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EXISTENCE THEOREMS FOR NONLINEAR INTEGRO-DIFFERENTIAL SYSTEMS WITH UPPER AND LOWER SOLUTIONS

BY

B. G. PACHPATTE

1. **Introduction.** In this paper we study the existence of solutions of the nonlinear integrodifferential system of the form

$$x'(t) = F\left(t, x(t), \int_0^t k[t, s, x(s)]ds\right), \qquad x(0) = x_0, \tag{1}$$

provided we know the existence of a lower solution v and upper solution w of (1) with $v \leqslant w$ in $I = [0, T]$, where $T > 0$ is finite but can be arbitrarily large. Here $x \in C[I, R^r]$, $k \in C[I \times I \times R^r, R^r]$, $F \in C[I \times R^r \times R^r, R^r]$ and R^r denote the r-dimensional Euclidean space. The problem (1) includes as special cases the initial value problems for ordinary differential equations and various types of nonlinear integrodifferential equations studied by many authors in the literature, see [8, 10, 15] and the references given therein. For equations of the form (1) some interesting results of the type which we consider here have been obtained by V.S. Ruchinskii [13, 14] by using Chaplygin's method [7]. The results presented in this paper are derived in a rather simple unified way, which makes further applications to more general boundary value problems easier. The main tool employed in our analysis is based on the monotone iterative method and the notion of upper and lower solutions. Besides their theoretical value, the inclusion statements established in this paper can be applied to obtain numerical error bounds for approximate solutions.

2. **Main Results.** In this section, we establish our main results on the existence of maximal and minimal solutions of (1) by using the monotone iterative method which in turn yields the upper and lower bounds on the exact solution of the equation (1). Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise.

Definition 1. *The function $W \in C^1[I, R^r]$ is called an upper solution of* (1) *if*

$$w'(t) \geqslant F\left(t, w(t), \int_0^t k[t, s, w(s)]\, ds\right), \qquad w(0) \geqslant x_0. \tag{2}$$

Similarly, a function $v \in C^1[I, R^r]$ is called a lower solution of (1) *if*

$$v'(t) \leqslant F\left(t, v(t), \int_0^t k[t, s, v(s)]ds\right), \qquad v(0) \leqslant x_0.$$

Definition 2. *The functions α and β, $\alpha, \beta \in C^1[I, R^r]$ are called minimal and maximal solutions of* (1) *respectively, if every other solution $x \in C^1[I, R^r]$ of* (1) *satisfies the relation $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for $t \in I$.*

We list for convenience the following assumptions:

(H_1) the functions $v, w \in C^1[I, R^r]$ with $v \leqslant w$ on I are lower and upper solutions of (1);

(H_2) for each i, $k_i[t, s, x]$ is monotone nondecreasing in x for fixed t and s, and $F_i(t, x, u)$ is monotone nondecreasing in u for fixed t and x;

(H_3) for each i, $F_i(t, x, u) - F_i(t, \overline{x}, u) \geqslant -M(x_i - \overline{x}_i)$ whenever $v(t) \leqslant \overline{x} \leqslant x \leqslant w(t)$ for $t \in I$, where $M \geqslant 0$ is a constant.

For any $\eta \in C^1[I, R^r]$ such that $v \leqslant \eta \leqslant w$ on I, consider the following linear system

$$x_i'(t) = F_i\left(t, \eta(t), \int_0^t k[t, s, \eta(s)]ds\right) - M(x_i(t) - \eta_i(t)), \quad x_i(0) = x_0, \tag{3}$$

which possesses a unique solution x defined on I. For each $\eta \in C^1[I, R^r]$ such that $v \leqslant \eta \leqslant w$ on I, define the mapping A by $A\eta = x$, where x is the unique solution of (3). This mapping will be used to define the sequences that converge to the minimal and maximal solutions of (1).

We next state and prove the following lemma which forms the basis of our main results.

Lemma 1. *Assume that the hypotheses* (H_1)–(H_3) *hold. Then*

(i) *$v \leqslant Av$, $w \geqslant Aw$ on I;*

(ii) *A is monotone operator on the segment*

$$[v, w] = \{x \in R^n : v(t) \leqslant x \leqslant w(t),\ t \in I\}.$$

Proof. To prove (i), set $Av = x$, where x is the unique solution of (3) corresponding to v. Setting $\Phi_i(t) = x_i(t) - v_i(t)$, we have $\Phi_i(0) = x_i(0) - v_i(0) \geqslant x_0 - x_0 = 0$ and

$$\Phi_i'(t) \geqslant F_i\left(t, v(t), \int_0^t k[t, s, v(s)]ds\right) - M(x_i(t) - v_i(t))$$

$$- F_i\left(t, v(t), \int_0^s k[t, s, v(s)]\,ds\right) = -M\Phi_i(t),$$

which implies $\Phi_i(t) \geqslant 0$, i.e. $x_i(t) \geqslant v_i(t)$ and hence $v(t) \leqslant x(t)$ for $t \in I$. Thus we have proved $v \leqslant Av$ for $t \in I$. Similar argument shows that $w \geqslant Aw$ for $t \in I$, proving (i).

To prove (ii), let $\zeta, \eta \in C[I, R^r]$, be such that $\zeta, \eta \in [v, w]$ and $\zeta \leqslant \eta$. Suppose that $x = A\zeta$ and $y = A\eta$. Then setting $\Phi_i(t) = x_i(t) - y_i(t)$, we have $\Phi_i(0) = x_i(0) - y_i(0) = x_0 - x_0 = 0$ and $\Phi_i'(t) = F_i\left(t, \zeta(t), \int_0^t k[t, s, \zeta(s)]ds\right) -$

$$- M(x_i(t) - \zeta_i(t)) - F_i\left(t, \eta(t), \int_0^t k[t, s, \eta(s)]ds\right) + M(y_i(t) - \eta_i(t)) \leqslant$$

$$\leqslant F_i\left(t, \zeta(t), \int_0^t k[t,s, \eta(s)]\,ds\right) - M(x_i(t) - \zeta_i(t)) - F_i\left(t, \eta(t), \int_0^t k[t, s, \eta(s)]ds\right) +$$

$$+ M(y_i(t) - \eta_i(t)) \leqslant M(\eta_i(t) - \zeta_i(t)) - M(x_i(t) - \zeta_i(t)) + M(y_i(t) - \eta_i(t)) = -M\Phi_i(t)$$

which implies $x_i(t) \leqslant y_i(t)$ i.e. $x(t) \leqslant y(t)$ for $t \in I$. It therefore follows that the mapping A is monotone on the segment $[v, w]$. This completes the proof of the lemma.

In view of Lemma 1, we can define the sequences $v_n = Av_{n-1}$, $w_n = Aw_{n-1}$, with $v_0 = v$ and $w_0 = w$. It is easy to observe that the sequences $\{v_n\}$ and $\{w_n\}$ are monotone nondecreasing, nonincreasing respectively, and $v \leqslant v_n \leqslant w_n \leqslant w$ on I. Furthermore, using standard arguments (see, [1], [6]) it follows that these sequences converge uniformly and monotonically to solutions α and β of (1).

Let x be any solution of (1) such that $x \in [v, w]$. Then, by the induction argument, it is easily seen that $x \leqslant w_n$ and $x \geqslant v_n$ for every $n = 0, 1, 2, \ldots$ Hence, we have $\alpha \leqslant x \leqslant \beta$. This shows that β is a maximal solution and α is a minimal solution of (1) on I. The above observation leads to the following:

Theorem 1. *Assume that the hypotheses* (H_1)–(H_3) *hold. Then the sequence $\{w_n\}$ converges uniformly from above to a maximal solution β of* (1) *while the sequence $\{v_n\}$ converges uniformly from below to a minimal solution α of* (1). *Furthermore, if x is any solution of* (1) *such that $x \in [v, w]$, then*

$$v \leqslant v_1 \leqslant \ldots \leqslant v_n \leqslant \ldots \leqslant \alpha \leqslant x \leqslant \beta \leqslant \ldots \leqslant w_n \leqslant \ldots \leqslant w_1 \leqslant w, \tag{4}$$

on I.

We note that, if the solutions of (1) are unique, then the assumptions of Theorem 1 imply that $\alpha(t) = x(t) = \beta(t)$ for $t \in I$. In this case, the iteration scheme (4) has the advantage that $\{v_n\}$ and $\{w_n\}$ converge to x on I and $v_n \leqslant x \leqslant w_n$. That is an approximate solution can be constructed to any degree of accuracy by making $|v_n - w_n|$ sufficiently small.

3. Application to Boundary Value Problem. Recently, in [11], J. Morchalo has studied the existence and uniqueness of solutions of the boundary value problem (*BVP* for short)

$$y''(t) = F\left(t, y(t), \int_0^t k[t, s, y(s)]ds)\right), \tag{5}$$

with the given boundary conditions

$$y(0) = y(b) = 0. \tag{6}$$

In [11] the results are obtained by using the general idea of Chaplygin [7] which yield the lower and upper bounds on the solutions of (5)–(6). In this section we apply the monotone method developed in Section 2 to more general boundary value problem associated with the systems of integrodifferential equations of the form

$$x' = f\left(t, x, y, \int_0^t k[t, s, x, y]ds\right), \quad y' = g\left(t, x, x, y, \int_0^t h[t, x, s, y]ds\right), \tag{7}$$

with the given boundary conditions

(8) $$x(a)=x_a,\ y(b)=y_b,$$

where $k,\ h\in C[J\times J\times R^r\times R^r,\ R^r]$, $f,g\in C[J\times R^r\times R^r\times R^r,\ R^r]$, $J=[a,b]$ and x_a and y_b are given vectors. In [2—5, 12, 16] the authors have studied the existence and uniqueness of the solutions of (7) when $k=h=0$ satisfying certain boundary conditions by using various techniques. In the special case when

$$x'=f\left(t,x,y,\int_0^t k[t,s,x,y]ds\right)=y,\quad y'=g\left(t,x,y,\int_0^t h[t,s,x,y]ds\right),$$

the BVP (7)—(8) reduces to the following boundary value problem

(9) $$x''=g\left(t,x,x',\int_0^t h[t,s,x,x']ds\right),$$

with boundary conditions

(10) $$x(a)=x_a,\qquad x'(b)=y_b,$$

which in turn is a further generalization of the BVP (5)—(6) and contains as a special case, the boundary value problems studied by many authors in the literature when $h=0$ (see, [4] and the references given therein).

Definition 3. *A pair of functions* $(\bar v,\bar w)$, $\bar v,\ \bar w\in C^1[J,R^r]$ *is called an upper solution of BVP* (7)—(8) *if*

$$\bar v'\geqslant f\left(t,\bar v,\bar w,\int_0^t k[t,s,\bar v,\bar w]ds\right),\qquad \bar v(a)\geqslant x_a,$$

$$\bar w'\leqslant g\left(t,\bar v,\bar w,\int_0^t h[t,s,\bar v,\bar w]ds\right),\qquad \bar w(a)\leqslant y_b$$

and similarly a pair of functions $(\underline{v},\underline{w})$, $\underline{v},\ \underline{w}\in C^1[J,R^r]$ *is called a lower solution of BVP* (7)—(8) *if*

$$\underline{v}'\leqslant f\left(t,\underline{v},\underline{w},\int_0^t k[t,s,\underline{v},\underline{w}]ds\right),\qquad \underline{v}(a)\leqslant x_a,$$

$$\underline{w}'\geqslant g\left(t,\underline{v},\underline{w},\int_0^t h[t,s,\underline{v},\underline{w}]ds\right),\qquad \underline{w}(b)\geqslant y_b.$$

Definition 4. *The pairs of functions* (α,β) *and* $(\bar\alpha,\bar\beta)$ *are called* ***minimal*** *and maximal solutions of BVP* (7) — (8) *respectively, if every other* ***solution*** (x,y) *of BVP* (7)—(8) *satisfies the relation*

$$\alpha(t)\leqslant x(t)\leqslant\bar\alpha(t),\qquad \beta(t)\leqslant y(t)\leqslant\bar\beta(t),$$

for all $t\in J$.

We consider the BVP (7)—(8) under the following hypotheses:

(G_1) the functions $\underline{u}=(\underline{v},\underline{w})$, $\bar u=(\bar v,\bar w)$, $\underline{v}$, $\underline{w}$, $\bar v$, $\bar w\in C[J,R^r]$ with $\underline{u}(t)\leqslant\bar u(t)$ for $t\in J$ are lower and upper solutions of BVP (7)—(8);

(G_2) for each i, $k_i[t,s,x,y]$ and $h_i[t,s,x,y]$ are monotone nondecreasing in x and y for fixed t and s, $f_i(t,x,y,z)$ is monotone nondecreasing in y and z for fixed t and x, $g_i(t,x,y,z)$ is monotone decreasing in x and z for fixed t and y;

(G_3) for each i, $f_i(t,x,y,z)-f(t,\bar x,y,z)\geqslant -M(x_i-\bar x_i)$, whenever $\underline{v}(t)\leqslant x\leqslant\bar x\leqslant\bar v(t)$ for $t\in J$ and $g_i(t,x,y,z)-g_i(t,x,\bar y,z)\leqslant -M(y_i-\bar y_i)$, whenever $\underline{w}(t)\leqslant\bar y\leqslant y\leqslant\bar w(t)$ for $t\in J$, where $M\geqslant 0$ is a constant.

For any $\mu=(\xi,\eta)$, $\xi,\eta\in C[J,R^r]$ such that $\underline{u}\leqslant\mu\leqslant\bar u$ on J, consider the following linear system

(11) $$\begin{aligned} x_i'&=f_i\left(t,\xi,\eta,\int_0^t k[t,s,\xi,\eta]ds\right)-M(x_i-\xi_i),\quad x_i(a)=x_a,\\ y_i'&=g_i\left(t,\xi,\eta,\int_0^t h[t,s,\xi,\eta]ds\right)-M(y_i-\eta_i),\quad y_i(b)=y_b,\end{aligned}$$

which possess a unique solution $u=(x,y)$ defined on J. For each $\mu=(\xi,\eta)$, $\xi,\eta\in C[J,R^r]$ such that $\underline{u}(t)\leqslant\mu(t)\leqslant\bar u(t)$ for $t\in J$, define the mapping B by $B\mu=u$, where $u=(x,y)$ is the unique solution of (11).

The following result concerning the mapping B holds.

Lemma 2. *Assume that the hypotheses* (G_1)—(G_3) *hold. Then*

(i) $\underline{u}\leqslant B\underline{u}$, $\bar u\geqslant B\bar u$;

(ii) *B is monotone operator on the segment*

$$[\underline{u},\bar u]=\{z=(z_1,z_2),\ z_1,z_2\in R^r:\underline{u}(t)\leqslant z\leqslant\bar u(t),\ t\in J\}.$$

The details of the proof of this lemma follow by an argument similar to that given in Lemma 1 with suitable modifications (see, also [9, 12]). We do not discuss it here.

In view of Lemma 2, we can define the sequences $\underline{u}_n=B\underline{u}_{n-1}$, $\bar u_n=B\bar u_{n-1}$, where $\underline{u}_n=(\underline{v}_n,\underline{w}_n)$, $\bar u_n=(\bar v_n,\bar w_n)$ with $\underline{u}_0=\underline{u}$, and $\bar u_0=\bar u$. Then by following the same argument as in section 2, it is easy to establish the following theorem.

Theorem 2. *Let the assumptions* (G_1)—(G_3) *hold. Then the sequence* $\{(\bar v_n,\bar w_n)\}$ *converges uniformly from above to a maximal solution* $(\bar\alpha,\bar\beta)$ *of* BVP (7) — (8) *while the sequence* $\{(\underline{v}_n,\underline{w}_n)\}$ *converges uniformly from below to a minimal solution* (α,β) *of* BVP (7)—(8). *Furthermore, if* (x,y) *is any solution of* BVP (7)—(8) *such that* $\underline{v}\leqslant x\leqslant\bar v$, $\underline{w}\leqslant y\leqslant\bar w$, *then*

$$\underline{v}\leqslant\underline{v}_1\leqslant\ldots\leqslant\underline{v}_n\leqslant\ldots\leqslant\alpha\leqslant x\leqslant\bar\alpha\leqslant\ldots\leqslant\bar v_n\leqslant\ldots\leqslant\bar v_1\leqslant\bar v,$$

$$\underline{w}\leqslant\underline{w}_1\leqslant\ldots\leqslant\underline{w}_n\ldots\leqslant\beta\leqslant y\leqslant\bar\beta\leqslant\ldots\leqslant\bar w_n\leqslant\ldots\leqslant\bar w_1\leqslant\bar w,$$

on J.

We note that the maximal and minimal solutions obtained in Theorem 2 are not necessarily the same. However, if the functions involved in (7) satisfy uniqueness conditions, then we obtain $\underline{u}(t)=u(t)=\bar u(t)$ for $t\in J$. For the uniqueness result of this type, we refer the interested readers to our recent paper [9, Theorem 2].

In our opinion, the main advantages of the approach presented in this paper is that our results yield the inclusion and existence statements for the solutions of equation (1) and BVP (7)—(8) at the same time. Althougth the results of this paper have a superficial similarity with the papers of Ruchinskii [13, 14] and Morchalo [11] it is easily seen that the assertions and the proofs of the results established in this paper are decidedly distinct from those of [11, 13, 14].

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AN EQUIVARIANT SHAPE THEORY

BY

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0. Introduction. Shape theory was first introduced by Karol Borsuk, [2], for compact metric spaces. The notion of „shape" was extended to the compact Hausdorff spaces by S. Mardesic and J. Segal [18], [19], and also by W. Holsztynski [12], and then to metric spaces by K. Borsuk [3] and R. Fox [11]. In 1972 S. Mardesic established shape theory for arbitrary topological spaces [20], and in 1975 K. Morita proved in [23] that the notion of shape map can be described using ANR's inverse homotopy systems. K. Morita also showed that the Čech homology and cohomology constructed by normal open coverings of topological spaces are shape invariants.

In 1980 F. W. Bauer [1] defined shape theory for an arbitrary category K of topological spaces with respect to a fine subcategory P of the category K, as the study of those invariants of a topological space X in K which are determined by the maps from X to the fine spaces of the subcategory P.

Recently we showed [26], [27] and [28], [29] that shape theory can also be extended to other categories by constructing, following the model of Mardesic-Morita, a shape theory for Boolean closure algebras and for spaces with two topologies.

In 1979 Yu. M. Smirnov proved [30] the possibility of the construction of a shape theory in the sense of K. Borsuk for a category K of G-spaces in the following situations: 1) G is locally compact and σ-compact, and K is the category of metrizable G-spaces; 2) G is compact, and K is the category of compact G-spaces.

In this paper we construct an equivariant shape theory for the category of arbitrary G-spaces, the group G being assumed finite.

We will know afterwards that the transformation groups theory includes a series of aspects. In this order G. Bredon [5], [6] and Th. Bröker [7] have constructed for discrete groups, the equivariant homology and cohomology with prescribed coefficients. In this case (of actions of discrete groups), the Cech cohomology based on invariant coverings was developed (see [4, p. 135]). Sören Illmann [14] has studied the equivariant singular homology and cohomology in the Eilenberg-Steenrod axiomatical sense. I. M. James & G. B. Segal [15], [16], T. Matumoto [22], G. Bredon [5] have developed the theory of the equivariant homotopy.