2	3	1	6	4	5	7	8	9
12	10	11	15	13	14	18	16	17	3	1	2	4	5	6	8	9	7
13	14	15	16	17	18	10	11	12	9	7	8	1	2	3	5	6	4
14	15	13	17	18	16	11	12	10	7	8	9	2	3	1	6	4	5
15	13	14	18	16	17	12	10	11	8	9	7	3	1	2	4	5	6
16	17	18	10	11	12	13	14	15	5	6	4	9	7	8	1	2	3
17	18	16	11	12	10	14	15	13	6	4	5	7	8	9	2	3	1
18	16	17	12	10	11	15	13	14	4	5	6	8	9	7	3	1	2

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Received 15.IV.1982

Department of mathematics
University of Ife
Ile-Ife, Nigeria

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX—2, s. I a Matematică

Addendum

ON THE CONSTRUCTION OF BOL-LOOPS

BY

A. R. T. SOLARIN and B. L. SHARMA

1. Theorem 3 in [1] is not true for $n>2$. The corrected form is as follows:

Theorem 3. *Let L be defined as all formal symbols*

$$z^i y^j x^k,\ i=0,1,\ j=0,1;\ k=0,1,2,\dots,n-1$$

where $z^2=y^2=x^n=e$ and

$$z^{\alpha_1}y^{\beta_1}x^{\gamma_1}\circ z^{\alpha_2}y^{\beta_2}x^{\gamma_2}=\begin{cases} z^{(\alpha_1+\alpha_2),\bmod 2}\,y^{(\beta_1+\beta_2),\bmod 2},\,x^{(\gamma_1+\gamma_2),\bmod n} & \text{if } \alpha_2\equiv 1,\ \text{mod } 2\\ z^{(\alpha_1+\alpha_2),\bmod 2}\,y^{(\beta_1+\beta_2)\bmod 2},\,x^{\nu} & \text{if } \alpha_2\equiv 1,\ \text{mod } 2\end{cases}$$

$$\textit{where}\ \nu=\begin{cases}(\gamma_1+\gamma_2),\ \text{mod } n \ \text{ if } \beta_2\equiv 0 \pmod 2,\\ (\gamma_1-(n-1)\gamma_2),\ \text{mod } n\ \text{ if } \beta_1\equiv 1 \pmod 2\end{cases}$$

then $L(0)$ is a Bol loop of order $4n$ ($n>2$ and even) with identity e.

Theorem 4. *Let $G(o)=C_{2n}\times C_2$ and the binary operation is defined as follows:*

(i) $$(x^\alpha,e)\circ(x^\beta,e)=(x^{\alpha+\beta},e)$$

(ii) $$(x^\alpha,e)\circ(x^\beta,a)=(x^{-\alpha+\beta},a)\ \textit{if } \beta \textit{ is odd}$$
$$=(x^{\alpha+\beta},a)\ \textit{if } \beta \textit{ is even}$$

(iii) $$(x^\alpha,a)\circ(x^\beta,e)=(x^{\alpha+\beta},a)$$

(iv) $$(x^\alpha,a)\circ(x^\beta,a)=(x^{-\alpha+\beta},e)\ \textit{if } \beta \textit{ is odd}$$
$$=(x^{\alpha+\beta},e);\ \textit{if } \beta \textit{ is even}$$

then $G(o)$ is a right Bol loop of order $4n(n\geqslant 2)$.

TABLE III

(Bol loop of Order 12)

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	6	4	5	8	9	7	11	12	10
3	1	2	5	6	4	9	7	8	12	10	11
4	6	5	2	1	3	10	12	11	7	9	8
5	4	6	1	3	2	11	10	12	8	7	9
6	5	4	3	2	1	12	11	10	9	8	7
7	8	9	10	11	12	1	2	3	4	5	6
8	9	7	12	10	11	2	3	1	5	6	4
9	7	8	11	12	10	3	1	2	6	4	5
10	12	11	8	7	9	4	6	5	1	3	2
11	10	12	7	9	8	5	4	6	2	1	3
12	11	10	9	8	7	6	5	4	3	2	1

[1] The paper „On the construction of Bol-Loops" has appeared „An. şt. Univ. Iaşi", XXVII, (1981) I a p. 13—17

This theorem gives a construction of Bol loops of order $4n$ for all n greater than or equal to two. The table below is a Boll loop of order 12 arising from theorem 4.

The table III in [1] is not correct.

Thus we have shown the existence of Bol loops of order $4n$ $(n\geqslant 2)$.

Department of Mathematics
University of Ife
Ile–Ife, Nigeria

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXX—2, s. I a Matematică

ON THE (p, q)-ORDER AND LOWER (p, q)-ORDER OF ENTIRE DIRICHLET SERIES

BY

S. K. VAISH, G. S. SRIVASTAVA and NARESH KUMAR

1. Let $f(s)$ be an entire function represented by the Dirichlet series

$$f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n),\ s=\sigma+it, \tag{1.1}$$

where λ_n's satisfy the following conditions:

$$0\leqslant\lambda_1<\lambda_2<\ldots<\lambda_n<\ldots\ ;\ \lambda_n\to\infty \text{ as } n\to\infty \text{ and} \tag{1.2}$$

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=D<\infty. \tag{1.3}$$

The (p, q)-order $\rho(p, q)$ and lower (p, q)-order $\lambda(p, q)$ for $f(s)$ having an index pair (p, q) are defined by [1]:

$$\lim_{\delta\to\infty}\frac{\sup.\log^{[p]}M(\sigma)}{\inf\log^{[q]}\sigma}=\frac{\rho(p, q)\equiv\rho}{\lambda(p, q)\equiv\lambda}, \tag{1.4}$$

where $M(\sigma)=\underset{-\infty<t<\infty}{\text{l.u.b.}}|f(\sigma+it)|$, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $0<\log^{[m-1]}x<\infty$, $\log^{[0]}x=x$, $m=1, 2, \ldots$.

It is known that [2, p, 59].

$$\rho=P(L(p, q)) \text{ if } \delta(p, q)=0 \text{ and } L(p, q)<\infty, \tag{1.5}$$

where

$$\delta\equiv\delta(p, q)=\limsup_{n\to\infty}\frac{\log n}{\lambda_n\exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/L}}, \tag{1.6}$$

$$L\equiv L(p, q)=\limsup_{n\to\infty}\frac{\log^{[p-1]}\lambda_n}{\log^{[q]}(\lambda_n^{-1}\log|a_n|^{-1})}, \tag{1.7}$$

and

$$P(L)\equiv P(L, p, q)=\begin{cases} L & \text{if } q+1<p<\infty\\ 1+L & \text{if } p=q+1=2\\ \max(1, L) & \text{if } 3\leqslant q+1=p<\infty\\ \infty & \text{if } p=q=\infty\end{cases} \tag{1.8}$$

Further, if $x_n=(\log|a_n/a_{n+1}|)/(\lambda_{n+1}-\lambda_n)$ forms a non-decreasing function of n $(>n_0)$, then for $(p, q)\neq(2, 1)$

$$\lambda=P(\theta), \tag{1.9}$$

where

$$\theta\equiv\theta(p, q)=\liminf_{n\to\infty}\frac{\log^{[p-1]}\lambda_{n-1}}{\log^{[q]}(\lambda_n^{-1}\log|a_n|^{-1})} \tag{1.10}$$

and $P(\theta)$ is defined similar to $P(L)$. (1.9) holds for $(p, q)=(2, 1)$ also, if further $\log\lambda_n\sim\log\lambda_{n+1}$ as $n\to\infty$.