This theorem gives a construction of Bol loops of order $4n$ for all n greater than or equal to two. The table below is a Boll loop of order 12 arising from theorem 4.

The table III in [1] is not correct.

Thus we have shown the existence of Bol loops of order $4n$ $(n \geqslant 2)$.

Department of Mathematics
University of Ife
Ile–Ife, Nigeria

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ON THE (p, q)-ORDER AND LOWER (p, q)-ORDER OF ENTIRE DIRICHLET SERIES

BY

S. K. VAISH, G. S. SRIVASTAVA and NARESH KUMAR

1. Let $f(s)$ be an entire function represented by the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n),\ s = \sigma + it, \tag{1.1}$$

where λ_n's satisfy the following conditions:

$$0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots\ ;\ \lambda_n \to \infty \text{ as } n \to \infty \text{ and} \tag{1.2}$$

$$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = D < \infty. \tag{1.3}$$

The (p, q)-order $\rho(p, q)$ and lower (p, q)-order $\lambda(p, q)$ for $f(s)$ having an index pair (p, q) are defined by [1]:

$$\lim_{\delta \to \infty} \frac{\sup . \log^{[p]} M(\sigma)}{\inf \log^{[q]} \sigma} = \frac{\rho(p, q) \equiv \rho}{\lambda(p, q) \equiv \lambda}, \tag{1.4}$$

where $M(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} |f(\sigma + it)|$, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $0 < \log^{[m-1]} x < \infty$, $\log^{[0]} x = x$, $m = 1, 2, \ldots$.

It is known that [2, p. 59].

$$\rho = P(L(p, q)) \text{ if } \delta(p, q) = 0 \text{ and } L(p, q) < \infty, \tag{1.5}$$

where

$$\delta \equiv \delta(p, q) = \limsup_{n \to \infty} \frac{\log n}{\lambda_n \exp^{[q-1]} (\log^{[p-2]} \lambda_n)^{1/L}}, \tag{1.6}$$

$$L \equiv L(p. q) = \limsup_{n \to \infty} \frac{\log^{[p-1]} \lambda_n}{\log^{[q]} (\lambda_n^{-1} \log |a_n|^{-1})}, \tag{1.7}$$

and

$$P(L) \equiv P(L, p, q) = \begin{cases} L & \text{if } q + 1 < p < \infty \\ 1 + L & \text{if } p = q + 1 = 2 \\ \max(1, L) & \text{if } 3 \leqslant q + 1 = p < \infty \\ \infty & \text{if } p = q = \infty \end{cases} \tag{1.8}$$

Further, if $x_n = (\log |a_n / a_{n+1}|) / (\lambda_{n+1} - \lambda_n)$ forms a non-decreasing function of n $(> n_0)$, then for $(p, q) \neq (2, 1)$

$$\lambda = P(\theta), \tag{1.9}$$

where

$$\theta \equiv \theta(p, q) = \liminf_{n \to \infty} \frac{\log^{[p-1]} \lambda_{n-1}}{\log^{[q]} (\lambda_n^{-1} \log |a_n|^{-1})} \tag{1.10}$$

and $P(\theta)$ is defined similar to $P(L)$. (1.9) holds for $(p, q) = (2, 1)$ also, if further $\log \lambda_n \sim \log \lambda_{n+1}$ as $n \to \infty$.

For the two entire functions $f_i(s)=\sum_{n=1}^{\infty} a_{i,n}\exp(s\lambda_{i,n})$, $(i=1, 2)$, we may similarly define $P(L_i)$ and $\delta_i(p, q)$ etc.

We say that $f(s)$ is of regular (p, q)-growth if $\rho(p, q)=\lambda(p, q)$.

In the present paper we have investigated various relations between (p, q)-order and lower (p, q)-order of two entire Dirichlet series. The results can easily be extended to the case of any finite numbers. Theorems 2 and 3 are the refinement and the generalization of the results due to Jain [1] and Kamthan [3], respectively.

Notations used.

$H=(\log^{[q]}|a_n|^{-1/\lambda_n})^{m}$, $0<m<\infty$,

$F=(\log^{[q]}|a_{1,n}|^{-1/\lambda_{1,n}})^{m_1}$, $0<m_1<\infty$, $\quad X=(\log^{[q]}|a_{2,n}|^{-1/\lambda_{2,n}})^{m_2}$, $0<m_2<\infty$,

and $x_{i,n}=\log|a_{i,n}/a_{i,n+1}|/[\lambda_{i,n+1}-\lambda_{i,n}]$

2. We now prove

Theorem 1. *Let* $f_i(s)=\sum_{n=1}^{\infty} a_{i,n}\exp(s\lambda_{i,n})$, $(i=1,2)$, *be two entire Dirichlet series of finite* (p, q)*-orders* ρ_i *and lower* (p, q)*-orders* λ_i. *Suppose that there exist sequences* $\{\lambda_n\}$ *and* $\{a_n\}$ *such that*

(i) $\log^{[p-1]}\lambda_n\sim\log^{[p-1]}\lambda_{1,n}\sim\log^{[p-1]}\lambda_{2,n}$, (ii) $\delta=\delta_1=\delta_2=0$,

(iii) $\log H\sim \log(FX)^{1/2}$ *and* (iv) x_n *and* $x_{i,n}$ *are non-decreasing functions of* n *for all sufficiently large* n. *Then* $f(s)=\sum_{n=1}^{\infty} a_n\exp(s\lambda_n)$ *is an entire Dirichlet series of* (p, q)*-order* $\rho(=P(L))$ *and lower* (p, q)*-order* $\lambda(=P(\theta))$, *such that*

$$\frac{2m}{\theta}\geq\frac{m_1}{\theta_1}+\frac{m_2}{L_2}\geq\frac{2m}{L}\ ;\ \frac{2m}{\theta}\geq\frac{m_1}{L_1}+\frac{m_2}{\theta_2}\geq\frac{2m}{L}, \tag{2.1}$$

provided $(p, q)\neq(2, 1)$.

Further (2.1) *holds for the index pair* $(p, q)=(2, 1)$ *also, if*

(v) $\log\lambda_n\sim\log\lambda_{n+1}$ *and* $\log\lambda_{i,n}\sim\log\lambda_{i,n+1}$.

Proof. Since

$$\liminf_{n\to\infty}\frac{\log H}{\log^{[p-1]}\lambda_n}=\liminf_{n\to\infty}\frac{\log(FX)^{1/2}}{\log^{[p-1]}\lambda_n}\geq\liminf_{n\to\infty}\frac{\log F}{2\log^{[p-1]}\lambda_{1,n}}+\liminf_{n\to\infty}\frac{\log X}{2\log^{[p-1]}\lambda_{2,n}}\ .$$

therefore,

$$\frac{2m}{L}\geq\frac{m_1}{L_1}+\frac{m_2}{L_2}\ . \tag{2.2}$$

Thus L is finite because m_1, m_2 are positive and L_1, L_2 are finite. Hence $f(s)$ is an entire Dirichlet series. Now for $(p, q)\neq(2, 1)$

$$\varlimsup_{n\to\infty}\frac{\log H}{\log^{[p-1]}\lambda_{n-1}}\geq\varlimsup_{n\to\infty}\frac{\log F}{2\log^{[p-1]}\lambda_{1,n}}+\varlimsup_{n\to\infty}\frac{\log X}{2\log^{[p-1]}\lambda_{2,n-1}}\geq$$

$$\geq\varlimsup_{n\to\infty}\frac{\log H}{\log^{[p-1]}\lambda_n}\ .$$

Hence,

$$\frac{2m}{\theta}\geq\frac{m_1}{L_1}+\frac{m_2}{\theta_2}\geq\frac{2m}{L}\ . \tag{2.3}$$

Similarly, we can prove that

$$\frac{2m}{\theta}\geq\frac{m_1}{\theta_1}+\frac{m_2}{L_2}\geq\frac{2m}{L}\ ,\ \text{for } (p, q)\neq(2, 1). \tag{2.4}$$

(2.3) and (2.4) give (2.1). On using condition (v), we can also prove that (2.1) holds for $(p, q)=(2, 1)$. Hence the theorem. The inequalities (2.1) lead to

Theorem 2. *If all the conditions are the same as in Theorem 1 and either of the two functions* $f_1(s)$ *or* $f_2(s)$ *is of regular* (p, q)*-growth, then*

$$\frac{2m}{L}=\frac{m_1}{L_1}+\frac{m_2}{L_2}\ . \tag{2.5}$$

Remark 1. This theorem generalizes a result of Jain [1] for $m_1=m_2=m=1$, $p=2$, $q=1$, which he obtained under the more restrictive condition that both $f_1(s)$ and $f_2(s)$ are of regular (2, 1)-growth.

Theorem 3. *If the conditions* (i) *to* (iv) *are the same as in Theorem 1, except* (iii), *which may be read as*

(iii) $\qquad\log H\sim\{(\log F)(\log X)\}^{1/2}$,

then $f(s)$ *is an entire Dirichlet series such that*

$$\left\{\frac{m_1 m_2}{\theta_1\,\theta_2}\right\}^{1/2}\geq\frac{m}{\theta}>\begin{matrix}\left\{\frac{m_1 m_2}{L_1\,\theta_2}\right\}^{1/2},\\[2ex] \left\{\frac{m_1 m_2}{\theta_1\,L_2}\right\}^{1/2}\end{matrix}>\frac{m}{L}\geq\left\{\frac{m_1 m_2}{L_1\,L_2}\right\}^{1/2} \tag{2.6}$$

provided $(p, q)\neq(2, 1)$.

Further (2.6) holds for $(p, q)=(2, 1)$ also, if the condition (v) of Theorem 1 satisfied.

Proof. Since $f_i(s)$ are entire functions, therefore, for a fixed large number n_0, $\log^{[q]}|a_{i,n}|^{-1/\lambda_{i,n}}>K$, $n>n_0$, where K is positive and arbitrarily large. Using (iii)′, we get $\log H>(m_1m_2)^{1/2}\log K$, or, $\liminf_{n\to\infty}\log^{[q]}|a_n|^{-1/\lambda_n}=\infty$.

Hence, $f(s)$ is an entire Dirichlet series.

Now, we have, for any $G>0$, $(p, q)\neq(2, 1)$ and sufficiently large n,

$$\left\{\begin{aligned}&\frac{\log H}{\log^{[p-1]}\lambda_n}>m\left\{\frac{1}{L}-G\right\},\\ &\frac{\log F}{\log^{[p-1]}\lambda_{1,n}}>m_1\left\{\frac{1}{L_1}-G\right\},\ \text{and}\\ &\frac{\log X}{\log^{[p-1]}\lambda_{2,n}}>m_2\left\{\frac{1}{L_2}-G\right\},\end{aligned}\right. \tag{2.7}$$

(2.7), (i) and (iii)′ give,

$$\frac{\log H}{\log^{[p-1]}\lambda_n}\geq\left\{m_1\,m_2\left(\frac{1}{L_1}-G\right)\left(\frac{1}{L_2}-G\right)\right\}^{1/2}.$$

Thus

$$\frac{m}{L} \geqslant \left\{\frac{m_1 m_2}{L_1 L_2}\right\}^{1/2}. \tag{2.8}$$

Again, from (1.10), we have, for any $G>0$ and $n>n_0$,

$$\left\{\begin{aligned} &\frac{\log H}{\log^{[p-1]}\lambda_{n-1}} < m\left\{\frac{1}{0}+G\right\}, \\ &\frac{\log F}{\log^{[p-1]}\lambda_{1,n-1}} < m_1\left\{\frac{1}{0_1}+G\right\}, \text{ and} \\ &\frac{\log X}{\log^{[p-1]}\lambda_{2,n-1}} < m_2\left\{\frac{1}{0_2}+G\right\}, \end{aligned}\right. \tag{2.9}$$

provided $(p,q)\neq(2,1)$. Now, proceeding as above, we obtain

$$\frac{m}{0} \leqslant \left\{\frac{m_1 m_2}{\theta_1 \ \theta_2}\right\}^{1/2}. \tag{2.10}$$

Also, we have

$$\limsup_{n\to\infty} \frac{\log H}{\log^{[p-1]}\lambda_{n-1}} \geqslant \left\{\liminf_{n\to\infty} \frac{\log F}{\log^{[p-1]}\lambda_{1,n}}\right\}^{1/2}\left\{\limsup_{n\to\infty} \frac{\log X}{\log^{[p-1]}\lambda_{2,n-1}}\right\}^{1/2} \geqslant$$

$$\geqslant \liminf_{n\to\infty} \frac{\log H}{\log^{[p-1]}\lambda_n},$$

or,

$$\frac{m}{0} \geqslant \left\{\frac{m_1 m_2}{L_1 \ \theta_2}\right\}^{1/2} \geqslant \frac{m}{L}. \tag{2.11}$$

Similarly, we can prove that

$$\frac{m}{0} \geqslant \left\{\frac{m_1 m_2}{\theta_1 \ L_2}\right\}^{1/2} \geqslant \frac{m}{L}. \tag{2.12}$$

Combining the inequalities (2.8), (2.10), (2.11) and (2.12), we get the desired result. We can also easily prove that (2.6) holds for $(p,q)=(2,1)$ under the condition (v) of Theorem 1.

Remark 2. (2.6) is the refinement and the generalization of the result $\rho(1)\leqslant\sqrt{\rho_1(1)\rho_2(1)}$ due to K a m t h a n [3, p. 81] for $m_1=m_2=m=1$, $p=2$ and $q=0$.

Theorem 4. *If all the conditions are same as in Theorem 3, then* $m/L=[m_1, m_2/L_1L_2]^{1/2}$; *provided either of the two functions* $f_1(s)$ *or* $f_2(s)$ *is of regular* (p,q)*-growth.*

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Department of Mathematics
University of Roorkee
Roorkee—247667 (U. P.),
India

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ON BEST VECTORIAL APPROXIMATION IN LINEAR SPACES

BY

VASILE POSTOLICĂ

The aim of this paper is to establish some conditions which characterize the elements of best vectorial approximation in a linear space with respect to a family of real functions defined on the space or are useful for the indication of these elements and to give new sufficient conditions for the existence of minimal elements for convex vectorial programs.

Let X be a real or a complex linear space, $\Phi=\{f_{\alpha\in A}\}$ a family of real functions defined on X, G a non-void subset of X, $x_0\in X$ and the function $\rho_\Phi: X\to R^A$ defined by

$$[\rho_\Phi(x)](\alpha)=f_\alpha(x), \quad x\in X, \quad \alpha\in A.$$

Let Y and D denote, respectively, a non-void set and a convex cone in R^A. The following definition is due to Yu [5]:

Definition 1. *A point* $y_0\in Y$ *is a* D*-extreme point of* Y *if there exists no* $y_1\in Y$, $y_1\neq y_0$ *such that* $y_0\in y_1+D$.

We shall denote the set of all D-extreme points of Y by Ext $[Y/D]$ and we recall that if $f, g\in R^A$; then

(i) $f\leqslant g$ means $f(\alpha)\leqslant g(\alpha)$ for every $\alpha\in A$;
(ii) $f<g$ means $f\leqslant g$ and there exists $\alpha\in A$ such that $f(\alpha)<g(\alpha)$;
(iii) $R^A_+=\{f\in R^A : f\geqslant 0\}$.

Also, throughout the paper we make the usual calculation convention: $\inf\varnothing=+\infty$, where $\varnothing$ denote the empty set.

If for every $g\in G$ we consider the set $S(g)=\{g'\in G:\ \rho_\Phi(x_0-g')=\rho_\Phi(x_0-g)\}$ then we make the following definition:

Definition 2. *An element* $g_0\in G$ *is said to be a best vectorial approximation of* x_0 *by the elements of* G *with respect to abbreviated* g_0 *is a* Φ*-b.v.a. of* x_0 *by* G) *if it satisfies one of the following equivalent conditions:*

(a) *There exists no element* $g\in G$ *such that*

$$\rho_\Phi(x_0-g)<\rho_\Phi(x_0-g_0);$$

(b) *For every* $g\in G\setminus S(g_0)$ *there exists* $f_\alpha\in\Phi$ *such that*

$$f_\alpha(x_0-g)>f_\alpha(x_0-g_0);$$

(c) $$\bigcap_{\alpha\in A}\{g\in G\setminus S(g_0):\ f_\alpha(x_0-g)\leqslant f_\alpha(x_0-g_0)\}=\varnothing;$$

(d) $$\rho_\Phi(x_0-g_0)\in \text{Ext}\,[\{\rho_\Phi(x_0-G)/R^A_+]\,;$$