Case 2. $\{g \in G : \rho_\alpha(g) < \rho_\alpha(g_0)\} = \emptyset$: then $g_0 \in \mathcal{S}_{\mathcal{S}_G(\rho_\alpha)}(f_\alpha)$ (because $g_0 \in \mathcal{U}_G(\rho_\Phi)$) and by the Theorem of Psenicnil and Rockafellar ([3], p. 1025)it follows that there exists $\varphi \in X^*$ such that

$$\varphi \in \partial f_\alpha(g_0) \text{ and } \inf_{g \in \mathcal{S}_G(\rho_\alpha)} \varphi(g) = \varphi(g_0). \tag{19}$$

Therefore $\varphi(g_0) \leqslant \inf \varphi(\mathcal{S}_G(\rho_\alpha))$ and $\varphi(g_\alpha) - \varphi(g_0) \leqslant f_\alpha(g_\alpha) - f_\alpha(g_0) \leqslant 0$ for every $g_\alpha \in \mathcal{S}_G(f_\alpha)$ which implies

$$\varphi(g_0) \geqslant \sup \varphi(\mathcal{S}_G(f_\alpha)). \tag{20}$$

By (19) and (20) we obtain (14). On the other hand we have

$g_0 \in \mathcal{S}_G(\rho_\alpha) \subseteq \mathcal{S}\{g \in G : \varphi(g) = \varphi(g_0)\}(\rho_\alpha)$. Let $g \in G$ with $\varphi(g) = \varphi(g_0)$.

By (19) we obtain $\varphi(g) - \varphi(g_0) \leqslant f_\alpha(g) - f_\alpha(g_0)$, that is $g_0 \in \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha)$ and the theorem is proved.

Corollary. *If for every $f_\alpha \in \Phi$ for which there exists $g \in G$ with $f_\alpha(g) < f_\alpha(g_0)$ we have $\mathcal{S}_G(f_\alpha) \neq \emptyset$, there exists $f_\beta \in \Phi$ such that $\mathcal{S}_G(f_\beta) \neq \emptyset$ and $\varphi \in X^*$ such that*

$$\sup \varphi(\mathcal{S}_G(f_\alpha)) \leqslant \varphi(g_0) \leqslant \inf \varphi(\mathcal{S}_G(f_\beta)) \tag{21}$$

and

$$g_0 \in \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\alpha) \cap \mathcal{S}_{\{g \in G : \varphi(g) = \varphi(g_0)\}}(f_\beta), \tag{22}$$

then g_0 is a minimal element for the convex vectorial program (G, ρ_Φ).

The proof follows the same line as the proof of the point (i) the above theorem for ρ_α interchanged with f_β and for this reason we have omitted it.

Remark 3. The Theorem 3 generalizes the Theorem 1 of [3] and it is clear that all our results established on the minimal elements can be applied without any difficulty to the elements of best vectorial approximation and to the elements of best simultaneous approximation.

REFERENCES

1. Barbu V., Precupanu, T. — *Convexity and optimization in Banach spaces.* Sijthof Noordhoff Publishing House of Romanian Academy, 1978
2. Bacopoulos, A., Godini, G., Singer, I. — *Infima of sets in the plane and applications to vectorial optimization.* Rev. Roum. Math. Pures et Appl., 23, 3, 1978
3. Godini, G. — *On vectorial optimization.* Rev. Roum. Math. Pures et Appl., 23, 7, 1978
4. Postolică, V. — *Strict convexity and best vectorial approximation in linear spaces.* An. şt. Univ. Iaşi (to appear)
5. Yu, P. L. — *On the cone extreme points.* Journ. of Optimization Theory and Applications, 14, 1978, p. 319—377

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Faculty of Mathematics
University of Iaşi
6500 — Iaşi, R. S. Romania

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LOWER BOUNDS FROM DISCRETE INEQUALITIES OF GOLLWITZER-LANGENHOP TYPE

BY

P. R. BEESACK

1. Introduction. In a recent issue of this journal, S. M. Singare and B. G. Pachpatte [7] proved a number of discrete analogues of Gronwall-type inequalities in two independent variables. Since then Pachpatte [6] gave similar proofs of discrete versions of other Gronwall-type integral inequalities due to Gollwitzer [3], Langenhop [4], and Pachpatte [5]. Unfortunately, five of the six theorems in [6] contain rather unnatural and restrictive conditions which appear to be required by the method of proof. In this paper we shall show how a simple transformation allows us to use the theorems of [7] to obtain better versions of the five theorems of [6] and, at the same time, drop the restrictive conditions. In the case of integral inequalities, this type of transformation was pointed out by the author in [1; pp. 4—5, 22—23]. For discrete inequalities the transformation produces a shift of one unit in the limits of summation.

2. Notation and Results From [7]. We use the same notation as in [7], but always with $N = \{0, 1, 2, \dots\}$. Only three of the theorems of [7] will be needed here, and we state only the special cases required as Theorems A, B, C.

Theorem A [7; th. 1]. *If $\Phi(i, j)$, $b(i, j)$ are real-valued nonnegative functions on N^2, $A \geqslant 0$ is constant, and $\Phi(i, j) \leqq A + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} b(\alpha, \beta)\Phi(\alpha, \beta)$ for $i, j \geqq 0$, then $\Phi(i, j) \leqq A \cdot \prod_{\alpha=0}^{i-1} \left[1 + \sum_{\beta=0}^{j-1} b(\alpha, \beta)\right]$ for $i, j \geqq 0$.*

In [7] the requirement $A > 0$ was made, but if $A = 0$ then the results hold with any $A = \varepsilon > 0$, and the conclusion follows by letting $\varepsilon \to 0+$.

Theorem B [7, th. 2]. *Let U, c be nonnegative functions on N^2, $M \geqq 0$ be constant, and let $W(u)$ be continuous and nondecreasing on $I = [0, \infty[$ with $W(u) > 0$ for $u > 0$. If*

$$U(i, j) \leqq M + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} c(\alpha, \beta) W(U(\alpha, \beta)) \quad \text{for} \quad i, j \geqslant 0$$

then for $0 \leqq i \leqq i_0$, $0 \leqq j \leqq j_0$, we have $U(i, j) \leqq \Omega^{-1}[\Omega(M) + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} c(\alpha, \beta)]$, where $\Omega(r) = \int_{r_0}^{r} ds/W(s)$ for $r > 0$, where $r_0 > 0$ is arbitrary, Ω^{-1} denotes the inverse function of Ω, and i_0, j_0 are chosen so that $\Omega(M) + \sum_{\alpha=0}^{i_0-1} \sum_{\beta=0}^{j_0-1} c(\alpha, \beta) \in \mathrm{Dom}\,(\Omega^{-1})$.

In [7], the case $M = 0$ could be handled by the same method ($M = \varepsilon \to 0+$) used above. Also in case $b \equiv 0$ in [7], a simple modification of the proof shows that W need not be submultiplicative, since (14) of [7] reduces to $u(x,y)\leqslant a(x,y)$. Note that Theorem B is a discrete version of Bihari's inequality [2].

Theorem C [7, th. 3]. *If Φ, b, c, are nonnegative functions on N^2, $A\geqslant 0$ is constant, and*

$$\Phi(i,j)\leqq A+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-1} b(\alpha,\beta)\left[\Phi(\alpha,\beta)+\sum_{\mu=0}^{\alpha-1}\sum_{\nu=0}^{\beta-1} c(\mu,\nu)\Phi(\mu,\nu)\right]$$

holds for $i,j\geqq 0$, then for all such i,j

$$\Phi(i,j)\leqq A\left[1+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-1} b(\alpha,\beta)\prod_{\mu=0}^{\alpha-1}\left[1+\sum_{\nu=0}^{\beta-1}\{b(\mu,\nu)+c(\mu,\nu)\}\right]\right].$$

3. **Main Results.** As noted in the Introduction, the Theorems we shall obtain here by applying a standard transformation to Theorems A, B, C will differ slightly in form from those of Pachpatte [6], but will not require such restrictive hypotheses and will give better conclusions.

Theorem 1. *Let Φ, a, b, u be nonnegative functions on N^2 such that*

$$u(s,t)\geqq \varphi(x,y)-a(s,t)\cdot\sum_{m=x+1}^{s}\sum_{n=y+1}^{t} b(m,n)\Phi(m,n) \tag{1}$$

holds for $0\leqslant x\leqslant s$, $0\leqslant y\leqslant t$. Then,

$$u(s,t)\geqq\varphi(x,y)\cdot\prod_{m=x+1}^{s}\left[1+a(s,t)\cdot\prod_{n=y+1}^{t} b(m,n)\right]^{-1} \tag{2}$$

for all such s,t,x,y.

Proof. If we make the transformation

$$x=s-i,\ y=t-j,\ m=s-\alpha,\ n=t-\beta, \tag{3}$$

$$\varphi(i,j)=\Phi(s-i,t-j),\quad B(i,j)=b(s-i,t-j). \tag{4}$$

in (1), we may rewrite (1) in the form

$$\Phi(i,j)\leqq u(s,t)+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-1} a(s,t)B(\alpha,\beta)\Phi(\alpha,\beta), \tag{1'}$$

valid for $0\leqslant i\leqslant s$, $0\leqslant j\leqslant t$, for *arbitrary fixed* $s,t\geqslant 0$. By Theorem A with $A=u(s,t)$, $b=a(s,t)B$, we obtain

$$\varphi(i,j)\leqq u(s,t)\prod_{\alpha=0}^{i-1}\left[1+a(s,t)\cdot\sum_{\beta=1}^{j-1} B(\alpha,\beta)\right], \tag{2'}$$

which reduces to (2) on using (4), (3).

Remark 1. Since $(1+\alpha)^{-1}\geqq 1-\alpha$ holds for all $\alpha>-1$, the lower bound in (2) is better than the (correspondingly modified) lower bound

$$u(s,t)\geqslant\varphi(x,y)\cdot\prod_{m=x+1}^{s}\left[1-a(s,t)\cdot\sum_{n=y+1}^{t} b(m,n)\right]$$

found in [6, th. 1], where the additional restriction is made that the factors in square brackets all be positive.

Theorem 2. *Let φ, a, b, u be nonnegative functions on N^2, and let $H(r)$ be positive, continuous, strictly increasing, convex, and submultiplicative on $(0,\infty)$ with $H(r)\to\infty$ as $r\to\infty$. Suppose also that p,q are positive functions on N^2 with $p(s,t)q(s,t)=1$. If*

$$u(s,t)\geqq\varphi(x,y)-a(s,t)H^{-1}\left[\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)H(\varphi(m,n))\right] \tag{5}$$

holds for $0\leqq x\leqq s$, $0\leqq y\leqq t$ then for all such x,y,s,t,

$$u(s,t)\geqq p(s,t)H^{-1}\left\{p^{-1}(s,t)H(\varphi(x,y))\prod_{m=x+1}^{s}\left[1+q(s,t)H(a(s,t)q^{-1}(s,t))\cdot\sum_{n=y+1}^{t}(m,n)\right]^{-1}\right\} \tag{6}$$

where H^{-1} denotes the inverse function of H, but $p^{-1}=1/p$, $q^{-1}=1/q$.

Proof. As in [6, th. 2], we rewrite (5) as $\varphi(x,y)\leqslant u(s,t)+a(s,t)H^{-1}[\Sigma\Sigma\, b(m,n)H(\varphi(m,n))]$, whence using the properties of H, we have $H(\varphi(x,y))\leqslant p(s,t)H(u\cdot p^{-1}(s,t))+q(s,t)H(a\cdot q^{-1}(s,t))\cdot\Sigma\Sigma\, b(m,n)H(\varphi(m,n))$ for $0\leqq x\leqq s$, $0\leqq y\leqq t$. Again, we make the change of variables corresponding to (3), (4), and the last inequality is transformed to

$$H(\Phi(i,j)\leqq p(s,t)H(up^{-1}(s,t))+q(s,t)H(a\cdot q^{-1}(s,t))\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-1} B(\alpha,\beta)H(\Phi(\alpha,\beta))$$

valid for $0\leqq i\leqq s$, $0\leqq j\leqq t$, for *arbitrary fixed* $s,t\geqq 0$. Again we may apply Theorem A to obtain

$$H(\Phi(i,j))\leqq p(s\cdot t)H(up^{-1}(s,t))\cdot\prod_{\alpha=0}^{i-1}\left[1+q(s,t)H(a\cdot q^{-1}(s,t))\sum_{\beta=0}^{j-1} B(\alpha,\beta)\right].$$

By (4), (3) this reduces to

$$H(\Phi(x,y)\leqq p(s,t)H(u\cdot p^{-1}(s,t))\prod_{m=x+1}^{s}\left[1+q(s,t)H(a\cdot q^{-1}(s,t)\sum_{n=y+1}^{t} b(m,n)\right],$$

or

$$H(u\cdot p^{-1}(s,t)\geqslant p^{-1}(s,t)H(\varphi(x,y))\prod_{x+1}^{s}\left[1+q(s,t)H(a\cdot q^{-1}(s,t))\sum_{y+1}^{t} b(m,n)\right]^{-1}.$$

Because H is increasing and submultiplicative, H^{-1} is increasing but supermultiplicative, so if we apply H^{-1} to the last inequality we obtain (6).

Remark 2. Because H^{-1} is increasing and $(1+a)^{-1}\geqq 1-a$ holds for all $a>-1$, we see that the lower bound in (6) is better (larger) than the corresponding lower bound in [6, th. 2], and does not involve the restrictive hypothesis given there. In [6], the unnecessary assumption $H(0)=0$ was also made.

Theorem 3. *Let the functions u, a, b, be nonnegative on N^2, and suppose that $W(u)$ is continuous and nondecreasing on $[0,\infty)$ with $W(u)>0$ for $u>0$. If*

$$u(s,t)\geqq u(x,y)-a(s,t)\sum_{m=x+1}^{s}\sum_{n=y+1}^{t} b(m,n)W(u(m,n)) \tag{7}$$

holds for $0\leqq x\leqq s$, $0\leqq y\leqq t$, then for $0\leqq x\leqq s\leqq s_0$, $0\leqq y\leqq t\leqq t_0$, we have

$$u(s,t)\geqq\Omega^{-1}\left[\Omega(u,(x,y))-a(s,t)\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)\right], \tag{8}$$

where $\Omega(r)=\int_{r_0}^{r} ds/W(s)$ for $r>0$, with arbitrary $r_0>0$, Ω^{-1} is the inverse function of Ω, and s_0, t_0 are chosen so that

$$\Omega(x,(x,y))-a(s,t)\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)\in \mathrm{Dom}\,(\Omega^{-1})\ for\ 0\leqq x\leqq s\leqq s_0,\quad 0\leqq y\leqq t\leqq t_0.$$

The proof follows by rewriting (7) in the form

$$u(u, v) \leq u(s,t) + a(s,t) \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n) W(u(m, n)), \tag{7'}$$

and making the change of variables (3), with B defined by (4) and also $U(i, j) = u(s-i, t-j)$ used in the first and last occurrence of u in (7'). The inequality (7') is thereby reduced to that of Theorem B with $M = u(s, t)$, $c = a(s, t)\, B$, and the conclusions of Theorem B transforms into

$$u(x, y) \leq \Omega^{-1}\left\{\Omega(u(s,t)) + a(s,t) \cdot \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n)\right\}$$

for $0 \leq x \leq s \leq s_1$, $0 \leq y \leq t \leq t_1$ say. Since both Ω, Ω^{-1} are increasing, this transforms into (8) for the indicated domain.

We observe that (except for the shift in the limits of summation) (8) is the same as the result given in [6; th. 3].

Theorem 4. *Let φ, a, b, c, u be nonnegative functions on N^2, such that*

$$u(s,t) \geq \varphi(x, y) - a(s,t) \cdot \sum_{m=x+1}^{s} \sum_{n=y+1}^{t} b(m, n)\left\{\varphi(m, n) + \right.$$
$$\left. + \sum_{\xi=m+1}^{s} \sum_{\rho=n+1}^{t} c(\xi, \rho)\varphi(\xi, \rho)\right\}, \tag{9}$$

for $0 \leq x \leq s$, $0 \leq y \leq t$. Then for all such x, y, s, t, we have

$$u(s,t) \geq \Phi(x, y)\left\{1 + a(s,t) \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n) \cdot \prod_{\xi=m+1}^{s}\left[1 + \right.\right.$$
$$\left.\left. + \sum_{\rho=n+1}^{t} (a(s,t) b(\xi, \rho) + c(\xi, \rho))\right]\right\}^{-1}. \tag{10}$$

Proof. As in Theorems 1, 2 we make the change of variables (3), (4) in (9), and also set $\xi = s - \mu$, $\rho = t - \nu$, $C(\mu, \nu) = c(s-\mu, t-\nu)$. Then (9) can be written as

$$\Phi(i, j) \leq u(s,t) + a(s,t) \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} B(\alpha, \beta)\left\{\Phi(\alpha, \beta) + \sum_{\mu=0}^{\alpha-1} \sum_{\nu=0}^{\beta-1} C(\mu, \nu)\Phi(\mu, \nu)\right\}$$

valid for $0 \leq i \leq s$, $0 \leq j \leq t$. To this we apply Theorem C with $A = u(s, t)$, $b = a(s, t) B$, $c = C$, and obtain

$$\Phi(i, j) \leq u(s,t)\left\{1 + a(s,t) \cdot \sum_{0}^{i-1} \sum_{0}^{j-1} B(\alpha, \beta) \cdot \prod_{\mu=0}^{\alpha-1}\left[1 + \sum_{\nu=0}^{\beta-1} (a(s,t) B(\mu, \nu) + C(\mu, \nu))\right]\right\}$$

for $0 \leq i \leq s$, $0 \leq j \leq t$. By the change of variables this reduces to

$$\varphi(x, y) \leq u(s,t)\left\{1 + a(s,t) \cdot \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n) \cdot \prod_{\xi=m+1}^{s}\left[1 + \right.\right.$$
$$\left.\left. + \sum_{\rho=n+1}^{t} (a(s,t) b(\xi, \rho) + c(\xi, \rho))\right]\right\}$$

for $0 \leq x \leq s$, $0 \leq y \leq t$. Hence (10) follows.

Remark 3. If we write (10) symbolically as

$$u \geq \varphi\{1 + a\Sigma b \cdot \Pi(1+d)\}^{-1} = \mu_1$$

and the corresponding (modified) lower given in [6, th. 4] as

$$u \geq \varphi\{1 + a\Sigma b \cdot \{\Pi(1-d)\}^{-1}\}^{-1} = \mu_2$$

then one easily sees that $\mu_1 \geq \mu_2$, so the bound (10) is again better. In addition the restriction $1 - d > 0$ is not required for (10).

Theorem 5. *Let φ, a, b, c, u be nonnegative on N^2, let H, p, q be as in Theorem 2, and suppose that*

$$u(s,t) \geq \varphi(x, y) - a(s,t) H^{-1}\left[\sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n)\, \{H(\Phi(m, n)) + \right.$$
$$\left. + \sum_{\xi=m+1}^{s} \sum_{\rho=n+1}^{t} c(\xi, \rho)\, H(\varphi(\xi, \rho))\}\right] \tag{11}$$

holds for $0 \leq x \leq s$, $0 \leq y \leq t$. Then for all such x, y, s, t, we have

$$u(s,t) \geq p(s,t) H^{-1}\left[p^{-1}(s,t) H(\Phi(x, y))\left\{1 + \right.\right.$$
$$+ q(s,t) H(a \cdot q^{-1}(s,t) \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n). \tag{12}$$
$$\left.\left. \cdot \prod_{\xi=m+1}^{s}\left[1 + \sum_{\rho=n+1}^{t} (q(s,t) H(a \cdot q^{-1}(s,t)) b(\xi, \rho) + c(\xi, \rho))\right]\right\}^{-1}\right].$$

The proof follows in the same way as that of Theorem 2, but by using Theorem C in place of Theorem A. More easilay, by writing (11) in the form $\varphi(x, y) \leq u(s, t) + a(s, t) H^{-1}[\ldots]$, and using the properties of H we obtain

$$H(\varphi(x, y)) \leq p(s,t) H(u \cdot p^{-1}(s,t)) + q(s,t) H(a \cdot q^{-1}(s,t)\, [\ldots].$$

That is,

$$p(s,t) H(u \cdot p^{-1}(s,t)) \geq H(\Phi(x, y)) - q(s,t) H(a \cdot q^{-1}(s,t)\, [\ldots].$$

We may now apply Theorem 4 with u, Φ, a replaced by $pH(u \cdot p^{-1})$, $H(\Phi)$, $qH(a \cdot q^{-1})$ respectively, to obtain

$$H(u \cdot p^{-1}(s,t)) \geq p^{-1}(s,t) H(\Phi(x, y))\left\{1 + q(s, t) H(a \cdot q^{-1}(s,t) \sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n) \cdot \right.$$
$$\left. \cdot \prod_{m+1}^{s}\left[1 + \sum_{\rho=n+1}^{t} (q(s,t) H(a \cdot q^{-1}(s, t)) b(\xi, \rho) + c(\xi, \rho))\right]\right\}^{-1}.$$

Since H^{-1} is increasing, this implies (12).

We observe that again the bound given by (12) is better than that given in [6, th. 5], and an unnecessary restriction is removed.

Theorem 6. *Let u, a, b, c* be nonnegative on N^2, *and let $G(r)$ be positive, continuous, strictly increasing, subadditive, and submultiplicative for $r > 0$ with $G(\infty) = \infty$. If*

$$u(s,t) \geq u(x, y) - a(s,t) G^{-1}\left[\sum_{x+1}^{s} \sum_{y+1}^{t} b(m, n)\left\{G(u(m, n)) + \right.\right.$$
$$\left.\left. + \sum_{\xi=m+1}^{s} \sum_{\rho=m+1}^{t} c(\xi, \rho) G(u(\xi, \rho))\right\}\right] \tag{13}$$

for $0 \leqq x \leqq s$, $0 \leqq y \leqq t$, *then for all such* x, y, s, t, *we have*

$$u(s,t) \geqq G^{-1}\Big[G(u(x,y))\Big\{1+G(a(s,t))\cdot\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)\cdot\prod_{\xi=m+1}^{s}\Big[1+\sum_{\rho=n+1}^{t}(G(a(s,t))\cdot b(\xi,\rho)+c(\xi,\rho))\Big]\Big\}^{-1}\Big]. \tag{14}$$

Proof. If we rewrite (13) in the form $u(x,y) \leqq u(s,t)+(s,t)G^{-1}[\ldots]$ and use the properties of G we obtain

$$G(u(s,t) \geqq G(u(x,y)) - G(a(s,t))\Big[\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)\Big\{G(u(m,n)) + \sum_{m+1}^{s}\sum_{n+1}^{t} c(\xi,\rho)\cdot G(u(\xi,\rho))\Big\}\Big]$$

for $0 \leqq x \leqq s$, $0 \leqq y \leqq t$. Hence by Theorem 4,

$$G(u(s,t)) \geqq G(u(x,y))\Big\{1+G(a(s,t))\sum_{x+1}^{s}\sum_{y+1}^{t} b(m,n)\prod_{\xi=m+1}^{s}\Big[1+\sum_{\rho=n+1}^{t}(G(a(s,t))b(\xi,\rho)+c(\xi,\rho))\Big]\Big\}^{-1}.$$

Since G^{-1} is increasing, (14) follows.

The same analysis as that given in Remark 3 shows that the lower bound in (14) is better than that given (with corresponding change in summation limits) in [6, Th. 6].

REFERENCES

1. Beesack, P. R. — *Gronwall Inequalities*, Carleton Mathematical Lecture Notes, No. 11, 1975
2. Bihari, I. — *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hung. 7, (1956), 71—94
3. Gollwitzer, H. E. — *A note on a functional inequality*, Proc. Amer. Math. Soc. 23, (1969), 642—647
4. Langenhop, C. E. — *Bounds on the norm of a solution of a general differential equation*, Proc. Amer. Math. Soc. 11 (1960), 795—799
5. Pachpatte, B. G. — *A note on an integral inequality*, J. Math. Phy. Sci. 9, (1975), 11—14
6. Pachpatte, B. G. — *On some fundamental discrete inequalities in two independent variables*, Tamkang J. Math. 12 (1981), 21—33
7. Singare, S. M. and Pachpatte, B. G. — *On some fundamental discrete inequalities of the Wendroff type*, An. şt. Univ. Iaşi, 26 (1980), 85—94

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Department of Mathematics and Statistics
Carleton University
Ottawa, Canada, K1S 5B6

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CONDITIONS ON THE BEHAVIOUR OF SOLUTIONS FOR A CERTAIN CLASS OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

BY

ANTHONY UYI AFUWAPE

1. Introduction. In [3], Barbălat considered equations of the forms:

$$x''' + ax'' + bx' + h(x) = p(t) \tag{1.1}$$

and

$$x''' + f(x'') + bx' + cx = p(t) \tag{1.2}$$

and proved, under certain conditions, using the frequency-domain method, that the equations have a unique bounded solution on **R** which is

(A) globally exponentially stable for $t\to\infty$;

and

(B) periodic (respectively almost periodic) whenever $p(t)$ is periodic (respectively almost periodic).

In this paper, we shall consider equations of the forms

$$x''' + f(x'') + bx' + h(x) = p(t) \tag{1.3}$$

and

$$x''' + f_1(x')x'' + bx' + h(x) = p(t), \tag{1.4}$$

where $b>0$, functions f, f_1, h and p are continuous in their arguments, to arrive at similar conclusions as in [3], but the results of which are particular cases of our result. The method of approach shall be the frequency-domain method using tho generalized Theorem of Yacubovich [3, p. 81].

2. Statements of our results. Our first result is concerned with equation (1.3).

Theorem 1. *Consider equation* (1.3) *with* $b>0$, *functions* f, h *and* p *continuous in their arguments, such that* $f(0)=0$, $h(0)=0$ *and* $p(t)$ *is bounded in* **R**.

Suppose there exist real parameters $\alpha>0$, $c>0$ *such that* $b>c/\alpha$ *and* μ_1, μ_2 *such that*

$$\alpha \leqslant \frac{f(z)-f(\bar z)}{z-\bar z} \leqslant \alpha+\mu_1 \text{ for all } z\neq\bar z \text{ in } \mathbf{R} \tag{2.2}$$

and

$$c \leqslant \frac{h(z)-h(\bar z)}{z-\bar z} \leqslant c+\mu_2 \text{ for all } z\neq\bar z \text{ in } \mathbf{R}, \tag{2.3}$$

where $\mu_1<\alpha b-c$ *and* $\mu_2<\alpha$. *Then equation* (1.3) *has a unique solution* $x(t)$ *which*: (*i*) *has* $x(t)$, $x'(t)$, $x''(t)$ *bounded in* **R** *and globally exponentially stable as* $t\to\infty$; *and* (*ii*) *is periodic* (*respectively almost periodic*) *whenever* $p(t)$ *is periodic* (*respectively almost periodic*).