$$\dot{x}_1=x_2,\ \dot{x}_2=-bx_1-\alpha x_2+x_3-\hat{f}(x_2),\ \dot{x}_3=-cx-\hat{h}(x_1)+p(t)\ ;$$

then we have

$$A=\begin{pmatrix}0 & 1 & 0\\ -b & -\alpha & 1\\ -c & 0 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix};\quad C=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix};\quad \text{and}\quad P(t)=\begin{pmatrix}0\\ 0\\ p(t)\end{pmatrix}$$

Following the proof of Theorem 1, we obtain the frequency-domain inequality, after simplification as

$$\pi(\omega)=\begin{pmatrix}\dfrac{\tau_1}{\mu_1}+\dfrac{\tau_1(c-\alpha\omega^2)}{|\Delta|^2} & \pi_2(\omega)\\ \pi_3(\omega) & \dfrac{\tau_2}{\mu_2}-\dfrac{\tau_2\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\end{pmatrix}>0 \tag{4.1}$$

where $\pi_2(\omega)=\overline{\pi}_3(\omega)=\dfrac{1}{2|\Delta|^2}\{\omega^2(\tau_1+\tau_2)(b-\omega^2)+i\omega(\tau_1-\tau_2)(c-\alpha\omega^2)\}$.

We note that both diagonal elements of inequality (4.1) and matrix $\pi(\omega)$ in (3.2) are the same. Thus for inequality (4.1) to be true for all ω in **R**, as in the proof of Theorem 1, it suffices to have $0\leqslant\mu_1<\alpha b-c$, $0\leqslant\mu_2<\alpha$ as well as that the determinant of $\pi(\omega)$ be positive. By simplification,

$$\det\ \pi(\omega)=\left\{\tau_1\tau_2\left[\frac{1}{\mu_1}+\frac{(c-\alpha\omega^2)}{|\Delta|^2}\right]\left[\frac{1}{\mu_1}-\frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right]-\frac{\tau_2^2K_1(\lambda,\omega^2)}{4|\Delta|^2}\right\}$$

where

$$K_1(\lambda,\omega^2)=\omega^2\{\omega^2(\lambda+1)^2(b-\omega^2)^2+(\lambda-1)^2(c-\alpha\omega^2)\}$$

and $\lambda=\tau_1/\tau_2$. For fixed values of $\tau_1>0$, $\tau_2>0$, the maximum value of $-\tau_2^2K_1(\lambda,\omega^2)/(4|\Delta|^2)$ zero for all ω in **R**. This value is attained at $\omega=0$. Hence we always have for all ω in **R** that

$$\det\ \pi(\omega)>\tau_1\tau_2\left\{\frac{1}{\mu_1}+\frac{(c-\alpha\omega^2)}{|\Delta|^2}\right\}\left\{\frac{1}{\mu_2}-\frac{\omega^2(c-\alpha\omega^2)}{|\Delta|^2}\right\}$$

which is always positive with the conditions already imposed on μ_1 and μ_2. This completes the proof of Theorem 2.

Remark (5). The proofs of Theorem 1 and 2 show that equations (1.3) and (1.4) are not „dual" (following [2, p. 166]) to each other. If they were, the frequency-domain inequalities (3.2) and (4.1) would have been equivalent. This did not occur in the case of [1] where we had two dual equations.

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p^{th}-ORDER CONDITIONAL STABILITY OF ASYMPTOTICALLY SELF-INVARIANT SETS FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

BY

OLUSOLA AKINYELE

§1. **Introduction.** In this paper, we study the non-linear abstract differential equation

$$\frac{du}{dt}=A(t)u+f(t,u),\qquad u(t_0)=u_0\in D(A(t_0)), \tag{1}$$

where $f\in C(R^+\times X, X)$, the space of continuous functions from $R^+\times X$ into X, and X is a Banach space. We assume further that $A(t)$ is a linear operator closed and generally unbounded with dense domain $D(A(t))$. In [1], we investigated the non-linear abstract differential equation (1) and introduced new concepts of p^{th}-order conditional stability properties of the trivial solution of (1). While broadening the scope of the applicability of vector Lyapunov functions, we obtained sufficient conditions for these Lyapunov stability properties to hold for the system (1). However, in many concrete problems in other fields of Mathematical applications, for example, adaptive control systems, one has to consider the stability of sets which are not self-invariant and so one cannot talk meaningfully of Lyapunov stability [cf. 4]. In order to handle such problems for the system (1), the results of [1] become inadequate because they assume the existence of a self-invariant set. To cope with problems of these types, the concept of asymptotically self-invariant sets have been introduced. In fact our work in this paper was motivated by the results of [1] and the concept of asymptotically self-invariant (ASI) sets. Here we extend our work in [2] to cover the concept of p^{th}-order conditional stability of an ASI set with respect to (1). We also obtain sufficient conditions for such stability concepts. In many instances, our results will include as special cases, known stability theorems for differential equations in Euclidean spaces and Banach spaces [cf. 1, §2, §3 Remarks]. In particular, we extend and improve results recently obtained by the author [1, 2] and Lakshminkantham and Ladas [3].

§ 2. **Preliminary definitions.** Let X and Y be Banach spaces and suppose $B(X, Y)$ is the Banach space of bounded linear transformations on X into Y with the appropriate norm $\|\cdot\|$. Let $\Phi\in B(X, Y)$ and define

$$B=\{u\in X:\|\Phi(u)\|=0\},$$

$$S(B,\varepsilon)=\{u\in X:\|\Phi(u)\|<\varepsilon\},\quad \overline{S(B,\varepsilon)}=\{u\in X:\|\Phi(u)\|\leqslant\varepsilon\}.$$

For each $t\in R^{+}=[0,\infty)$, $A(t)$ denotes a linear operator on X with domain denoted by $D(A(t))$ depending on t, and for sufficiently small $h>0$, the operator $R[h;A(t)]$ is defined by

$$R[h;A(t)]=[I-hA(t)]^{-1}.$$

A solution of the system (1) is a strongly differentiable function $u(t,t_0,u_0)$ such that $u(t)\in D(A(t))$ and which satisfies (1) for all $t\geqslant t_0$. We assume conditions which guarantee the existence of solutions $u(t,t_0,u_0)$ of (1) for all $t\geqslant t_0$.

Define

$\mathcal{K}=\{\Phi\in C([0,\rho),R^{+}):\Phi(0)=0$ and $\Phi(r)$ is strictly monotone increasing in $r\}$.

$\mathcal{L}=\{\sigma\in C([0,\infty),R^{+}):\sigma(t)$ is monotone decreasing in t and $\sigma(t)\to 0$ as $t\to\infty\}$.

Definition 2.1. *A set $A\subseteq X$ is said to be asymptotically self-invariant (ASI) with respect to (1) if given any monotone decreasing sequence $\{\varepsilon_q\}$, $\varepsilon_q\to 0$ as $q\to\infty$, $\exists$ a monotone increasing sequence $\{t(\varepsilon_q)\}$, $t_q(\varepsilon)\to\infty$ as $q\to\infty$ such that $u_0\in A$ for $t_0\geqslant t_q(\varepsilon)$ implies $u(t,t_0,u_0)\in S(A,\varepsilon_q)$, $t\geqslant t_0$; $q=1,2,\ldots\ldots$*

Denote by H any closed subspace of X such that $B\subset H$. We shall assume that the set B is ASI for the moment with respect to the system (1). We introduce the definitions of p^{th}-Order Conditional Stability of the set B with respect to (1). In view of definition 2.1 of [1] and our definitions of conditional stability of an ASI set in [2], we can formulate the p^{th}-Order Conditional Stability definitions for an ASI set B with respect to (1). A simple definition is the following:

Definition 2.2. *The ASI set B with respect to the system (1) is p^{th}-Order Conditional equistable, if for each $\varepsilon>0$ and $t_0\in R^{+}$, there exist a positive number A, $t_1(\varepsilon)\to\infty$ as $\varepsilon\to 0$, $\delta=\delta(t_0,\varepsilon)$ and $t_0\geqslant t_1(\varepsilon)$ continuous with respect to t_0 for each ε, such that*

$$u(t,t_0,u_0)\in S(B,\varepsilon)\ \text{and}\ u^{(m)}(t,t_0,u_0)\in S(B,A_\varepsilon)$$

for $t\geqslant t_0\geqslant t_1(\varepsilon)$ provided $u_0\in \overline{S(B,\delta)}\cap H$ where $m=1,2,\ldots,p$, and $u^{(m)}(t,t_0,u_0)$ denotes the m^{th} strong derivative of $u(t,t_0,u_0)$. We note that $u^{(0)}(t,t_0,u_0)\equiv\equiv u(t,t_0,u_0)$.

Other definitions of stability can be formulated very easily from definition 2.2 and similar definitions of [1]. These matters are straightforward and we omit details.

Remark. The idea of p^{th}-order conditional stability of the trivial solution of the system (1) was introduced in [1]. If we take $B=\{0\}$ and assume that the set $u=0$ is self invariant with respect to (1), then our definitions reduce to the p^{th}-order conditional stability of the trivial solution of (1). Thus definition 2.2 generalizes the concept of p^{th}-order conditional stability to ASI sets.

§ 3. p^{th}-Order Conditional Stability of an ASI set. We now state results giving sufficient conditions for the stability properties formulated in §2 to hold.

Theorem 3.1. *Assume that*

(i) *$g\in C(R^{+}\times R^{n}_{+},R^{n})$ and $g(t,x)$ is quasi-monotone nondecreasing in x for each $t\in R^{+}$.*

(ii) *$V\in C(R^{+}\times S(B,\rho),R^{n}_{+})$, $V(t,u)$ is locally Lipschitzian in u and*

$$b(\|\Phi(u)\|)\leqslant\sum_{i=1}^{n}V_i(t,u),\ \text{for}\ (t,u)\in R^{+}\times S(B,\rho)$$

where $b\in\mathcal{K}$ and $\sum_{i=1}^{n}V_i(t,u)=\sigma(t)$ if $u\in B$ and $\sigma\in\mathcal{L}$.

(iii) *$f\in C(R^{+}\times S(B,\rho),X)$ and for $(t,u)\in R^{+}\times S(B,\rho)$*

$$D^{+}V(t,u)\leqslant g(t,V(t,u));$$

(iv) *any solution $u(t,t_0,u_0)$ of the system (1) is p-times strongly differentiable for $t\geqslant t_0\geqslant 0$ and there exists a continuous function $\alpha(r)$ monotonically increasing in r for $r\in[0,\rho)$, $\alpha(0)=0$ and such that if $(t,u^{(m)})\in R^{+}\times S(B,\rho)$ $m=0,1,2,\ldots p$, and $\alpha_0=\max\{\alpha(\|\Phi(u^{(m)}(t,t_0,u_0))\|),\ m=0,1,2,\ldots p)\}$, then*

$$\sum_{i=1}^{n}V_i(t,u(t,t_0,u_0))\geqslant\alpha_0.$$

(v) *$V_i(t,u)=0$, $i=1,2,\ldots,k$, $k<n$ and for $u\in H$;*

(vi) *for each $t\in R^{+}$ and $h>0$ (h sufficiently small) the operator $R[h;A(t)]$ exists as a bounded operator defined on X and for each $u\in X$, $\lim_{h\to 0}R[h;A(t)]u=u$;*

(vii) *for any function $\beta(t,r)$, continuous on $R^{+}\times R^{+}$ decreasing in t for each r, increasing in r for each fixed t, such that $\lim_{t\to\infty}\lim_{r\to 0}\beta(t,r)=0$, we have*

$$\sum_{i=1}^{n}x_i(t,t_0,x_0)\leqslant\beta\left(t_0,\sum_{i=1}^{n}x_{i0}\right)$$

provided $x_{i0}=0$ $(i=1,2,\ldots k)$ where $x(t,t_0,x_0)$ is any solution of the auxilliary equation;

$$\frac{dx}{dt}=g(t,x),\quad x(t_0)=x_0\geqslant 0\ \ t_0\geqslant 0,\tag{2}$$

(viii) *for $a(t,r)$ defined and continuous on $R^{+}\times R^{+}$, monotonic increasing in r for each fixed t and monotonic decreasing in t for each fixed r such that $\lim_{t\to\infty}\lim_{r\to 0}a(t,r)=0$ and $\sum_{i=1}^{n}V_i(t,u)\leqslant a(t,\|\Phi(u)\|)$ for $(t,u)\in R^{+}\times S(B,\rho)$.*

Then the conditional stability of the ASI set $x=0$ of the auxilliary differential equation (2) implies that the set B is an ASI set and moreover p^{th}-order conditional equistable with respect to the system (1).

Proof. Clearly Theorem 3.1 of [2] implies that the set B is ASI with respect to (1). Moreover using arguments similar to that of Theorem 3.3 of [2], it follows that $\exists t_3(\varepsilon)\geqslant 0$ and $\delta_1\geqslant 0$ such that $u_0\in\overline{S(B,\delta_1)}\cap H$ for $t\geqslant t_0\geqslant t_3(\varepsilon)$ implies $u(t,t_0;u_0)\in S(B,\varepsilon)$.

By (iv), $\exists\alpha(r)$ such that for $(t, u)\in R^+\times S(B, \rho)$, $\alpha_0\leqslant\sum_{i=1}^{n} V_i(t, u(t, t_0, u_0))$, where α_0 is as defined in (iv). Hence given any $\alpha(\varepsilon)>0$ and $t_0\in R^+$, $\exists t_4(\varepsilon)$, $t_4(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $\delta_0=\delta_0(t_0, \varepsilon)$, $t_0\geqslant t_4(\varepsilon)$ such that for $t\geqslant t_0\geqslant t_4(\varepsilon)$,

$$\sum_{i=1}^{n} x_i(t ; t_0, x_0)<\alpha(\varepsilon)$$

provided $\sum_{i=1}^{n} x_{i0}\leqslant\delta_0$ and $x_{i0}=0$ $i=1, 2, \ldots k$. Choose $\hat{\delta}=\min\{\delta_0, \delta_1\}$ and set $\sum_{i=1}^{n} x_{i0}=a(t_0, \|\Phi(u)\|)$, then by hypothesis $\exists$, $t_5(\varepsilon)\geqslant 0$ and $\hat{\delta}_0=\hat{\delta}_0(t_0, \varepsilon)\geqslant 0$ with $t_0\geqslant t_5(\varepsilon)$ such that

$$a(t_0, \|\Phi(u_0)\|)\leqslant\hat{\delta} \text{ and } \|\Phi(u_0)\|\leqslant\hat{\delta}_0.$$

Let $\delta^*=\min\{\hat{\delta}_0, \hat{\delta}\}$ and $t(\varepsilon)=\max\{t_3(\varepsilon), t_4(\varepsilon), t_5(\varepsilon)\}$, then $u_0\in S(B, \delta^*)\cap H$ and by (iv) $u^{(m)}(t, t_0, u_0)$ exists for $m=1, 2, \ldots, p$. In addition, choose A such that $\alpha(\varepsilon)\leqslant\alpha(A\varepsilon)$ with $A\varepsilon\in[0, \rho)$, then we claim that for A, δ^* and $t(\varepsilon)$, $u^{(m)}(t, t_0, u_0)\in S(B, A\varepsilon)$ provided $u_0\in S(B, \delta^*)\cap H$ for $t\geqslant t_0\geqslant t(\varepsilon)$. Suppose the assertion is not true, then $\exists$ an integer l, $0<l\leqslant p$ and $t_1, t_2, >t_0\geqslant\geqslant t(\varepsilon)$ such that

$$\|\Phi(u^{(l)}(t_2, t_0, u_0))\|=A\varepsilon, \quad \|\Phi(u^{(l)}(t_1, t_0, u_0))\|=\delta^*$$

and

$$\delta^*\leqslant\|\Phi(u^{(l)}(t, t_0, u_0))\|\leqslant A\varepsilon \text{ for } t\in[t_0, t_2].$$

Applying Theorem 3.1 of [1], we obtain

$$\alpha(A\varepsilon)\leqslant\alpha(\|\Phi(u^{(l)}(t_2, t_0, u_0))\|)\leqslant\alpha_0\leqslant\sum_{i=1}^{n} V_i(t_2, u(t_2))\leqslant\sum_{i=1}^{n} r_i(t_2, t_0, x_0)<\alpha(\varepsilon)\leqslant\alpha(A\varepsilon)$$

which is a contradiction. Hence the proof is complete.

Theorem 3.2. *Assume that the hypotheses (i), (iv), (v), (vi) and (vii) of the last theorem hold. In addition, let*

(i) $V\in C(R^+\times S(B, \rho), R^n_+)$, *$V(t, u)$ is locally Lipschitzian in u and*

$$b(\|\Phi(u)\|)\leqslant\sum_{i=1}^{n} V_i(t, u)\leqslant a(\|\Phi(u)\|)$$

for $0<r<\|\Phi(u)\|<\rho$ and $t\geqslant\theta(r)$ where a, $b\in\mathcal{K}$ and the function $\theta(r)\geqslant 0$ is monotone decreasing in r for $0<r<\rho$.

(ii) *$f\in C(R^+\times S(B, \rho), X)$ and for $(t, u)\in R^+\times S(B, \rho)$, $t\geqslant\theta(r)$ with $0<$ $<r<\|\Phi(u)\|<\rho$, $D^+V(t, u)\leqslant g(t, V(t, u))$.*

Then the conditional uniform stability of the ASI *set $x=0$ of (2) implies that the set B is* ASI *and p^{th}-order conditionally uniform stable with respect to* (1).

Proof. The conditional uniform stability of the ASI set $x=0$ implies that δ_1 and δ_0 are independent of t_0, so $\hat{\delta}$ is independent of t_0. If we choose $\sum_{i=1}^{n} x_{i0}=a(\|\Phi(u_0)\|$ then we can choose $\hat{\delta}_0$ independent of t_0 and consequently δ^* of last theorem is independent of t_0. These modifications and a parallel argument to that of Theorem 3.1 concludes the proof.

Theorem 3.3. *Assume that all the hypothesis of Theorem 3.1 hold. Then the conditional equi-asymptotic stability of the* ASI *set $x=0$ of (2) implies that the* ASI *set B is p^{th}-order conditionally equi-asymptotically stable with respect to* (1).

Proof. The set B is ASI with respect to (1) and it is p^{th}-order conditionally equistable by Theorem 3.1. To complete the proof it remains only to show that it is p^{th}-order conditional quasi-equi-asymptotically stable. Let $0<\varepsilon<\rho$, then using arguments similar to Theorem 3.1, $\exists t_1(\varepsilon)>0$, $\delta=\delta(t_0)$ and $T_1=T_1(t_0, \varepsilon)>0$ such that $u_0\in\overline{S(B, \delta)}\cap H$ implies for $t_0\geqslant t_1(\varepsilon)$ and $t\geqslant t_0+T_1$, $t_1(\varepsilon)\to\infty$, $\varepsilon\to 0$, $u(t, t_0, u_0)\in S(B, \varepsilon)$.

Given $\alpha(\varepsilon)>0$ and $t_0\in R^+$, then $\exists\hat{\delta}=\hat{\delta}(t_0)$, $t_2(\varepsilon)\geqslant 0$, $t_2(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $T_2=T_2(t_0, \varepsilon)$ such that $\sum_{i=1}^{n} x_i(t, t_0, x_0)<\alpha(\varepsilon)$ for $t_0\geqslant t_2(\varepsilon)$ and $t\geqslant t_0+T_2$ provided that

$$\sum_{i=1}^{n} x_{i0}\leqslant\hat{\delta} \text{ with } x_{i0}=0 \ i=1, 2, \ldots, k.$$

Choose $\delta_1=\min\{\delta, \hat{\delta}\}$ and set $\sum_{i=1}^{n} x_{i0}=a(t_0, \|\Phi(u_0)\|)$ then, by hypothesis, $\exists t_3(\varepsilon)$, $t_3(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $\delta_2=\delta(t_0, \varepsilon)$ such that

$$a(t_0, \|\Phi(u_0)\|)\leqslant\delta_1 \text{ and } \|\Phi(u_0)\|\leqslant\delta_2.$$

Choose $\delta_3=\min\{\delta_1, \delta_2\}$, $t(\varepsilon)=\max\{t_1(\varepsilon), t_2(\varepsilon), t_3(\varepsilon)\}$, $T=\max\{T_1, T_2\}$ and A such that $\alpha(\varepsilon)\leqslant\alpha(A\varepsilon)$, then we prove that with $t(\varepsilon)$, δ_3, T and A,

$$u^{(m)}(t, t_0, u_0)\in S(B, A\varepsilon)$$

for $t\geqslant t_0+T$, $t_0\geqslant t(\varepsilon)$ and for $m=1, 2, \ldots p$ provided $u_0\in\overline{S(B, \delta_3)}\cap H$. Suppose the assertion is not true then for some integer l, with $0<l\leqslant p$, there exists a sequence $\{t_j\}$, $t_j\geqslant t(\varepsilon)$, $t_j\geqslant t_0+T$ and $t_j\to\infty$ as $j\to\infty$ such that $\|\Phi(u^{(l)}(t_j, t_0, u_0))\|\geqslant A\varepsilon$ and $\|\Phi(u(t, t_0, u_0))\|<\varepsilon$ for $t\geqslant t_0+T$ provided $u_0\in\overline{S(B, \delta_3)}\cap H$. Hence by Theorem 3.1 of [1],

$$\alpha(A\varepsilon)\leqslant\alpha(\|\Phi(u^{(l)}(t_j : t_0, u_0))\|)\leqslant\alpha_0\leqslant\sum_{i=1}^{n} V_i(t_j, u(t_j, t_0, u_0))\leqslant\sum_{i=1}^{n} r_i(t_j, t_0, x_0).$$

Therefore,

$$\alpha(A\varepsilon)\leqslant\sum_{i=1}^{n} V_i(t_j, u(t_j, t_0, u_0))\leqslant\sum_{i=1}^{n} r_i(t_j, t_0, x_0)<\alpha(\varepsilon)\leqslant\alpha(A\varepsilon)$$

which is a contradiction. Hence the result.

Remark 2. If $B=\{0\}$ and we assume that the set $u=0$ is self-invariant with $\|\Phi(u)\|=\|u\|$ where $\|\cdot\|$ is a norm on X and $x=0$ of the system (2) is self-invariant, then Theorems 3.1, 3.2, and 3.3 reduce to Theorems 3.4, 3.5 and 3.6 of [1]. Thus our results in this section generalize results of section 3 of [1] to asymptotically self-invariant sets. Moreover in view of the remarks of [1], our results contain as special cases many other stability results for differential equation in Euclidean spaces as well as Banach spaces.

Theorem 3.4. *Assume that the hypothesis of Theorem 3.3 hold. Then the conditional uniform asymptotic stability of the ASI set $x=0$ of (1) implies the p^{th}-order conditional uniform asymptotic stability of the ASI set B with respect to the system (1).*

Proof. In Theorem 3.3, we can choose δ_3 and T of the proof independent of t_0 and the result follows.

Remark 3. If $p=0$ in Theorems 3.1, 3.2, 3.3 and 3.4 then the results reduce to similar results for conditional stability of an ASI set in [2]. If $p\neq 0$, then we have the p^{th}-order conditional stability results of an ASI set which extend results of [1] to invariant sets. Moreover in view of the remarks of [1] we see clearly that our work in this paper extend in many ways known results of stability of asymptotically invariant sets of differential equations in Euclidean spaces to differential equations in Banach spaces. Our work therefore improves the results of [1] and [2].

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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS OF HYPERBOLIC DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** Consider the hyperbolic differential equations of the type:

$$u_{xy}=f(x,y,u,u_x,u_y),\ (x,y)\in R_0 \tag{1}$$

subject to the conditions

$$u(x,0)=\Phi_1(x),\ u(0,y)=\psi_1(y),\ u(0,0)=u_0, \tag{2}$$

where $f\in C[R_0\times R^3,\ R]$ and R_0 is the rectangle defined as

$$R_0=(0\leqslant x<\infty,\ 0\leqslant y<\infty).$$

Let $u(x,y)$ — the solution of (1), (2) — exist for $(x,y)\in R_0$. We shall assume that $f(x,y,0,0,0)\equiv 0$ so that (1) (2) possesses the trivial solution.

In this paper we establish a hyperbolic differential inequality and obtain a very general comparison theorem in the framework of Lyapunov function for the above hyperbolic differential equation.

Using this inequality we establish conditions for the stability and boundedness of solutions of the hyperbolic differential equations (1), (2).

2. **Definitions.** We now formulate some definitions of stability and boundedness of solutions of (1), (2).

Definition 1. *A trivial solution of the equation (1), (2) is said to be stable if for given $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, $(0.0)\in R_0$ there exists $\delta_i=\delta_i(\varepsilon_i)>0$ $(i=1,2,3)$* such that

$$|u(0,0)|\leqslant\delta_1,\quad |u_x(0,0)|\leqslant\delta_2,\quad |u_y(0,0)|\leqslant\delta_3$$

implies $|u(x,y)|<\varepsilon_1$, $|u_x(x,y)|<\varepsilon_2$, $|u_y(x,y)|<\varepsilon_3$ *for* $x,y\geqslant 0$.

Definition 2. *A trivial solution of the equation (1), (2) is said to be asymptotically stable if it is stable and for $\delta_1,\delta_2,\delta_3>0$*

$$|u(0,0)|\leqslant\delta_1,\quad |u_x(0,0)|\leqslant\delta_2,\quad |u_y(0,0)|\leqslant\delta_3 \ \textit{imply}$$

$$\lim_{\substack{x\to\infty\\ y\to\infty}}|u(x,y)|=0,\quad \lim_{\substack{x\to\infty\\ y\to\infty}}|u_x(x,y)|=0,\quad \lim_{\substack{x\to\infty\\ y\to\infty}}|u_y(x,y)|=0.$$

Definition 3. *A solution of the equation (1), (2) is said to be bounded if there exist constants M_1, M_2, M_3 such that*