Theorem 3.4. *Assume that the hypothesis of Theorem 3.3 hold. Then the conditional uniform asymptotic stability of the ASI set $x=0$ of (1) implies the p^{th}-order conditional uniform asymptotic stability of the ASI set B with respect to the system (1).*

Proof. In Theorem 3.3, we can choose δ_3 and T of the proof independent of t_0 and the result follows.

Remark 3. If $p=0$ in Theorems 3.1, 3.2, 3.3 and 3.4 then the results reduce to similar results for conditional stability of an ASI set in [2]. If $p\neq 0$, then we have the p^{th}-order conditional stability results of an ASI set which extend results of [1] to invariant sets. Moreover in view of the remarks of [1] we see clearly that our work in this paper extend in many ways known results of stability of asymptotically invariant sets of differential equations in Euclidean spaces to differential equations in Banach spaces. Our work therefore improves the results of [1] and [2].

REFERENCES

1. Akinyele, O. — *Vector Lyapunov functions and p^{th}-Order conditional stability and boundedness.* Annali Mat. Pura ed. Appl. (4), vol. 121, 351—372 (1979)
2. Akinyele, O. — *Conditional stability of asymptotically self-invariant sets for differential equations in Banach spaces*, to appear
3. Ladas, G. E., and Lakshmikantham, V. — *Differential equations in abstract spaces*, Academic Press, New York (1972)
4. Lakshmikantham, V. and Leela, S. — *Differential and Integral inequalities, theory and applications*, Vol. I, Academic press, New York (1979)

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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS OF HYPERBOLIC DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** Consider the hyperbolic differential equations of the type:

$$u_{xy}=f(x,y,u,u_x,u_y),\ (x,y)\in R_0 \tag{1}$$

subject to the conditions

$$u(x,0)=\Phi_1(x),\ u(0,y)=\psi_1(y),\ u(0,0)=u_0, \tag{2}$$

where $f\in C[R_0\times R^3, R]$ and R_0 is the rectangle defined as

$$R_0=(0\leqslant x<\infty,\ 0\leqslant y<\infty).$$

Let $u(x,y)$ — the solution of (1), (2) — exist for $(x,y)\in R_0$. We shall assume that $f(x,y,0,0,0)\equiv 0$ so that (1) (2) possesses the trivial solution.

In this paper we establish a hyperbolic differential inequality and obtain a very general comparison theorem in the framework of Lyapunov function for the above hyperbolic differential equation.

Using this inequality we establish conditions for the stability and boundedness of solutions of the hyperbolic differential equations (1), (2).

2. **Definitions.** We now formulate some definitions of stability and boundedness of solutions of (1), (2).

Definition 1. *A trivial solution of the equation* (1), (2) *is said to be stable if for given $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, $(0.0)\in R_0$ there exists $\delta_i=\delta_i(\varepsilon_i)>0$ $(i=1,2,3)$* such that

$$|u(0,0)|\leqslant\delta_1,\quad |u_x(0,0)|\leqslant\delta_2,\quad |u_y(0,0)|\leqslant\delta_3$$

implies $|u(x,y)|<\varepsilon_1$, $|u_x(x,y)|<\varepsilon_2$, $|u_y(x,y)|<\varepsilon_3$ *for* $x,y\geqslant 0$.

Definition 2. *A trivial solution of the equation* (1), (2) *is said to be asymptotically stable if it is stable and for $\delta_1,\delta_2,\delta_3>0$*

$$|u(0,0)|\leqslant\delta_1,\quad |u_x(0,0)|\leqslant\delta_2,\quad |u_y(0,0)|\leqslant\delta_3 \text{ imply}$$

$$\lim_{\substack{x\to\infty\\ y\to\infty}}|u(x,y)|=0,\quad \lim_{\substack{x\to\infty\\ y\to\infty}}|u_x(x,y)|=0,\quad \lim_{\substack{x\to\infty\\ y\to\infty}}|u_y(x,y)|=0.$$

Definition 3. *A solution of the equation* (1), (2) *is said to be bounded if there exist constants M_1, M_2, M_3 such that*

$|u(x,y)| \leq M_1$, $|u_x(x,y)| \leq M_2 \cdot |u_y(x,y)| \leq M_3$ for all $x, y, \geq 0$.

Now consider the auxiliary hyperbolic differential equation.

$$z_{xy} = g(x, y, z, z_x, z_y) \tag{3}$$

with

$$z(x,0) = \Phi_2(x),\ z(0,y) = \psi_2(y),\ z(0,0) = z_0, \tag{4}$$

where $g \in C[R_0 \times R^3, R]$, $g(x, y, P_1, P_2, P_3)$ is non-decreasing with respect to P_1, P_2, P_3. Under these conditions, maximal solution exist for (3), (4) see [1]. Let $z(x, y)$ be any solution of (3), (4).

Concerning the auxiliary equation (3), (4), we need the corresponding definitions for stability and boundedness that may be defined analogously. For our subsequent discussion we need the following classes of functions.

Definition 4. *A function $b(r)$ is said to belong to the class K if $b \in C[R_+, R_+]$, $b(0)=0$, $b(r)$ is strictly increasing in r.*

Definition 5. *A function $a(x, y, r)$ is said to belong to S if $a \in C[R_0 \times \times R_+, R_+]$, $a(x, y, 0)=0$, $a(x, y, r)$ is increasing in r for each fixed $(x, y) \in \in R_0$.*

3. **Hyperbolic differential inequalities.** We shall establish a result that will be useful in the theory of hyperbolic differential equations.

Theorem 1. *Assume that*

i. *$g \in C[R_0 \times R^3, R]$, $g(x, y, p_1, p_2, p_3)$ is non-decreasing in p_1, p_2, p_3 for each fixed $(x, y) \in R_0$ and $r(x, y)$ is the maximal solution of (3), (4);*

ii. *$m(x, y) \in C(R_0, R]$ and $m(x, y)$ satisfies*

$$m_{xy} \leq g(x, y, m, m_x, m_y),\ (x, y) \in R_0 \tag{5}$$

iii. $m_x(x, 0) \leq r_x(x, 0)$, $m_y(0, y) \leq r_y(0, y)$, $m(0, 0) \leq r(0, 0)$.

Then we have on R_0

$$m(x, y) \leq r(x, y),\ m_x(x, y) \leq r_x(x, y),\ m_y(x, y) \leq r_y(x, y).$$

Proof. From (5) and the fact that $r(x, y)$ is the solution of (3), (4), we have

$$m_{xy} - g(x, y, m, m_x, m_y) \leq r_{xy} - g(x, y, r, r_x, r_y). \tag{*}$$

Let us assume $p\Phi = \Phi_{xy} - g(x, y, \Phi, \Phi_x, \Phi_y)$ we have from (*) $pm \leq pr$ and the rest of the proof follows similar to Theorem 1 [2, p. 256].

4. **Comparison Theorem.** We shall develop a result which furnishes a very general comparison theorem for hyperbolic differential equations. This is achieved by using the hyperbolic differential inequalities that are established in Theorem 1 and by introducing the concept of Lyapunov function. This result play an important role in studying the qualitative behaviour of solutions of (1), (2).

Let the function $V \in C[R_0 \times R, R]$, and $V(x, y, u)$ be locally lipschitzian in $(x, y, u) \in R_0 \times R$ and define

$$D_1 V(x, y, u) = \lim_{h \to 0} [(1/h)\ (V(x+h, y, u(x+h, y)) - V(x, y, u(x, y)))],$$

$$D_2 V(x, y, u) = \lim_{h \to 0} [(1/h)\ (V(x, y+h, u(x, y+h)) - V(x, y, u(x, y)))].$$

$$D_{12} V(x, y, u) = \lim_{h \to 0} [(1/h^2)\ (V(x+h, y+h, u(x+h, y) + u(x, y+h) - u(x, y) + h^2 f(x, y, u, u_x, u_y)) - V(x+h, y, u(x+h, y)) - V(x, y+h, u(x, y+h)) + V(x, y, u(x, y)))]$$

where D_1 and D_2 are derivatives with respect to the first and second variable and D_{12} is the mixed derivative for $(x, y, u) \in R_0 \times R$.

From here on, we shall assume that the function g in (3) and the function V satisfy the following hypotheses:

(H_1) $g \in C[R_0 \times R^3, R]$, $g(x, y, p_1, p_2, p_3)$ is non-decreasing in p_1, p_2, p_3 for each fixed $(x, y) \in R_0$.

(H_2) $r(x, y)$ is the maximal solution of (3), (4) existing in R_0.

(H_3) $g(x, y, 0, 0, 0,) \equiv 0$ for $(x, y) \in R_0$.

(H_4) $V(x, y, u) \in C\ [R_0 \times R, R]$ $D_1 V(x, y, u)$, $D_2 V(x, y, u)$ exists in R_0 and $V(x, y, u)$ is locally lipschitzian in $(x, y, u) \in R_0 \times R$.

Further for $(x, y, u) \in R_0 \times R$.

$$D_{12} V(x, y, u) = g(x, y, V, D_1 V, D_2 V).$$

(H_5) For $(x, y, u) \in R_0 \times R$

i. $b_1(|u|) \leq V(x, y, u) \leq a_1(x, y, |u|)$

ii. $b_2(|u_x|) \leq D_1 V(x, y, u) \in a_2(x, y, |u_x|)$

iii. $b_3(|u_y|) \leq D_2 V(x, y, u) \leq a_3(x, y, |u_y|)$

where a_i $(i=1, 2, 3) \in S$ & b_i $(i=1, 2, 3) \in K$.

(H_6) $f(x, y, 0, 0, 0,) \equiv 0$ for $(x, y) \in R_0$.

We shall state and prove the following comparison theorem.

Theorem 2. *Let the hypotheses (H_1) to (H_6) be satisfied. Assume that for any solution $u(x, y)$ of (1), (2)*

$$V(0, 0, u(0, 0)) \leq r(0, 0),\ D_1 V(0, 0, u(0, 0)) \leq r_x(0, 0)$$
$$D_2 V(0, 0, u(0, 0) \leq r_y(0, 0). \tag{6}$$

then

$$V(x, y, u) \leq r(x, y),\ D_1 V(x, y, u) \leq r_x(x, y)$$
$$D_2 V(x, y, u) \leq r_y(x, y) \text{ for all } (x, y) \in R_0.$$

Proof: Set $m(x, y) = V(x, y, u(x, y))$. Then, we have

$$m_x(x, y) = \lim_{h \to 0} [(1/h)(m(x+h, y) - m(x, y)] = \lim_{h \to 0} [(1/h)(V(x+h, y, u(x+h, y)) - V(x, y, u(x, y))) = D_1 V(x, y, u(x, y)). \tag{7}$$

Similarly $m_y(x, y) = D_2 V(x, y, u(x, y))$.

$(m(x+h, y+h) - m(x+h, y) - m(x, y+h) + m(x, y)) = (V(x+h, y+h, u(x+h, y+h)) - V(x+h, y, u(x+h, y)) - V(x, y+h, u(x, y+h)) + V(x, y, u(x, y)))$

$$= V(x+h,\ y+h,\ u(x+h,\ y+h)) - V(x+h,\ y+h,\ u(x+h,\ y)+u(x,\ y+h)-u(x,\ y)+h^2 f(x,\ y,\ u,\ u_x,\ u_y)) + V(x+h,\ y+h,\ u(x,\ y+h)+u(x+h,\ y)-u(x,\ y)+h^2 f(x,\ y,\ u,\ u_x,\ u_y)) - V(x+h,\ y,\ u(x+h,\ y)) - V(x,\ y+h,\ u(x,\ y+h)) + V(x,\ y,\ u(x,\ y)) \leqslant K\,|u(x+h,\ y+h)-u(x+h,\ y)-u(x,\ y+h)+u(x,\ y)-h^2 f(x,\ y,\ u,\ u_x,\ u_y)| + V(x+h,\ y+h,\ u(x,\ y+h)+u(x+h,\ y)-u(x,\ y)+h^2 f(x,\ y,\ u,\ u_x,\ u_y)) - V(x,\ y+h,\ u(x,\ y+h)) - V(x+h,\ y,\ u(x+h,\ y)) + V(x,\ y,\ u(x,\ y)).$$

Where K is a local Lipschitz constant. This together with (1), (5) and (7), yields the inequality $m_{xy} \leqslant g(x,y,m,m_x,m_y)$. Now from (6) we get

$$m(0,0) \leqslant r(0,0),\ m_x(x,0) \leqslant r_x(x,0) \text{ and } m_y(0,y) \leqslant r_y(0,y).$$

Hence by Theorem 1 we obtain the desired result.

5. Stability results. Theorem 3. *Let the hypotheses (H_1) to (H_5) hold; then the stability (asymptotic stability) of trivial solution of (3), (4) implies the stability (asymptotic stability) of trivial solution of (1) (2).*

Proof. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, $x, y \geqslant 0$ be given. Assume that the trivial solution of (3), (4) is stable. Then for $b_1(\varepsilon_1)$, $b_2(\varepsilon_2)$, $b_3(\varepsilon_3)$, and $x, y \geqslant 0$, there exist $\delta_i = \delta_i(\varepsilon_i)$ $(i=1,2,3)$ such that

$$|Z(0,0)| \leqslant \delta_1,\ |Z_x(0,0)| \leqslant \delta_2,\ |Z_y(0,0)| \leqslant \delta_3$$

imply

$$|Z(x,y)| < b_1(\varepsilon_1),\ |Z_x(x,y)| < b_2(\varepsilon_2),\ |Z_y(x,y)| < b_3(\varepsilon_3).$$

Let us choose $Z(0,0)$, $Z_x(0,0)$, $Z_y(0,0)$ so that

$$V(0,0,u(0,0)) \leqslant Z(0,0),\ D_1V(0,0,u(0,0)) \leqslant Z_x(0,0),$$
$$D_2V(0,0,u(0,0)) \leqslant Z_y(0,0) \text{ and}$$
$$Z(0,0) = a_1(0,0\,|u(0,0)|),\ Z_x(0,0) = a_2(0,0,|u_x(0,0)|)$$

(8) $$Z_y(0.0) = a_3(0,0,|u_y(0,0)|) \text{ for } (0,0) \in R_0.$$

Since $a_i(0,0,\cdot) \in S$ $(i=1,2,3)$ we can find $\delta_1', \delta_2', \delta_3' > 0$ such that $|u(0,0)| \leqslant \delta_1'$, $|u_x(0,0)| \leqslant \delta_2'$, $|u_y(0,0)| \leqslant \delta_3'$.
Now we claim that the trivial solution of (1), (2) is stable. That is to prove $|u(x,y)| < \varepsilon_1$, $|u_x(x,y)| < \varepsilon_2$, $|u_y(x,y)| < \varepsilon_3$ $x, y \geqslant 0$. If this is not true, there exists $t_0 = x_0 + y_0$, such that $|u(x,y)| > \varepsilon_1$, $t > t_0$ and $u(x_0, y_0) = \varepsilon_1$ at $t = t_0$, where $t = x+y$. On the other hand we have the inequality

(9) $$V(x,y,u(x,y)) \leqslant r(x,y).$$

Now by (H_3) and (9) we have

(10) $$b_1(|u(x,y)|) \leqslant V(x,y,u(x,y)) \leqslant r(x,y)$$

which gives

(11) $$b_1(\varepsilon_1) = b_1(|u(x_0,y_0)|) \leqslant V(x_0,y_0,u(x_0,y_0)) \leqslant |r(x_0,y_0)| < b_1(\varepsilon_1)$$

a contradiction. Similarly we can prove $|u_x(x,y)| < \varepsilon_2$, $|u_y(x,y)| < \varepsilon_3$ and hence the theorem.

Next we prove that the asymptotic stability of the trivial solution of (3), (4) implies the asymptotic stability of trivial solution of (1), (2). It is enough to prove for $\delta_1, \delta_2, \delta_3 > 0$,

$$|u(0,0)| \leqslant \delta_1,\quad |u_x(0,0)| \leqslant \delta_2,\quad |u_y(0,0)| \leqslant \delta_3 \text{ imply,}$$

(12) $$\lim_{\substack{x\to\infty\\ y\to\infty}} |u(x,y)| = 0,\ \lim_{\substack{y\to\infty\\ y\to\infty}} |u_x(x,y)| = 0,\ \lim_{\substack{y\to\infty\\ x\to\infty}} |u_y(x,y)| = 0$$

Since the trivial solution of (3), (4) is asymptotically stable, given $\delta_1', \delta_2', \delta_3' > 0$,

$$|z(0,0)| \leqslant \delta_1',\ |z_x(0,0)| \leqslant \delta_2',\ |z_y(0,0)| \leqslant \delta_3'$$

implies

(13) $$\lim_{\substack{x\to\infty\\ y\to\infty}} |z(x,y)| = 0,\ \lim_{\substack{x\to\infty\\ y\to\infty}} |z_x(x,y)| = 0,\ \lim_{\substack{x\to\infty\\ y\to\infty}} |z_y(x,y)| = 0.$$

As before choose $z(0,0)$, $z_x(0,0)$, $z_y(0,0)$ so that (8) holds and choose $\delta_1, \delta_2, \delta_3 > 0$ such that $|u(0,0)| \leqslant \delta_1$, $|u_x(0,0)| \leqslant \delta_2$, $|u_y(0,0)| \leqslant \delta_3$. we claim that (12) holds, then otherwise there exists a sequence $\{t_n\}$

$$t_n = x_n + y_n,\ t_n \to \infty \text{ such that}$$
$$|u(x,y)| \geqslant \eta_1,\ |u_x(x,y)| \geqslant \eta_2,\ |u_y(x,y)| \geqslant \eta_3,\ t \geqslant t_n,\ \eta_1, \eta_2, \eta_3 > 0.$$

This together with (10), (13) will establish the validity of (12). This completes the proof of the theorem.

Theorem 4. *Let the hypotheses (H_1), (H_2), (H_3) and (H_5) hold. Then the boundedness of solutions of (3), (4) implies the boundedness of solutions of (1), (2).*

Proof. Let M_1, M_2, M_3 and $x, y \geqslant 0$ be given. Assume that the solution of (3), (4) is bounded. Then for $b_1(M_1)$, $b_2(M_2)$, $b_3(M_3)$ and for $x, y \geqslant 0$.

(M_3) $$|z(x,y)| \leqslant b_1(M_1),\ |z_x(x,y)| \leqslant b_2(M_2),\ |z_y(x,y)| \leqslant b_3.$$

Now from the hypothesis (H_3) and Theorem 2 we have

$$b_1(|u(x,y)|) \leqslant V(x,y,u(x,y)) \leqslant r(x,y) \leqslant b_1(M_1)$$
$$b_2(|u_x(x,y)|) \leqslant D_1(x,y,u(x,y)) \leqslant r_x(x,y) \leqslant b_2(M_2)$$
$$b_3(|u_y(x,y)|) \leqslant D_2V(x,y,u(x,y)) \leqslant r_y(x,y) \leqslant b_3(M_3).$$

From the definition of b_1 we get

$$b_1^{-1}[b_1(|u(x,y)|)] \leqslant b_1^{-1}[b_1(M_1)],\ |u(x,y)| \leqslant M_1.$$

Similarly we can prove

$$|u_x(x,y)| \leqslant M_2 \text{ and } |u_y(x,y)| \leqslant M_3 \text{ for } x, y \geqslant 0.$$

Hence the proof of the theorem.

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REFERENCES

1. Agarwal, R. P. and Thandapani, E. — *A comparison result for the Goursat problem*, Matscience Report 97, (1979), 115—120
2. Lakshmikantham, V. and Leela, S. — *Differential and integral inequalities*, vol. 2, Academic press, New York, 1969
3. Lakshmikantham, V. — *On the uniqueness and boundedness of solutions of hyperbolic differential equations*, Proc. Camb. PHIL. Soc. 58 (1962), 583—587
4. Shanahan, J. P. — *On uniqueness questions for hyperbolic differential equations*, Pacific J. Math, 10 (1960), 677—688
5. Walter, W. — *Differential and integral inequalities*, Springer-Verlag, New York, 1970

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EXISTENCE THEOREMS FOR NONLINEAR INTEGRO-DIFFERENTIAL SYSTEMS WITH UPPER AND LOWER SOLUTIONS

BY

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1. **Introduction.** In this paper we study the existence of solutions of the nonlinear integrodifferential system of the form

$$x'(t)=F\left(t,\ x(t),\ \int_0^t k[t,s,x(s)]ds\right),\qquad x(0)=x_0, \tag{1}$$

provided we know the existence of a lower solution v and upper solution w of (1) with $v\leqslant w$ in $I=[0,T]$, where $T>0$ is finite but can be arbitrarily large. Here $x\in C[I,R^r]$, $k\in C[I\times I\times R^r,R^r]$, $F\in C[I\times R^r\times R^r,R^r]$ and R^r denote the r-dimensional Euclidean space. The problem (1) includes as special cases the initial value problems for ordinary differential equations and various types of nonlinear integrodifferential equations studied by many authors in the literature, see [8, 10, 15] and the references given therein. For equations of the form (1) some interesting results of the type which we consider here have been obtained by V.S. Ruchinskii [13, 14] by using Chaplygin's method [7]. The results presented in this paper are derived in a rather simple unified way, which makes further applications to more general boundary value problems easier. The main tool employed in our analysis is based on the monotone iterative method and the notion of upper and lower solutions. Besides their theoretical value, the inclusion statements established in this paper can be applied to obtain numerical error bounds for approximate solutions.

2. **Main Results.** In this section, we establish our main results on the existence of maximal and minimal solutions of (1) by using the monotone iterative method which in turn yields the upper and lower bounds on the exact solution of the equation (1). Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise.

Definition 1. *The function $W\in C^1[I,R^r]$ is called an upper solution of* (1) *if*

$$w'(t)\geqslant F\left(t,\ w(t),\ \int_0^t k[t,s,w(s)]\,ds\right),\qquad w(0)\geqslant x_0. \tag{2}$$

Similarly, a function $v\in C^1[I,R^r]$ is called a lower solution of (1) *if*