for $0 \leqq x \leqq s$, $0 \leqq y \leqq t$, *then for all such* x, y, s, t, *we have*

$$u(s,t) \geqq G^{-1}\Bigg[G(u(x,y))\Bigg\{1+G(a(s,t)) \cdot \sum_{x+1}^{s} \sum_{y+1}^{t} b(m,n) \cdot \prod_{\xi=m+1}^{s} \Bigg[1+\sum_{\rho=n+1}^{t} (G(a(s,t)) \cdot b(\xi,\rho)+c(\xi,\rho))\Bigg]\Bigg\}^{-1}\Bigg]. \tag{14}$$

Proof. If we rewrite (13) in the form $u(x,y) \leqq u(s,t)+(s,t)G^{-1}[\ldots]$ and use the properties of G we obtain

$$G(u(s,t) \geqq G(u(x,y)) - G(a(s,t))\Bigg[\sum_{x+1}^{s} \sum_{y+1}^{t} b(m,n)\Bigg\{G(u(m,n)) + \sum_{m+1}^{s} \sum_{n+1}^{t} c(\xi,\rho) \cdot G(u(\xi,\rho))\Bigg\}\Bigg]$$

for $0 \leqq x \leqq s$, $0 \leqq y \leqq t$. Hence by Theorem 4.

$$G(u(s,t)) \geqq G(u(x,y))\Bigg\{1+G(a(s,t)) \sum_{x+1}^{s} \sum_{y+1}^{t} b(m,n) \prod_{\xi=m+1}^{s} \Bigg[1+ \sum_{\rho=n+1}^{t} (G(a(s,t))b(\xi,\rho)+c(\xi,\rho))\Bigg]\Bigg\}^{-1}.$$

Since G^{-1} is increasing, (14) follows.

The same analysis as that given in Remark 3 shows that the lower bound in (14) is better than that given (with corresponding change in summation limits) in [6, Th. 6].

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CONDITIONS ON THE BEHAVIOUR OF SOLUTIONS FOR A CERTAIN CLASS OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

BY

ANTHONY UYI AFUWAPE

1. Introduction. In [3], Barbălat considered equations of the forms:

$$x'''+ax''+bx'+h(x)=p(t) \tag{1.1}$$

and

$$x'''+f(x'')+bx'+cx=p(t) \tag{1.2}$$

and proved, under certain conditions, using the frequency-domain method, that the equations have a unique bounded solution on **R** which is

(A) globally exponentially stable for $t \to \infty$;

and

(B) periodic (respectively almost periodic) whenever $p(t)$ is periodic (respectively almost periodic).

In this paper, we shall consider equations of the forms

$$x'''+f(x'')+bx'+h(x)=p(t) \tag{1.3}$$

and

$$x'''+f_1(x')x''+bx'+h(x)=p(t), \tag{1.4}$$

where $b>0$, functions f, f_1, h and p are continuous in their arguments, to arrive at similar conclusions as in [3], but the results of which are particular cases of our result. The method of approach shall be the frequency-domain method using the generalized Theorem of Yacubovich [3, p. 81].

2. Statements of our results. Our first result is concerned with equation (1.3).

Theorem 1. *Consider equation* (1.3) *with* $b>0$, *functions* f, h *and* p *continuous in their arguments, such that* $f(0)=0$, $h(0)=0$ *and* $p(t)$ *is bounded in* **R**.

Suppose there exist real parameters $\alpha>0$, $c>0$ *such that* $b>c/\alpha$ *and* μ_1, μ_2 *such that*

$$\alpha \leqslant \frac{f(z)-f(\bar{z})}{z-\bar{z}} \leqslant \alpha+\mu_1 \quad \textit{for all } z \neq \bar{z} \textit{ in } \mathbf{R} \tag{2.2}$$

and

$$c \leqslant \frac{h(z)-h(\bar{z})}{z-\bar{z}} \leqslant c+\mu_2 \quad \textit{for all } z \neq \bar{z} \textit{ in } \mathbf{R}, \tag{2.3}$$

where $\mu_1<\alpha b-c$ *and* $\mu_2<\alpha$. *Then equation* (1.3) *has a unique solution* $x(t)$ *which*: (*i*) *has* $x(t)$, $x'(t)$, $x''(t)$ *bounded in* **R** *and globally exponentially stable as* $t \to \infty$; *and* (*ii*) *is periodic* (*respectively almost periodic*) *whenever* $p(t)$ *is periodic* (*respectively almost periodic*).

Remarks. (1) If $f(x'')=\alpha x''$ in (1.3) and $\mu_1=0$ in (2.2), then Theorem 1 reduces to proposition 1 in [3].

(2) If $h(x)=cx$ in (1.3) and $\mu_2=0$, then Theorem 1 reduces to proposition 5 in [3].

(3) The extension of [3] given in Theorem 1 above is not trivial, since by the introduction of two non-linear terms, besides obtaining a real inequality from the frequency-domain inequality as in [3], we obtain a matrix inequality.

Our second result is concerned with equation (1.4).

Theorem 2. *Consider equation* (1.4) *where constant* b, *functions* h *and* p *satisfy the hypotheses of Theorem 1.*

Suppose $f_1(x')$ *is continuous, with* $f_1(0)=0$ *and there exist real parameters* $\alpha>0$, $c>0$, μ_1 *such that* $b>c/\alpha$ *and* $\alpha\leqslant\int_0^y f_1(z)dz/y\leqslant\alpha+\mu_1$ *for all* z *in* $\mathbf{R}$. *Then if* $\mu_1<\alpha b-c$ *and* $\mu_2<\alpha$ *the conclusions in Theorem* 1 *are valid for equation* (1.4).

Remark (4). If $h(x)\equiv cx$ in (1.4) and $\mu_2=0$, then Theorem 2 reduces to proposition 6 in [3].

3. **Proof of Theorem 1.** Let $f(z)=\alpha z+\hat{f}(z)$ and $h(z)=cz+\hat{h}(z)$ where $\hat{f}(z)$ and $\hat{h}(z)$ are strictly the non-linear parts of the functions $f(z)$ and $h(z)$ respectively. Set $\dot{x}_1=x_2$, $\dot{x}_2=x_3$. To employ the generalized Theorem of Yacubovich (as in [3, p. 81]), it is convenient to write equation (1.3) in the vector form

$$\dot{X}=AX-B\varphi(\sigma)+P(t),\quad \sigma=C^*X \tag{3.1}$$

where

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix};\quad A=\begin{pmatrix}0&1&0\\0&0&1\\-c&-b&-\alpha\end{pmatrix};\quad B=\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix};\quad C=\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}\ \text{and}\ P(t)=\begin{pmatrix}0\\0\\p(t)\end{pmatrix}.\ \text{Thus}$$

$$\sigma=\begin{pmatrix}x_1\\x_3\end{pmatrix}\ \text{and}\ \varphi(\tau)=\begin{pmatrix}\hat{h}(x_1)\\\hat{f}(x_3)\end{pmatrix}.$$

Choose matrices $M=\operatorname{diag}(\tau_j)$ and $D=\operatorname{diag}(\mu_j)$ for $(j=1,\ 2)$ where $\tau_j>0$. Then the frequency-domain inequality $\pi(\omega)\equiv MD+\operatorname{Re}\{DG(i\omega)\}>0$ for all ω in $\mathbf{R}$ where $G(i\omega)=C^*(i\omega I-A)^{-1}B$ becomes

$$\pi(\omega)\equiv\begin{pmatrix}\dfrac{\tau_1}{\mu_1}+\dfrac{\tau_1}{|\Delta|^2}(c-\tau\omega^2) & \pi_2(\omega)\\ \pi_3(\omega) & \dfrac{\tau_2}{\mu_2}-\dfrac{\tau_2\omega^2}{|\Delta|^2}(c-\alpha\omega^2)\end{pmatrix}>0 \tag{3.2}$$

for all ω in $\mathbf{R}$, where

$$\pi_2(\omega)=\bar{\pi}_3(\omega)=\frac{1}{2|\Delta|^2}[(\tau_1-\tau_2\omega^2)(c-\alpha\omega^2)-i\omega(\tau_1+\tau_2\omega^2)(b-\omega^2)]$$

and $\Delta\equiv\det(i\omega I-A)=(c-\alpha\omega^2)+i\omega(b-\omega^2)$ ($\bar{\pi}_3(\omega)$ being the complex conjugate of $\pi_3(\omega)$). Inequality (3.2) will be true for all ω in $\mathbf{R}$ if either

$$\tau_1\left[\frac{1}{\mu_1}+\frac{c-\alpha\omega^2}{|\Delta|^2}\right]>0\ \text{and}\ \det\pi(\omega)>0\ \text{for all}\ \omega\ \text{in}\ \mathbf{R} \tag{3.3}$$

or

$$\tau_2\left[\frac{1}{\mu_2}-\frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right]>0\ \text{and}\ \det\pi(\omega)>0\ \text{for all}\ \omega\ \text{in}\ \mathbf{R}. \tag{3.4}$$

We note that for $\omega^2\leqslant c/\alpha$, the first inequality in (3.3), $1/\mu_1+c-\alpha\omega^2$, $/|\Delta|^2>0$, is always positive. Thus, for this inequality to be true for all ω in $\mathbf{R}$, we only need to verify that it is true for all $\omega^2>c/\alpha$. This follows from Lemma 1 of [3]. In fact, if we let $v=\omega^2$ and define the strictly convex function $g(v)$ by $g(v)=\alpha v+v(b-v)^2/(\alpha v-c)$, the minimum value $g(v_0)$ of $g(v)$ is attained at $v_0=\omega_0^2$, where v_0 satisfies $c/\alpha<v_0<b$ and is the unique real root of the equation

$$A(v)=2\alpha v^3+(\alpha^3-2\alpha b-3c)v^2+2c(2b-\alpha^2)v+c(\alpha c-b^2)\equiv 0.$$

To see this, we note that $A(c/a)=-(c/a^2)(ab-c)^2<0$, and $A(b)=a(ab-c)^2>0$. Hence $c/a<v_0<b$. Furthermore, $g(v_0)$ satisfies $c<g(v_0)<ab$, since $g(v)>$ $>av+(v-b)^2/a>av$ for all $v>c/a$, giving $g(v_0)>av_0>c$. On the other hand, we have $g(v_0)<g(b)=ab$.

In a similar way, from Lemma 3 of [3], the first inequality in (3.4),

$$\frac{1}{\mu_2}-\frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}>0$$

is valid for all ω in R, so long as

$$\mu_2<\alpha<\{\min_{0<v<c/\alpha} g_1(v)\},$$

where $g_1(v)$ is the strictly convex function defined for $0<v<c/\alpha$ as

$$g_1(v)=\frac{c}{v}-\frac{(b-v)^2}{(c-\alpha v)}.$$

Let $\min\limits_{0<v<c/\alpha} g_1(v)=g_1(v_0)$, $c/\alpha<v_0<b$. Then v_0 is the unique real root of the equation $A_1(v)\equiv v^2(b-v)(\alpha v+\alpha b-2c)-c(c-\alpha v)^2=0$. The determinant of $\pi(\omega)$, in (3.2), when evaluated and simplified gives

$$\det\pi(\omega)\equiv\tau_1\tau_2\left\{\left[\frac{1}{\mu_1}+\frac{(c-\alpha\omega^2)}{|\Delta|^2}\right]\left[\frac{1}{\mu_2}-\frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right]-\frac{\tau_2^2}{4|\Delta|^2}K(\lambda,\omega^2)\right\}$$

where $K(\lambda,\omega^2)\equiv[(\lambda-\omega^2)^2(c-\alpha\omega^2)^2+\omega^2(\lambda+\omega^2)^2(b-\omega^2)^2]$ and $\lambda=\tau_1/\tau_2$. Since, for all ω in $\mathbf{R}$, $K(\lambda,\omega^2)>0$, then $-\tau_2^2/(4|\Delta|^2)K(\lambda,$ $\omega^2)<0$ for all ω in $\mathbf{R}$. Moreover, the maximum of $-\tau_2^2/(4|\Delta|^2)K(\lambda,\omega^2)$ is attained at $\omega^2=0$ and has the value $-\tau_1^2/(4c^2)$. Since $\tau_1>0$, $\tau_2>0$, for fixed values of τ_1, τ_2 we have

$$\det\pi(\omega)>\tau_1\tau_2\left\{\left(\frac{1}{\mu_1}+\frac{1}{c}\right)\frac{1}{\mu_2}-\frac{\tau_1^2}{4c^2}\right\}$$

and this is always positive whenever $\mu_2\leqslant 4c/\tau_1^2$. Thus the requirements of the generalized Theorem of Yacubovich are all satisfied and the conclusions of Theorem 1 follow.

4. **Proof of Theorem 2.** Let $\int_0^y f_1(s)ds=\alpha y+\hat{f}_1(y)$ and $h(x)=cx+\hat{h}(x)$. To write equation (1.4) in vector form, we set

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -bx_1 - \alpha x_2 + x_3 - \hat{f}(x_2),\ \dot{x}_3 = -cx - \hat{h}(x_1) + p(t);$$

then we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -b & -\alpha & 1 \\ -c & 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix};\quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix};\quad \text{and}\quad P(t) = \begin{pmatrix} 0 \\ 0 \\ p(t) \end{pmatrix}$$

Following the proof of Theorem 1, we obtain the frequency-domain inequality, after simplification as

$$\pi(\omega) = \begin{pmatrix} \frac{\tau_1}{\mu_1} + \frac{\tau_1(c-\alpha\omega^2)}{|\Delta|^2} & \pi_2(\omega) \\ \pi_3(\omega) & \frac{\tau_2}{\mu_2} - \frac{\tau_2\omega^2(c-\alpha\omega^2)}{|\Delta|^2} \end{pmatrix} > 0 \tag{4.1}$$

where $\pi_2(\omega) = \bar{\pi}_3(\omega) = \frac{1}{2|\Delta|^2}\{\omega^2(\tau_1+\tau_2)(b-\omega^2) + i\omega(\tau_1-\tau_2)(c-\alpha\omega^2)\}$.

We note that both diagonal elements of inequality (4.1) and matrix $\pi(\omega)$ in (3.2) are the same. Thus for inequality (4.1) to be true for all ω in $\mathbf{R}$, as in the proof of Theorem 1, it suffices to have $0 \leq \mu_1 < \alpha b - c$, $0 \leq \mu_2 < \alpha$ as well as that the determinant of $\pi(\omega)$ be positive. By simplification,

$$\det\ \pi(\omega) = \left\{\tau_1\tau_2\left[\frac{1}{\mu_1} + \frac{(c-\alpha\omega^2)}{|\Delta|^2}\right]\left[\frac{1}{\mu_1} - \frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right] - \frac{\tau_2^2 K_1(\lambda,\omega^2)}{4|\Delta|^2}\right\}$$

where

$$K_1(\lambda,\omega^2) = \omega^2\{\omega^2(\lambda+1)^2(b-\omega^2)^2 + (\lambda-1)^2(c-\alpha\omega^2)^2\}$$

and $\lambda = \tau_1/\tau_2$. For fixed values of $\tau_1 > 0$, $\tau_2 > 0$, the maximum value of $-\tau_2^2 K_1(\lambda,\omega^2)/(4|\Delta|^2)$ zero for all ω in $\mathbf{R}$. This value is attained at $\omega = 0$. Hence we always have for all ω in $\mathbf{R}$ that

$$\det\ \pi(\omega) > \tau_1\tau_2\left\{\frac{1}{\mu_1} + \frac{(c-\alpha\omega^2)}{|\Delta|^2}\right\}\left\{\frac{1}{\mu_2} - \frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right\}$$

which is always positive with the conditions already imposed on μ_1 and μ_2. This completes the proof of Theorem 2.

Remark (5). The proofs of Theorem 1 and 2 show that equations (1.3) and (1.4) are not „dual" (following [2, p. 166]) to each other. If they were, the frequency-domain inequalities (3.2) and (4.1) would have been equivalent. This did not occur in the case of [1] where we had two dual equations.

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p^{th}-ORDER CONDITIONAL STABILITY OF ASYMPTOTICALLY SELF-INVARIANT SETS FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

BY

OLUSOLA AKINYELE

§1. **Introduction.** In this paper, we study the non-linear abstract differential equation

$$\frac{du}{dt} = A(t)u + f(t, u), \qquad u(t_0) = u_0 \in D(A(t_0)), \tag{1}$$

where $f \in C(R^+ \times X, X)$, the space of continuous functions from $R^+ \times X$ into X, and X is a Banach space. We assume further that $A(t)$ is a linear operator closed and generally unbounded with dense domain $D(A(t))$. In [1], we investigated the non-linear abstract differential equation (1) and introduced new concepts of p^{th}-order conditional stability properties of the trivial solution of (1). While broadening the scope of the applicability of vector Lyapunov functions, we obtained sufficient conditions for these Lyapunov stability properties to hold for the system (1). However, in many concrete problems in other fields of Mathematical applications, for example, adaptive control systems, one has to consider the stability of sets which are not self-invariant and so one cannot talk meaningfully of Lyapunov stability [cf. 4]. In order to handle such problems for the system (1), the results of [1] become inadequate because they assume the existence of a self-invariant set. To cope with problems of these types, the concept of asymptotically self-invariant sets have been introduced. In fact our work in this paper was motivated by the results of [1] and the concept of asymptotically self-invariant (ASI) sets. Here we extend our work in [2] to cover the concept of p^{th}-order conditional stability of an ASI set with respect to (1). We also obtain sufficient conditions for such stability concepts. In many instances, our results will include as special cases, known stability theorems for differential equations in Euclidean spaces and Banach spaces [cf. 1, §2, §3 Remarks]. In particular, we extend and improve results recently obtained by the author [1, 2] and Lakshminkantham and Ladas [3].

§ 2. **Preliminary definitions.** Let X and Y be Banach spaces and suppose $B(X, Y)$ is the Banach space of bounded linear transformations on X into Y with the appropriate norm $\|\cdot\|$. Let $\Phi \in B(X, Y)$ and define