$$\frac{m}{L} \geqslant \left\{\frac{m_1 m_2}{L_1 L_2}\right\}^{1/2}. \tag{2.8}$$

Again, from (1.10), we have, for any $G>0$ and $n>n_0$,

$$\begin{cases} \dfrac{\log H}{\log^{[p-1]}\lambda_{n-1}} < m\left\{\dfrac{1}{\theta}+G\right\}, \\ \dfrac{\log F}{\log^{[p-1]}\lambda_{1,n-1}} < m_1\left\{\dfrac{1}{\theta_1}+G\right\}, \text{ and} \\ \dfrac{\log X}{\log^{[p-1]}\lambda_{2,n-1}} < m_2\left\{\dfrac{1}{\theta_2}+G\right\}, \end{cases} \tag{2.9}$$

provided $(p, q)\neq(2, 1)$. Now, proceeding as above, we obtain

$$\frac{m}{\theta} \leqslant \left\{\frac{m_1 m_2}{\theta_1 \ \theta_2}\right\}^{1/2}. \tag{2.10}$$

Also, we have

$$\limsup_{n\to\infty} \frac{\log H}{\log^{[p-1]}\lambda_{n-1}} \geqslant \left\{\liminf_{n\to\infty} \frac{\log F}{\log^{[p-1]}\lambda_{1,n}}\right\}^{1/2}\left\{\limsup_{n\to\infty} \frac{\log X}{\log^{[p-1]}\lambda_{2,n-1}}\right\}^{1/2} \geqslant$$

$$\geqslant \liminf_{n\to\infty} \frac{\log H}{\log^{[p-1]}\lambda_n},$$

or,

$$\frac{m}{\theta} \geqslant \left\{\frac{m_1 m_2}{L_1 \ \theta_2}\right\}^{1/2} \geqslant \frac{m}{L}. \tag{2.11}$$

Similarly, we can prove that

$$\frac{m}{\theta} \geqslant \left\{\frac{m_1 m_2}{\theta_1 \ L_2}\right\}^{1/2} \geqslant \frac{m}{L}. \tag{2.12}$$

Combining the inequalities (2.8), (2.10), (2.11) and (2.12), we get the desired result. We can also easily prove that (2.6) holds for $(p, q)=(2,1)$ under the condition (v) of Theorem 1.

Remark 2. (2.6) is the refinement and the generalization of the result $\rho(1) \leqslant \sqrt{\rho_1(1)\rho_2(1)}$ due to Kamthan [3, p. 81] for $m_1=m_2=m=1$, $p=2$ and $q=0$.

Theorem 4. *If all the conditions are same as in Theorem 3, then* $m/L = [m_1 \cdot m_2/L_1 L_2]^{1/2}$; *provided either of the two functions* $f_1(s)$ *or* $f_2(s)$ *is of regular* (p, q)-*growth.*

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ON BEST VECTORIAL APPROXIMATION IN LINEAR SPACES

BY

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The aim of this paper is to establish some conditions which characterize the elements of best vectorial approximation in a linear space with respect to a family of real functions defined on the space or are useful for the indication of these elements and to give new sufficient conditions for the existence of minimal elements for convex vectorial programs.

Let X be a real or a complex linear space, $\Phi=\{f_{\alpha\in A}\}$ a family of real functions defined on X. G a non-void subset of X, $x_0\in X$ and the function ρ_Φ : $X\to R^A$ defined by

$$[\rho_\Phi(x)](\alpha)=f_\alpha(x), \quad x\in X, \quad \alpha\in A.$$

Let Y and D denote, respectively, a non-void set and a convex cone in R^A. The following definition is due to Yu[5] :

Definition 1. *A point* $y_0\in Y$ *is a* D-*extreme point of* Y *if there exists no* $y_1\in Y$, $y_1\neq y_0$ *such that* $y_0\in y_1+D$.

We shall denote the set of all D-extreme points of Y by Ext $[Y/D]$ and we recall that if $f, g\in R^A$; then

(i) $f\leqslant g$ means $f(\alpha)\leqslant g(\alpha)$ for every $\alpha\in A$;
(ii) $f<g$ means $f\leqslant g$ and there exists $\alpha\in A$ such that $f(\alpha)<g(\alpha)$;
(iii) $R^A_+=\{f\in R^A : f\geqslant 0\}$.

Also, throughout the paper we make the usual calculation convention: inf $\varnothing=+\infty$, where $\varnothing$ denote the empty set.

If for every $g\in G$ we consider the set $S(g)=\{g'\in G : \rho_\Phi(x_0-g')=\rho_\Phi(x_0-g)\}$ then we make the following definition :

Definition 2. *An element* $g_0\in G$ *is said to be a best vectorial approximation of* x_0 *by the elements of* G *with respect to abbreviated* g_0 *is a* Φ-*b.v.a. of* x_0 *by* G) *if it satisfies one of the following equivalent conditions :*

(a) *There exists no element* $g\in G$ *such that*

$$\rho_\Phi(x_0-g)<\rho_\Phi(x_0-g_0) ;$$

(b) *For every* $g\in G\setminus S(g_0)$ *there exists* $f_\alpha\in\Phi$ *such that*

$$f_\alpha(x_0-g)>f_\alpha(x_0-g_0) ;$$

(c) $$\bigcap_{\alpha\in A} \{g\in G\setminus S(g_0) : f_\alpha(x_0-g)\leqslant f_\alpha(x_0-g_0)\}=\varnothing ;$$

(d) $$\rho_\Phi(x_0-g_0)\in \text{Ext}\,[\{\rho_\Phi(x_0-G)/R^A_+]\, ;$$

If $x_0=0$, then g_0 is called *minimal element for the vectorial program* (G, ρ_Φ) and we shall denote by $\mathcal{U}_G(\rho_\Phi)$ the set of all minimal elements for the vectorial program (G, ρ_Φ).

Definition 3. *An element $g_0 \in G$ is said to be a best simultaneous approximation of x_0 by the elements of G with respect to Φ abbreviated g_0 is a Φ-b.s.a. of x_0 by G) if*

$$f_\Phi(x_0-g_0) \leq \rho_\Phi(x_0-g)$$

for every $g \in G$.

Remark 1. The definitions remain valid if we replace R with an ordered linear space and it is easy to see that every Φ-b.s.a. of x by the elements of a non-void set $G \subseteq X$ is a Φ-b.v.a. of x_0 by G. We have studied other connections between the elements of best vectorial approximation and the elements of best simultaneous approximation in [4].

Let $M \subseteq R^A$ (R^A endowed with the usual product topology and with the usual partial ordering) be a non-void set and let $p \in R^A$. Following [2] we say that $p \in \text{INF } M$ if $p \in \overline{M}$ and there exists no $m \in M$ such that $m < p$.

Some immediately connections between $\rho_\Phi(G)$, $\text{INF } \rho_\Phi(G)$ and $\rho_\Phi(\mathcal{U}_G(\rho_\Phi))$ are collected in

Remark 2. It is easy to see that we have:

(i) $g_0 \in G$ is a minimal element for the vectorial program (G, ρ_Φ) if and only if

$$\rho_\Phi(g_0) \in \text{INF } \rho_\Phi(G);$$

(ii) $\rho_\Phi(\mathcal{U}_G(\rho_\Phi)) \subseteq \text{Fr } \rho_\Phi(G) \subseteq \overline{\rho_\Phi(G)} \subseteq \{x \in R^A : x \geq m\}$ where $m=(m_i) \in R^A$, with

$$m_i = \inf \{a_i : (a_i) \in \rho_\Phi(G)\}; \tag{1}$$

(iii) If $m \in \rho_\Phi(G)$, then $\rho_\Phi(\mathcal{U}_G(\rho_\Phi)) = \{m\}$;

Let us denote

$$D_{p_i} = \{(p_{i'}, p_i) \in R^A : p_{i'} \in R, \ \forall i' \in A / \{i\}\}, \ i \in A, \tag{2}$$

$$\gamma_i[(p_{i'})] = \inf \{a_i : (a_i, p_{i'}) \in \overline{\rho_\Phi(G)}\}, \ i \in A / \{i' \in A : i' \neq i\}, \tag{3}$$

$$\alpha_i[(p_{i'})] = \inf\{a_i : (a_i, a_{i'}) \in \rho_\Phi(G) \text{ and } (a_{i'}) \leq (p_{i'})\}, \ i \in A \setminus \{i' \in A : i' \neq i\} \tag{4}$$

$$\gamma_i = \inf \{a_i : (a_i, m_{i'}) \in \rho_\Phi(G), \ i' \in A, \ i' \neq i\}, \ i \in A. \tag{5}$$

Theorem 1. *For every $(p_i) \in \text{INF } \rho_\Phi(G)$ we have*

$$\gamma_i[(p_{i'})] \leq p_i \leq \alpha[(p_i)], \quad \forall i \in A \setminus \{i' \in A : i' \neq i\}. \tag{6}$$

Proof. Indeed, if $(p_i) \in \text{INF } \rho_\Phi(G)$, then $(p_i) \in \overline{\rho_\Phi(G)}$ and by (3) we obtain

$$\gamma_i[(p_{i'})] \leq p_i, \ \forall i \in A \setminus \{i' \in A : i' \neq i\}. \tag{7}$$

On the other hand, if there exists $(p_i) \in \text{INF } \rho_\Phi(G)$ and $i \in A \setminus \{i' \in A : i' \neq i\}$ such that $p_i > \alpha_i[(p_{i'})]$, then from (4) it follows that there exists $(a_i, a_{i'}) \in \rho_\Phi(G)$ with $(a_{i'}) \leq (p_{i'})$ and $a_i < p_i$ which implies $(a_i) < (p_i)$ in contradiction with $(p_i) \in \text{INF } \rho_\Phi(G)$. Taking into account (7) we obtain (6) and

the theorem is proved. The following Remarks and the Theorem 2 show some connections between $\text{INF } \rho_\Phi(G)$ and the definitions (1)—(5), with the mention that we have denoted the component i of $\rho_\Phi(x)$ $(x \in X)$ with

Remarks. 1) *For every $i \in A$ the set $\rho_\Phi(G) \cap [\bigcap_{\substack{i' \in A \\ i' \neq i}} D_{p_{i'}} \cap \text{INF } \rho_\Phi(G)]$ is either empty or the singleton $(\alpha_i[(p_{i'})], p_{i'})$.*

Indeed, if the announced set non-void and $\rho_\Phi(g_0) \in \rho_\Phi(G) [\bigcap_{\substack{i' \in A \\ i' \neq i}} D_{p'_{i'}} \cap \text{INF } \rho_\Phi(G)]$, then by Theorem 1 we have

$$\gamma_i[(\rho_{i'}(g_0))] \leq \rho_i(g_0) \leq \alpha_i[\rho_{i'}(g_0))], \quad \forall_i \in A \setminus \{i' \in A : i' \neq i\} \tag{8}$$

which combined with $\alpha_i[(\rho_{i'}(g_0))] \leq \rho_i(g_0)$ implies $\rho_i(g_0) = \alpha_i[(\rho_{i'}(g_0))]$, for every $i \in A \setminus \{i' \in A : i' \neq i\}$.

2) *If $g_0 \in G$ is a minimal element for the vectorial program (G, ρ_Φ), then there exists $i \in A$ and $p_i \in R$ such that*

$$g_0 \in \{g \in G : \forall g' \in G \text{ with } \rho_i(g') \leq p_i \Rightarrow (\rho_{i'}(g')) \nless (\rho_{i'}(g)), \ i' \in A \setminus \{i\}\}. \tag{9}$$

Indeed, if g_0 is a minimal element for the vectorial program (G, ρ_Φ) then it is sufficient to observe that (9) follows taking $p_i = \rho_i(g_0)$ for every $i \in A$

3) *If there exists $i \in A$ and $p_i \in R$ such that*

$$g_0 \in \{g \in G : \rho_i(g) \leq p_i \text{ and there exists no } \bar{g} \in G \text{ with } i' \in A \setminus \{i\} \ (\rho_{i'}(g)) < (\rho_{i'}(g)) \tag{10}$$

and

$$\rho_i / \{g \in G : \rho_i(g) \leq p_i\} = \text{constant}, \tag{11}$$

then g_0 is a minimal element for the vectorial program (G, ρ_Φ).

If g_0 satisfies the announced conditions and there exists $g' \in G$ such that $\rho_\Phi(g') < \rho_\Phi(g_0)$, then $\rho_{i'}(g') = \rho_{i'}(g_0)$ for every $i' \in A \setminus \{i\}$ and $\rho_i(g') < \rho_i(g_0) \leq \leq p_i$ which contradicts (11). The result follows.

4) *If there exists $i \in A$ $(a_{i'}) \in R^{A \setminus \{i\}}$ such that $(a_{i'}) < (p_{i'})$ and*

$$(\alpha_i[(p_{i'})], a_{i'}) \in \rho_\Phi(G), \textit{ then } \rho_\Phi(G) \cap [\bigcap_{\substack{i' \in A \\ i' \neq i}} D_{p_{i'}} \cap \text{INF } \rho_\Phi(G)] = \emptyset.$$

Indeed, if $(p_i, p_{i'}) \in \rho(G)_\Phi \cap [\bigcap_{\substack{i' \in A \\ i' \neq i}} D_{p_i} \text{ INF } \rho_\Phi(G)]$, then by Theorem 1 and by $(p_i, p_{i'}) \in \rho_\Phi(G)$ we obtain $p_i < \alpha_i[(p_i)]$ which contradicts 1).

5) *If for every $g \in G$ with $(\rho_{i'}(g)] < (p_{i'})$ we have $(\alpha_i[(p_i)], \rho_i(g)) \notin \rho_\Phi(G)$ then*

$$\bigcap_{\substack{i' \in A \\ i' \neq i}} (D p_{i'} \cap \text{INF } \rho_\Phi(G)) = \overline{\rho_\Phi(G)} \cap \{(p_i, p_{i'}) : \gamma_{i'}[(p_i)] \leq p_i \leq \alpha_i[(p_{i'})]\} = K; \ i' \in A \setminus \{i\}. \tag{12}$$

Indeed, if $(p_i, p_{i'}) \in \bigcap_{\substack{i' \in A \\ i' \neq i}} (D_{p_i} \cap \text{INF } \rho_\Phi(G))$, then $(p_i, p_{i'}) \in \overline{\rho_\Phi(G)}$ and by virtue of

Theorem 1 it follows $\gamma_i[(p_{i'})] \leqslant p_i \leqslant \alpha_i[(p_{i'})]$ $i' \in A \setminus \{i\}$, hence $(p_i, p_{i'}) \in K$. Thus we have proved the inclusion „$\subseteq$". Assume now that $(p_i, p_{i'}) \in K$. If there exists $g \in G$ such that $\rho_\Phi(G) < (p_i, p_{i'})$, then $\rho_i(g) \geqslant \alpha_i[(p_{i'})] \geqslant p_i \geqslant \rho_i(g)$ $i' \in A \setminus \{i\}$, hence $\rho_i(g) = \alpha_i[(p_{i'})] = p_i$, $(\alpha_i[(p_{i'})], \rho_i(g)) \in \rho_\Phi(G)$ and $(\rho_i, (g)) < (p_{i'})$ in contradiction with our hypotheses. Therefore we have also the inclusion „$\supseteq$" in (12) and thus (12) is proved.

6) $$\gamma_i \geqslant m_i, \ \forall i \in A ;$$

7) $$\rho_\Phi(U_G(\rho_\Phi)) \subseteq \{(\rho_i(g)) : g \in G \text{ and } m_i \leqslant \rho_i(g) \leqslant \gamma_i, \ \forall i \in A\} ;$$

Indeed, if $\rho_\Phi(g_0) \in \rho_\Phi(\mathcal{U}_G(\rho_\Phi))$, then $m_i \leqslant \rho_i(g)$ for every $i \in A$ and if there exists $i \in A$ such that $\gamma_i = +\infty$, then it is clear that $m_i \leqslant \rho_i(g) \leqslant \gamma_i$.

Assume now that $i \in A$, $\gamma_i < +\infty$ and $\rho_i(g_0) > \gamma_i$; then, by (5) it follows that there exists $a_i \in \rho_i(G)$ with $(a_i, m_{i'}) \in \rho_\Phi(G)$ and $\gamma_i \leqslant a_i < \rho_i(g_0)$. Hence $(a_i, m_{i'}) \in \rho_\Phi(G)$ and $(a_i, m_{i'}) < \rho_\Phi(g_0)$ which contradicts the vectorial minimality of g_0. The result follows and it is easy to see that, in general, the inclusion in 7) is strict.

8) $$(\gamma_i, m_{i'}) \in \rho_\Phi(\mathcal{U}_G(\rho_\Phi)) \text{ if and only if } (\gamma_i, m_{i'}) \in \rho_\Phi(G) ;$$

Theorem 2. *We have*

(13) $$\text{INF } \rho_\Phi(G) = \{(p_i) \in \overline{\rho_\Phi(G)} : \gamma_i[(p_{i'})] \leqslant p_i \leqslant \alpha_i[(p_{i'})], \ \forall i \in A \setminus \{i' \in A : i' \neq i\}\}.$$

Proof. By virtue of Theorem 1 it follows that for every $(p_i) \in$ INF $\rho_\Phi(G)$ we have $(p_i) \in \overline{\rho_\Phi(G)}$ and $\gamma_i[(p_{i'})] \leqslant p_i \leqslant \alpha_i[(p_{i'})]$, $\forall i \in A \setminus \{i' \in A : i' \neq i\}$.

Let now $(p_i) \in \overline{\rho_\Phi(G)}$ with $\gamma_i[(p_{i'})] \leqslant p_i \leqslant \alpha_i[(p_{i'})]$ for every $i \in A \setminus \{i' \in A ; i' \neq i\}$ and let us assume that there exists $g \in G$ such that $\rho_\Phi(g) < (p_i)$. Hence $\alpha_i[(p_{i'})] \leqslant \rho_i(g) \leqslant p_i \leqslant \alpha_i[(p_{i'})]$ for every $i \in A \setminus \{i' \in A : i' \neq i\}$ which implies $\rho_i(g) = p_i$, $i \in A$ in contradiction with our assumption and the theorem is proved.

Now we establish two new sufficient conditions for the existence of minimal elements for convex vectorial programs. For this, in accordance with [3], we shall denote by $\mathcal{S}_b(f) = \{b_0 \in B : f(b_0) \leqslant f(b), \ \forall b \in B\}$ for every non-empty set $B \subseteq X$ and every function $f : X \to Y$ with $(Y, \leqslant)$ an arbitrary partially ordered linear space and we assume that X is a real Hausdorff locally convex space.

If X^* is its conjugate (i.e. the linear topological space of all continuous linear functionals on X), G is a non-void convex subset of X, $\Phi = \{f_\alpha : \alpha \in A\}$ is a family of finite convex functionals defined on X, $g_0 \in G$ and for each $\alpha \in A$ we consider the function $\rho_\alpha : X \to R^{A \setminus \{\alpha\}}$ defined by $[\rho_\alpha(x)](\beta) = f_\beta(x)$, $x \in X$, $\beta \in A \setminus \{\alpha\}$, then

Theorem 3. (i) *If for every $\alpha \in A$ for which there exists $g \in G$ with $f_\alpha(g) < f_\alpha(g_0)$ we have $\mathcal{S}_G(f_\alpha)$, $\mathcal{S}_G(\rho_\alpha) \neq \emptyset$ and there exists $\varphi \in X^*$ such that*

(14) $$\sup \varphi(\mathcal{S}_G(f_\alpha)) \leqslant \varphi(g_0) \leqslant \inf \varphi(\mathcal{S}_G(\rho_\alpha))$$

and

(15) $$g_0 \in \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha) \cap \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(\rho_\alpha)$$

then g_0 is a minimal for the convex vectorial program (G, ρ_Φ);

(ii) *If $\mathcal{S}_G(\rho_\Phi) \neq \emptyset$, then $\mathcal{S}_G(\rho_\Phi) = \mathcal{U}_G(\rho_\Phi)$ and for this point the hypothesis of convexity of the functionals $f_\alpha \in \Phi$, $\alpha \in A$ is not necessary;*

(iii) *If $\mathcal{S}_G(\rho_\Phi) = \emptyset$ and if g_0 is a minimal element for the convex vectorial program (G, ρ_Φ), then there exists $\alpha \in A$ such that $\{g \in G : f_\alpha(g) < f_\alpha(g_0)\} \neq \emptyset$ and if $\{g \in G : \rho_\alpha(g) < \rho_\alpha(g_0)\} = \emptyset$ implies $g_0 \in \mathcal{S}_G(\rho_\alpha)$, then there exists $\varphi \in X^*$ such that (14), (15) hold.*

Proof. (i) Suppose $g_0 \notin \mathcal{U}_G(\rho_\Phi)$. Then, there exists $\bar{g} \in G$ such that $\rho_\Phi(\bar{g}) < \rho_\Phi(g_0)$, hence there exists $\alpha \in A$ with

(16) $$f_\alpha \bar{g}(\bar{g}) < f_\alpha(g_0).$$

Then, by (15) it follows $\varphi(\bar{g}) \neq \varphi(g_0)$. Suppose $\varphi(\bar{g}) < \varphi(g_0)$; in this case we have $g_0 \notin \mathcal{S}_G(\rho_\alpha)$ since, otherwise, it follows $\rho_\alpha(g_0) \leqslant \rho_\alpha(g)$, $\forall g \in G$, hence $\rho_\alpha(g_0) \leqslant \rho_\alpha(\bar{g}) \leqslant \rho_\alpha(g_0)$, which implies $\rho_\alpha(g_0) = \rho_\alpha(\bar{g})$. Therefore $\bar{g} \in \mathcal{S}_G(\rho_\alpha)$ and by (14) we obtain $\varphi(g_0) \leqslant \varphi(\bar{g})$ in contradiction with our assumption.

Let $g_\alpha \in \mathcal{S}_G(\rho_\alpha)$. By (14), $\varphi(g_0) \leqslant \varphi(g_\alpha)$ and if $\varphi(g_0) = \varphi(g_\alpha)$, then by (15) it follows $\rho_\alpha(g_0) \leqslant \rho_\alpha(g_\alpha)$, hence $g_0 \in \mathcal{S}_G(\rho_\alpha)$ which is not possible.

Therefore $\varphi(\bar{g}) < \varphi(g_0) < \varphi(g_\alpha)$. Let $\lambda \in (0,1)$ such that $\varphi(\lambda\bar{g} + (1-\lambda)g_\alpha) = \varphi(g_0)$. Since G is convex it follows that $\lambda\bar{g} + (1-\lambda)g_\alpha \in \{g \in G : \varphi(g) = \varphi(g_0)\}$, hence we have $\rho_\alpha(\lambda g + (1-\lambda)\rho_\alpha) \leqslant \lambda\rho_\alpha(\bar{g}) + (1-\lambda)\rho_\alpha(g_\alpha) < \rho_\alpha(g_0)$ (since $\rho_\alpha(\bar{g}) \leqslant \rho_\alpha(g_0)$ and $\rho_\alpha(g_\alpha) < \rho_\alpha(g_0)$) which contradicts (15). Therefore $g_0 \in \mathcal{U}_G(\rho_\Phi)$.

In a similar manner, taking into account the first part of (14) we obtain $g_0 \in \mathcal{U}_G(\rho_\Phi)$ if $\varphi(\bar{g}) > (g_0)$.

(ii) By virtue of Remark 1 we have $\mathcal{S}_G(\rho_\Phi) \subseteq \mathcal{U}_G(\rho_\Phi)$. Let now $g_0 \in \mathcal{U}_G(\rho_\Phi)$ and assume that $g_0 \notin \mathcal{S}_G(\rho_\Phi)$. Then, there exists $f_\alpha \in \Phi$ and $g^* \in G$ such that

(17) $$f_\alpha(g^*) < f_\alpha(g_0).$$

Let $\bar{g} \in \mathcal{S}_G(\rho_\Phi)$. We have $f_\alpha(\bar{g}) < f_\alpha(g_0)$ since, otherwise, $f_\alpha(\bar{g}) = f_\alpha(g_0)$ implies $f_\alpha(g_0) \leqslant f_\alpha(g)$, $\forall g \in G$, hence $f_\alpha(g_0) \leqslant f_\alpha(g^*)$ which contradicts (17).

Therefore $f_\alpha(\bar{g}) < f_\alpha(g_0)$ and $f_\beta(\bar{g}) \leqslant f_\beta(g_0)$, $\forall \beta \in A \setminus \{\alpha\}$ which implies $\rho_\Phi(\bar{g}) < \rho_\Phi(g_0)$, in contradiction with $g_0 \in \mathcal{U}_G(\rho_\Phi)$ and (ii) is proved.

(iii) Let $\mathcal{S}_G(\rho_\Phi) = \emptyset$. Then, there exists $f_\alpha \in \Phi$ such that $\{g \in G : f_\alpha(g) < f_\alpha(g_0)\}$ is non-empty.

Case 1. $\{g \in G : \rho_\alpha(g) < \rho_\alpha(g_0)\} \neq \emptyset$; we have Int $\{g \in G : f_\alpha(g) \leqslant f_\alpha(g_0)\} \cap$ Int $\{g \in G : \rho_\alpha(g) \leqslant \rho_\alpha(g_0)\} = \emptyset$ and by virtue of the Theorem 1.6 ([1], p.77) there exists $\varphi \in X^* \setminus \{0\}$ such that

(18) $$\sup \varphi(\{g \in G : f_\alpha(g) \leqslant f_\alpha(g_0)\}) = \varphi(g_0) = \inf \varphi(\{g \in G : \rho_\alpha(g) \leqslant \rho_\alpha(g_0)\})$$

and because $\mathcal{S}_G(f_\alpha) \subseteq \{g \in G : f_\alpha(g) \leqslant f_\alpha(g_0)\}$ and $\mathcal{S}_G(\rho_\alpha) \subseteq \{g \in G : \rho_\alpha(g) \leqslant \rho_\alpha(g_0)\}$ it follows that the condition (14) holds.

Assume that $g_0 \notin \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha)$. Then there exists $g_1 \in G$ with $\varphi(g_1) = \varphi(g_0)$ and $f_\alpha(g_1) < f_\alpha(g_0)$. Hence $\varphi(g) \leqslant \varphi(g_0)$ for every $g \in G$, because in the contrary case, there exists $\bar{g} \in G$ such that $\varphi(\bar{g}) > \varphi(g_0)$ and by (18) we obtain $f_\alpha(g_1) < f_\alpha(g_0) < f_\alpha(\bar{g})$, hence there exists $\lambda_1 \in (0,1)$ such that $f_\alpha(\lambda_1 g_1 + (1-\lambda_1)g) = f_\alpha(g_0)$ which contradicts (18) because $\varphi(\lambda_1 g_1 + (1-\lambda_1)\bar{g}) > \varphi(g_0)$. The second part of (15) is proved similarly.

Case 2. $\{g \in G : \rho_\alpha(g) < \rho_\alpha(g_0)\} = \varnothing$: then $g_0 \in \mathcal{S}_{\mathcal{S}_{G(\rho_\alpha)}}(f_\alpha)$ (because $g_0 \in \in \mathcal{U}_G(\rho_\Phi)$) and by the Theorem of Psenicnil and Rockafellar ([3], p. 1025)it follows that there exists $\varphi \in X^*$ such that

$$\varphi \in \partial f_\alpha(g_0) \text{ and } \inf_{g \in \mathcal{S}_{G(\rho_\alpha)}} \varphi(g) = \varphi(g_0). \tag{19}$$

Therefore $\varphi(g_0) \leqslant \inf \varphi(\mathcal{S}_G(\rho_\alpha))$ and $\varphi(g_\alpha) - \varphi(g_0) \leqslant f_\alpha(g_\alpha) - f_\alpha(g_0) \leqslant 0$ for every $g_\alpha \in \mathcal{S}_G(f_\alpha)$ which implies

$$\varphi(g_0) \geqslant \sup \varphi(\mathcal{S}_G(f_\alpha)). \tag{20}$$

By (19) and (20) we obtain (14). On the other hand we have

$g_0 \in \mathcal{S}_G(\rho_\alpha) \subseteq \mathcal{S}\{g \in G : \varphi(g) = \varphi(g_0)\} (\rho_\alpha)$. Let $g \in G$ with $\varphi(g) = \varphi(g_0)$.

By (19) we obtain $\varphi(g) - \varphi(g_0) \leqslant f_\alpha(g) - f_\alpha(g_0)$, that is $g_0 \in \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha)$ and the theorem is proved.

Corollary. *If for every $f_\alpha \in \Phi$ for which there exists $g \in G$ with $f_\alpha(g) < f_\alpha(g_0)$ we have $\mathcal{S}_G(f_\alpha) \neq \varnothing$, there exists $f_\beta \in \Phi$ such that $\mathcal{S}_G(f_\beta) \neq \varnothing$ and $\varphi \in \in X^*$ such that*

$$\sup \varphi(\mathcal{S}_G(f_\alpha)) \leqslant \varphi(g_0) \leqslant \inf \varphi(\mathcal{S}_G(f_\beta)) \tag{21}$$

and

$$g_0 \in \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha) \cap \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\beta), \tag{22}$$

then g_0 is a minimal element for the convex vectorial program (G, ρ_Φ).

The proof follows the same line as the proof of the point (i) the above theorem for ρ_α interchanged with f_β and for this reason we have omitted it.

Remark 3. The Theorem 3 generalizes the Theorem 1 of [3] and it is clear that all our results established on the minimal elements can be applied without any difficulty to the elements of best vectorial approximation and to the elements of best simultaneous approximation.

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LOWER BOUNDS FROM DISCRETE INEQUALITIES OF GOLLWITZER-LANGENHOP TYPE

BY

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1. Introduction. In a recent issue of this journal, S. M. Singare and B. G. Pachpatte [7] proved a number of discrete analogues of Gronwall-type inequalities in two independent variables. Since then Pachpatte [6] gave similar proofs of discrete versions of other Gronwall-type integral inequalities due to Gollwitzer [3], Langenhop [4], and Pachpatte [5]. Unfortunately, five of the six theorems in [6] contain rather unnatural and restrictive conditions which appear to be required by the method of proof. In this paper we shall show how a simple transformation allows us to use the theorems of [7] to obtain better versions of the five theorems of [6] and, at the same time, drop the restrictive conditions. In the case of integral inequalities, this type of transformation was pointed out by the author in [1; pp. 4—5, 22—23]. For discrete inequalities the transformation produces a shift of one unit in the limits of summation.

2. Notation and Results From [7]. We use the same notation as in [7], but always with $N = \{0, 1, 2, \ldots\}$. Only three of the theorems of [7] will be needed here, and we state only the special cases required as Theorems A, B, C.

Theorem A [7; th. 1]. *If $\Phi(i, j)$, $b(i, j)$ are real-valued nonnegative functions on N^2, $A \geqslant 0$ is constant, and $\Phi(i, j) \leqq A + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} b(\alpha, \beta)\Phi(\alpha, \beta)$ for $i, j \geqq 0$, then $\Phi(i, j) \leqq A \cdot \prod_{\alpha=0}^{i-1} \left[1 + \sum_{\beta=0}^{j-1} b(\alpha, \beta)\right]$ for $i, j \geqq 0$.*

In [7] the requirement $A > 0$ was made, but if $A = 0$ then the results hold with any $A = \varepsilon > 0$, and the conclusion follows by letting $\varepsilon \to 0+$.

Theorem B [7, th. 2]. *Let U, c be nonnegative functions on N^2, $M \geqq 0$ be constant, and let $W(u)$ be continuous and nondecreasing on $I = [0, \infty)$ with $W(u) > 0$ for $u > 0$. If*

$$U(i, j) \leqq M + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} c(\alpha, \beta) W(U(\alpha, \beta)) \quad \text{for} \quad i, j \geqslant 0$$

then for $0 \leqq i \leqq i_0$, $0 \leqq j \leqq j_0$, we have $U(i, j) \leqq \Omega^{-1}[\Omega(M) + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} c(\alpha, \beta)]$, where $\Omega(r) = \int_{r_0}^{r} ds/W(s)$ for $r > 0$, where $r_0 > 0$ is arbitrary, Ω^{-1} denotes the inverse function of Ω, and i_0, j_0 are chosen so that $\Omega(M) + \sum_{\alpha=0}^{i_0-1} \sum_{\beta=0}^{j_0-1} c(\alpha, \beta) \in \mathrm{Dom}\,(\Omega^{-1})$.

In [7], the case $M = 0$ could be handled by the same method ($M = \varepsilon \to \to 0+$) used above. Also in case $b \equiv 0$ in [7], a simple modification of the proof shows that W need not be submultiplicative, since (14) of [7] reduces