Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, Matematică, 1985, f. 2

HYPERHARMONIC CONES AND CONES OF HYPERHARMONICS

BY

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Introduction. E. Popa has defined a cone of hyperharmonics by using H-integrals ([5], p. 53) and a morphism of H-cones ([5], p. 57). A hyperharmonic cone has been introduced in [4], p. 9, and a hyperharmonic morphism in [4], p. 38. The main result in this paper is that any cone of hyperharmonics is a hyperharmonic cone and any morphism of H-cones is a hyperharmonic morphism.

1. **Preliminaries.** We recall the definition of a hyperharmonic cone ([4], p. 9) and a hyperharmonic morphism ([4], p. 38).

Definition 1.1. *An ordered convex cone* $(W, +, \leqslant)$ *is called a hyperharmonic cone if the following axioms hold:*

(H_1) *for any non-empty upper directed family* $F \subset W$ *there exists the least upper bound* $\vee F$ *satisfying*

$$\vee(x+F)=x+\vee F$$

for all $x \in W$,

(H_2) *for any non-empty family* $F \subset W$ *there exists the greatest lower bound* $\wedge F$ *satisfying*

$$\wedge(x+F)=x+\wedge F$$

for all $x \in W$,

(H_3) *for all* $u, v_1, v_2 \in W$ *such that* $u \leqslant v_1+v_2$ *there exist elements* $u_1, u_2 \in W$ *satisfying* $u_1 \leqslant v_1$, $u_2 \leqslant v_2$ *and* $u=u_1+u_2$.

Definition 2.1. *Let* V *and* W *be hyperharmonic cones. A mapping* $\varphi : V \to W$ *is called continuous in order from below if for any non-empty upper directed family* $F \subset V$,

$$\varphi(\vee F)=\bigvee_{v\in F}\varphi(v).$$

A mapping $\varphi : V \to W$ is called a *hyperharmonic morphism* if φ is additive and continuous in order from below.

For other notations and concepts not defined or explained in this text we refer to [4]. We also assume that the reader is familiar with the theory of H-cones, see *e.g.* [2], and especially the vector lattice properties of H-cones (see [2], p. 39—40). We also apply the algebraic potential theory [1].

In this paper, we frequently use the following notations:

$$A \wedge B=\{a \wedge b \mid a \in A,\ b \in B\}, \qquad A \vee B=\{a \vee b \mid a \in A,\ b \in B\},$$

$$\{a\} \wedge B=a \wedge B, \qquad \{a\} \vee B=a \vee B.$$

2. **Hyperharmonic cones and cones of hyperharmonics.** We first prove (Proposition 2.2) that every H-cone S can be extended to a hyperharmonic cone W satisfying the axiom (H_1) (see [4], p. 51).

(H_1) *for any non-empty, upper directed family* $F \subset W$ *and any* $x \in W$, $\bigvee (x \wedge F) = x \wedge (\bigvee F)$.

We remind that H-cones satisfy (H_1) for all non-empty upper directed and bounded families ([2], (c) p. 40). In the proof of Proposition 2.2 we use some ideas applied earlier by D o n n e r ([3], p. 6—10). Since H-cones are not distributive and the substraction is not generally defined, the result and the proof of D o n n e r ([3], 1.4 Theorem) are not directly applicable.

Lemma 2.1. *Let S be an H-cone. If $F \subset S$ and $G \subset S$ are non-empty upper directed families and $s \in S$, then*

$$\bigvee (s \wedge F) \leqq \bigvee (s \wedge G)$$

if and only if

$$\bigvee (s \wedge (t+F)) \leqq \bigvee (s \wedge (t+G))$$

for all $t \in S$.

Proof. Suppose that

$$\bigvee (s \wedge F) \leqq \bigvee (s \wedge G).$$

Let t belong to S. Using the Riesz decomposition property in S, we easily get $(u+v) \wedge w = (u \wedge w + v \wedge w) \wedge w$ for all $u, v, w \in S$. Since S is and H-cone, we obtain

$$\bigvee (s \wedge (t+F)) = \bigvee (s \wedge (s \wedge t + s \wedge F)) = s \wedge (\bigvee (s \wedge t + s \wedge F)) = s \wedge (s \wedge t +$$
$$+ \bigvee (s \wedge F)) \leqq s \wedge (s \wedge t + \bigvee (s \wedge G)) = s \wedge (\bigvee (s \wedge t + s \wedge G)) = \bigvee (s \wedge (t+G))$$

for all $t \in S$.

Proposition 2.2. *Every H-cone S can be extended to a hyperharmonic cone W satisfying (H_1) such that S is solid in order and dense in order in W. Moreover, this extension is unique up to an isomorphism.*

Proof. Let S be an H-cone and denote the family of all non-empty upper directed subsets of S by Ω. Define the relation $\sim$ on Ω by setting

$$F \sim G \Leftrightarrow \{\bigvee (s \wedge F) = \bigvee (s \wedge G) \text{ for all } s \in S\}.$$

Clearly this is an equivalence relation. Denote the equivalence classes relative to $\sim$ by $[F]$, $F \in \Omega$, and further $[\mathcal{F}] = \{[F] | F \in \mathcal{F} \subset \Omega\}$. If $s \in S$, we denote $[\{s\}] = [s]$. We define the addition and the multiplication with strictly positive real numbers by

$$[F] + [G] = [F+G],\ F, G \in \Omega,\quad \alpha[F] = [\alpha F],\ \alpha \in \mathbb{R}_+ \setminus \{0\},\ F \in \Omega.$$

Plainly, the multiplication with strictly positive real numbers is well defined. We also see using Lemma 2.1 that the addition is well defined.

We define a partial ordering $\leqq$ in $[\Omega]$ by

$$[F] \leqq [G] \text{ if } \bigvee (s \wedge F) \leqq \bigvee (s \wedge G) \text{ for all } s \in S.$$

Assume $F, G \in \Omega$ be such that $[F] \leqq [G]$. Let $A \in \Omega$. By Lemma 2.1, we get

$$\bigvee (s \wedge (a+F)) \leqq \bigvee (s \wedge (a+G))$$

for all $a \in A$. Therefore we obtain $\bigvee s \wedge (A+F) \leqq \bigvee (s \wedge (A+G))$, which means that

$$[A] + [F] = [A+F] \leqq [A+G] = [A] + [G].$$

The remaining properties to show that $([\Omega], +, \leqq)$ is an ordered convex cone are trivial.

To prove $([\Omega], +, \leqq)$ is a hyperharmonic cone, we need the following results:

$$s \leqq t \Leftrightarrow [s] \leqq [t] \text{ for all } s, t \in S, \tag{2.1}$$

$$[F] = \bigvee_{s \in F} [s] \text{ for all } F \in \Omega, \tag{2.2}$$

$$[F] \vee [G] = [F \vee G] \text{ for all } F, G \in \Omega, \tag{2.3}$$

$$[F] \wedge [G] = [F \wedge G] \text{ for all } F, G \in \Omega. \tag{2.4}$$

The equivalence in (2.1) is trivial. To prove (2.2), let $F \in \Omega$. Plainly, we have $[F] \geqq [s]$ for all $s \in F$. Suppose $G \in \Omega$ to be such that $[G] \geqq [s]$ for all $s \in F$. This means $\bigvee (w \wedge G) \geqq w \wedge s$ for all $s \in F$, which implies $\bigwedge (w \wedge G) \geqq$ $\geqq \bigvee_{s \in F} (w \wedge s)$. Hence $[F] = \bigvee_{s \in F} [s]$.

To prove (2.3), we need to show that $[s \vee t] = [s] \vee [t]$ for all $s, t \in S$. From this we get easily (2.3) by (2.2). Plainly, $[s \vee t] \geqq [s]$ and $[s \wedge t] \geqq [t]$. Let $F \in \Omega$ be a non-empty upper directed family such that $[F] \geqq [s]$ and $[F] \geqq [t]$. Then

$$\bigvee (w \wedge F) \geqq w \wedge s,\quad \bigvee (w \wedge F) \geqq w \wedge t$$

for all $w \in S$. Setting $w = s \vee t$, we obtain $\bigvee ((s \vee t) \wedge F) \geqq s \wedge t$. Since the converse is trivial, we have $\bigvee ((s \vee t) \wedge F) = s \vee t$. From this it follows

$$\bigvee (w \wedge F) \geqq (s \vee t) \wedge \bigvee (w \wedge F) = w \wedge \bigvee ((s \vee t) \wedge F) = (s \wedge t) \wedge w$$

for all $w \in S$, which implies $[F] \geqq [s \vee t]$. Therefore $[s] \vee [t] = [s \vee t]$.

To prove (2.4), let $F, G \in \Omega$. Trivially, $[F] \geqq [F \wedge G]$ and $[G] \geqq [F \wedge G]$. Suppose $A \in \Omega$ to be such that $[A] \leqq [F]$ and $[A] \leqq [G]$. This means $\bigvee (s \wedge A) \leqq$ $\leqq \bigvee (s \wedge F)$, $\bigvee (s \wedge A) \leqq \bigvee (s \wedge G)$, for all $s \in S$. Therefore we have

$$\bigvee (s \wedge A) \leqq (\bigvee (s \wedge F)) \wedge (\bigvee (s \wedge G)) = \bigvee (s \wedge F \vee \bigvee (s \wedge G)) = \bigvee (s \wedge F \wedge G),$$

and thus $[A] \leqq [F \wedge G]$. Therefore (2.4) holds.

To prove that $([\Omega], +, \leqq)$ is a hyperharmonic cone, we use [4], Theorem 2.3. To show (H_1), let $([G_\alpha])_{\alpha \in I}$ be a non-empty upper directed family in $[\Omega]$. Denote

$$G = \bigcup_{\substack{n \in N \\ \alpha_i \in I}} G_{\alpha_1} \vee \ldots \vee G_{\alpha_n}. \tag{2.5}$$

Then G is upper directed and hence $G \in \Omega$. By induction and (2.3), we obtain

$$[G_{\alpha_1} \vee \ldots \vee G_{\alpha_n}] = [G_{\alpha_1}] \vee \ldots \vee [G_{\alpha_n}] \tag{2.6}$$

for all $n \in N$, $\alpha_i \in I$, $i = 1, .., n$. Plainly, $[G] \geqq [G_\alpha]$ for all $\alpha \in I$. Suppose

$F\in\Omega$ to be such that $[F]\geqslant[G_\alpha]$ for all $\alpha\in I$. Then $[F]\geqslant[G_{\alpha_1}]\wedge\ldots\wedge[G_{\alpha_n}]$ for all $n\in N$, $\alpha_i\in I$, $i=1,\ldots,n$, which implies $[F]\geqslant[G]$ by (2.6). Hence

$$[G]=\bigvee_{\alpha\in I}[G_\alpha]. \tag{2.7}$$

Let $a\in S$ and assume $\alpha,\beta\in I$. Then there exists $\gamma\in I$ such that $[G_\gamma]\geqslant[G_\alpha]$ and $[G_\gamma]\geqslant[G_\beta]$. By (2.3), we get $[G_\gamma]\geqslant[G_\alpha]\vee[G_\beta]=[G_\alpha\vee G_\beta]$. Therefore

$$\bigvee_{\alpha\in I}[G_\alpha+a]\geqslant[G_\gamma+a]\geqslant[G_\alpha\vee G_\beta+a].$$

By induction, we obtain $\bigvee_{\alpha\in I}[G_\alpha+a]\geqslant[G+a]=[G]+[a]$. Since the converse is trivial, we have

$$[a]+\bigvee_{\alpha\in I}[G_\alpha]=[a]+[G]=\bigvee_{\alpha\in I}[G_\alpha+a]=\bigvee_{\alpha\in I}([G_\alpha]+[a]).$$

Let $A\in\Omega$. Then we have $\bigvee_{\alpha\in I}([G_\alpha]+[A])\geqslant[f]+[a]$ for all $f\in G_\alpha$, $\alpha\in I$, $a\in A$. From this it follows by (2.1)

$$\bigvee_{\alpha\in I}([G_\alpha]+[A])\geqslant\bigvee_{a\in A}\Big(\bigvee_{\substack{f\in G_\alpha\\ \alpha\in I}}([f]+[a])\Big)\geqslant\bigvee_{a\in A}([a]+[G])=\bigvee_{a\in A}[a+G]=[A]+[G].$$

Since the converse is trivial, $\vee$ is translation-invariant in $[\Omega]$.

Let $([G_\alpha])_{\alpha\in I}$ be an arbitrary family in $[\Omega]$. The set of lower bounds of $([G_\alpha])_{\alpha\in I}$ is plainly non-empty and upper directed by (2.3). Hence $\bigwedge_{\alpha\in I}[G_\alpha]$ exists. To show that $[\Omega]$ is a hyperharmonic structure, we first prove that the set

$$V=\{[X]\in[\Omega]\mid[F]\leqslant[A]+[X]\}$$

has the least element for all $F, A\in\Omega$. Let $F, A\in\Omega$. Then

$$[f]\leqslant[F]\leqslant[A]+[X]=[A+X]$$

for all $[X]\in V$ and all $f\in F$. This means that

$$w\wedge f\leqslant\vee(w\wedge(A+X))$$

for all $w\in S$ and all $f\in F$. Setting $w=f$, we get for any $f\in F$

$$f\leqslant\vee(f\wedge(A+X))\leqslant\vee(f\wedge A)+\vee(f\wedge X).$$

which implies $S_{\vee(f\wedge A)}f\leqslant\vee(f\wedge X)$ for all $f\in F$. Therefore $w\wedge S_{\vee(f\wedge A)}f\leqslant w\wedge\vee(f\wedge X)=\vee(w\wedge f\wedge X)\leqslant\vee(w\wedge X)$ for all $w\in S$ and $f\in F$. Hence $[X]\geqslant[S_{\wedge(f\wedge A)}f]$ for all $f\in F$. The family $M=\{S_{\vee(f\wedge A)}f\mid f\in F\}$ is upper directed. Indeed, suppose $f_1, f_2\in F$ and let $g\in F$ be such that $g\geqslant f_1$, $g\geqslant f_2$. Since $g\leqslant S_{\vee(g\wedge A)}g+\vee(g\wedge A)$ and $(u+v)\wedge s\leqslant u\wedge s+v\wedge s$ for all $u, v, s\in S$ by $(H3)$, we get

$$\begin{aligned}f_i=f_i\wedge g&\leqslant f_i\wedge(S_{\vee(g\wedge A)}g+\vee(g\wedge A))\leqslant S_{\vee(g\wedge A)}g+f_i\wedge(\vee(g\wedge A))=\\&=S_{\vee(g\wedge A)}g+\vee(f_i\wedge g\wedge A)=S_{\vee(g\wedge A)}g+\vee(f_i\wedge A)\end{aligned}$$

for all $i=1,2$. Hence $S_{\vee(g\wedge A)}g\geqslant S_{\vee(f_i\wedge A)}f_i$ for $i=1,2$ and therefore the family

M is upper directed. By (2.2), we get $[X]\geqslant[M]$ for all $[X]\in V$. Since $f\leqslant S_{\vee(f\wedge A)}f+\wedge(f\wedge A)$ for all $f\in F$, we have $[f]\leqslant[M]+[A]$ for all $f\in F$ by (2.1) and (2.2). Hence $[F]\leqslant[M]+[A]$ which means that $[M]\in V$. Now we have proved $\wedge V=[M]$.

To show finally that $[M]<[F]$, we have to prove the translation invariance of $\wedge$. Suppose that $[F]$ is a lower bound of $([G_\alpha+A])_{\alpha\in I}$. Then $[F]\leqslant[G_\alpha]+[A]$ for all $\alpha\in I$. This implies $[G_\alpha]\geqslant[M]$ for all $\alpha\in I$ and further $[M]\leqslant\bigwedge_{\alpha\in I}[G_\alpha]$. Therefore we obtain

$$[F]\leqslant[A]+[M]\leqslant[A]+\bigwedge_{\alpha\in I}[G_\alpha],$$

since $[M]\in V$. Moreover, $[A]+\bigwedge_{\alpha\in I}[G_\alpha]$ is clearly a lower bound of $([G_\alpha]+[A])_{\alpha\in I}$. Thus $\wedge$ is translation-invariant in $[\Omega]$.

Now we continue the proof of $[M]<[F]$. Since S is a mixed lattice semigroup by ([4], Theorem 3.14), we remind that $S_x u+Q_x u=u$ for all $u, x\in S$ ([1], (17.26)). Denote for $f\in F$

$$w_f=\bigwedge_{\substack{h\geqslant f\\ h\in F}}[Q_{\vee(f\wedge A)}h].$$

Then we get $[S_{\vee(f\wedge A)}f]+\bigwedge_{h\geqslant f}[Q_{\vee(f\wedge A)}h]\leqslant\bigwedge_{h\geqslant f}[h]=[F]$, for all $f\in F$. Therefore $[M]+\bigvee_{f\in F}[w_f]\leqslant[F]$, since $\vee$ is translation-invariant. The converse inequality follows from the inequality

$$[M]+[Q_{\vee(h\wedge A)}h]\geqslant[S_{\vee(h\wedge A)}h]+[Q_{\vee(h\wedge A)}h]=[h]=[S_{\vee(f\wedge A)}h]+[Q_{\vee(f\wedge A)}h]\geqslant[f]$$

for all $h\geqslant f$ by taking first the infimum $\wedge$ and then the supremum $\vee$. Consequently, we have $[M]+\bigvee_{f\in F}[w_f]=[F]$ which means $[M]<[F]$. Hence $[\Omega]$ is a hyperharmonic cone.

To show (H_1), let $([G_\alpha])_{\alpha\in I}$ be a non-empty upper directed family in $[\Omega]$ and let $[F]$ belong to $[\Omega]$. Denote G as before in (2.5). We have to show that

$$[G\wedge F]=[G]\wedge[F]=\bigvee_{\alpha\in I}([G_\alpha]\wedge[F])=\bigvee_{\alpha\in I}[G_\alpha\wedge F]$$

by (2.4). Let $g_\alpha\in G_\alpha$ and $g_\beta\in G_\beta$ for $\alpha,\beta\in I$. Then there exists $\gamma\in I$ such that $[G_\gamma]\geqslant[G_\alpha]\geqslant[g_\alpha]$ and $[G_\gamma]\geqslant[G_\beta]\geqslant[g_\beta]$. From (2.3) we obtain $[G_\gamma]\geqslant[g_\alpha\vee g_\beta]$. Using (2.4), we further get

$$\bigvee_{\alpha\in I}[G_\alpha\wedge F]\geqslant[G_\gamma\wedge F]\geqslant[(g_\alpha\vee g_\beta)\wedge F].$$

By induction and (2.2), we obtain $\bigvee_{\alpha\in I}[G_\alpha\wedge F]\geqslant[G\wedge F]$. Since the converse is trivial, (H_1) holds in $[\Omega]$.

Suppose $[F]\leqslant[s]$ for $F\in\Omega$ and $s\in S$. Then $\vee(w\wedge F)\leqslant w\wedge s\leqslant s$ for all $w\in S$ which implies

$$\vee(w\wedge F)=s\wedge\vee(w\wedge F)=w\wedge\vee(s\wedge F).$$

for all $w\in S$. This means that $[F]=[\vee(s\wedge F)]$. Consequently the set $[S]$ is solid in order. By (2.2), it is also dense in order in $[\Omega]$.

To prove uniqueness, we may identify $[s]$ by s and denote $[\Omega]=W$. Then S is dense in order in W. Let W^* be another hyperharmonic cone satisfying (H_1) such that S is solid and dense in order in W^*. Define a mapping $\varphi : W^* \to W$ by

$$\varphi(x)=\bigvee_{\substack{s\leq x\\ s\in S}} s,$$

where $\bigvee$ is taken in W. Let $x, y\in W^*$. Since W^* is a hyperharmonic cone, for any $s\leqq x+y$, $s\in S$, there exist s_1, $s_2\in W^*$ such that $s=s_1+s_2$, $s_1\leqq x$, $s_2\leqq y$. Since S is solid in W^*, we get $s_1\in S$ and $s_2\in S$. From this it follows

$$\varphi(x+y)=\bigvee_{\substack{s\leqq x+y\\ s\in S}} s=\bigvee_{\substack{s\leqq x\\ s\in S}} s+\bigvee_{\substack{s\leqq y\\ s\in S}} s=\varphi(x)+\varphi(y).$$

Hence φ is additive. Plainly, φ is increasing. Assume $\varphi(x)\leqq\varphi(y)$ for $x, y\in W^*$. Then $s\leqq y$ for all $s\in S$ with $s\leqq x$. Since $s\leqq\bigvee\{t\in S \mid t\leqq y\}$ for all $s\in S$ with $s\leqq x$, by (H_1) we get $s=\bigvee\{t\wedge s \mid t\leqq y, t\in S\}$ for all $s\in S$ with $s\leqq x$. Further, S is dense in order in W^* and therefore $x=\bigvee\{s\in S \mid s\leqq x\}=\bigvee_{s\leqq x}\{t\wedge s \mid t\leqq y, t\in S\}$ where $\bigvee$ is taken in W^*. Hence $x\leqq y$. This proves that φ is an isomorphism from W^* to W.

Proposition 2.3. *Let C be an H-cone. Denote by C^0 the set of maps $\mu : C\to\overline{\mathbb{R}}_+$ such that*

$$\mu(s)=\sup_{i\in I}\mu_i(s), \text{ for all } s\in C,$$

for some upper directed family $(\mu_i)_{i\in I}$ of H-integrals in C ([5], Definition 1.1). Then C^0 is a hyperharmonic cone.

Proof. By Proposition 2.2, an H-cone C can be extended to a hyperharmonic cone W satisfying (H_1) such that C is solid and dense in order in W. The set Hom $(W, \overline{\mathbb{R}}_+)$ is a hyperharmonic cone by [4], Corollary 6.16. Furthermore, every $\mu\in C^0$ can be extended to a hyperharmonic morphism from W to $\overline{\mathbb{R}}_+$ by [4], Theorem 6.20 and every H-integral $\mu : C\to\overline{\mathbb{R}}_+$ can be extended to a cancellable hyperharmonic morphism from W to $\overline{\mathbb{R}}_+$ by [4], Theorem 6.24.

We use [4], Theorem 2.3, to prove that C^0 is a hyperharmonic cone.

To show (H_1), let $(\mu_i)_{i\in I}$ be a non-empty upper directed family in C^0. Then $\bigvee_{i\in I}\mu_i=\sup_{i\in I}\mu_i$ by [5], Proposition 1.2. Plainly $\sup_{i\in I}\mu_i$ is translation-invariant in C^0.

Let $(\mu_i)_{i\in I}$ be a non-empty family in C^0. By [5], Proposition 1.2, $\bigwedge_{i\in I}\mu_i$ exists in C^0. To prove that C^0 is a hyperharmonic structure, let $\varphi=\sup_{i\in I}\varphi_i$ and $\psi=\sup_{j\in I}\psi_j$ for non-empty upper directed families of H-integrals $(\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in I}$. Denote φ_i^* and ψ_j^* the extensions of φ_i and ψ_j, respectively, to cancellable hyperharmonic morphisms from W to $\overline{\mathbb{R}}_+$ ([4], Theorem 6.24). Further denote $\varphi^*=\sup_{i\in I}\varphi_i^*$ and $\psi^*=\sup_{j\in I}\psi_j^*$. Since $(\varphi_i^*)_{i\in I}$ and $(\psi_j^*)_{j\in I}$ are upper directed, φ^* and ψ^* are hyperharmonic morphisms such that $\varphi^*|C=\varphi$ and $\psi^*|C=\psi$ ([4], Theorem 6.15). By [4], Theorem 6.15 and Lemma 3.21 e), we have $S_{\psi^*}\varphi^*=\sup_{i\in I}S_{\psi^*}\varphi_i^*$. Since $S_{\psi^*}\varphi_i^*\leqq\varphi_i^*$ and φ_i^* is cancellable, $S_{\psi^*}\varphi_i^*$ is cancellable in Hom $(W, \overline{\mathbb{R}}_+)$ for all $i\in I$. Therefore

$S_{\psi^*}\varphi_i^*|C$ is an H-integral for all $i\in I$ by [4], Theorem 6.24, which implies that $S_{\psi^*}\varphi^*|C$ belongs to C^0. If $\mu\in C^0$ and $\varphi\leqq\mu+\psi$, then the extension μ^* of μ to a hyperharmonic morphism from W to $\overline{\mathbb{R}}_+$ satisfies $\varphi^*\leqq\mu^*+\psi^*$, since C is dense in order in W. Thus we have $\mu^*\geqq S_{\psi^*}\varphi^*$ and therefore $\mu=\mu^*|C\geqq S_{\psi^*}\varphi^*|C$. This completes the proof of $S_\psi\varphi=S_{\psi^*}\varphi^*|C$. Finally, we have to prove that $S_\psi\varphi\prec\varphi$. Since Hom $(W, \overline{\mathbb{R}}_+)$ is a hyperharmonic structure by [4], Theorem 2.3, we have $S_{\psi^*}\varphi_i^*+Q_{\psi^*}\varphi_i^*=\varphi_i^*$ for all $i\in I$. Let us denote $w_i=\bigwedge_{k\geqq i}Q_{\psi^*}\varphi_k^*$. Since $\{S_{\psi^*}\varphi_i^* \mid i\in I\}$ is upper directed by [4], Lemma 3.2 (e), we have

$$w_i+\bigvee_{k\in I}S_{\psi^*}\varphi_k^*=w_i+\bigvee_{k\geqq i}S_{\psi^*}\varphi_k^*=\bigvee_{k\geqq i}(S_{\psi^*}\varphi_k^*+w_i)\leqq$$

$$\bigvee_{k\geqq i}(S_{\psi^*}\varphi_k^*+Q_{\psi^*}\varphi_k^*)=\bigvee_{k\geqq i}\varphi_k^*=\bigvee_{k\in I}\varphi_k^*=\varphi^*$$

for all $i\in I$. From this we get

$$\bigvee_{i\in I}w_i+\bigvee_{i\in I}S_{\psi^*}\varphi_k^*\leqq\varphi^*. \tag{2.8}$$

Since

$$\varphi_i^*\leqq\varphi_k^*=Q_{\psi^*}\varphi_k^*+S_{\psi^*}\varphi_k^*\leqq Q_{\psi^*}\varphi_k^*+\bigvee_{k\geqq i}S_{\psi^*}\varphi_k^*$$

for all $k\geqq i$, k, $i\in I$ and $\wedge$ is translation-invariant, we obtain $\varphi_i^*\leqq w_i+\bigvee_{k\geqq i}S_{\psi^*}\varphi_k^*$ for all $i\in I$. The converse inequality to (2.8) follows from this. Further, $w_i\leqq Q_{\psi^*}\varphi_i^*\leqq\varphi_i^*$ and therefore w_i is cancellable for all $i\in I$. Since $(\bigvee_{i\in I}w_i)|C=\sup_{i\in I}(w_i|C)$, the element $(\bigvee_{i\in I}w_i)|C$ belongs to C^0 and

$$(\bigvee_{i\in I}w_i)|C+S_\psi\varphi=\varphi.$$

This means $S_\psi\varphi\prec\varphi$. This proves that C^0 is a hyperharmonic structure. By [4], Theorem 2.3, the proof is complete.

Theorem 2.4. *Let C be an H-cone and C^* the set of H-integrals on C. The cone of hyperharmonics $\tilde{C}=(C^*)^0$ defined by Popa ([5], p. 54) is a hyperharmonic cone. Moreover any morphism of H-cones defined by Popa [5], Definition 2.1, is a hyperharmonic morphism.*

Proof. By [2], Theorem 2.2.6, C^* is an H-cone. Hence $(C^*)^0$ is a hyperharmonic cone by Proposition 2.3. Using this we immediately get the remaining assertion.

Remark. All hyperharmonic cones are not isomorphic to the cone of hyperharmonics, since the set of cancellable elements is not always dense in order ([4], Example 3.12).

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Received 15.IX.1984

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, Matematică, 1985, f. 2

LOCALLY H-CLOSED SPACES

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We characterize LHC spaces, study some of their properties and focus on a special class of LHC spaces, the Hausdorff spaces for which any H-closed extension is obtained by adding only a finite set of points. We also establish a theorem for the identification of the LHC extensions of a Hausdorff space.

LHC spaces were studied by O b r e a n u [6], C.T. Liu [4], P o r t e r [8, 9], B l a s z c z y k [1] and O v s e p j a n [7].

As in [3], we denote by X a Hausdorff space, by $\mathrm{Sp}(X)$ the set of filters on X, by $\mathscr{X}$ the set of filters on X with an open base and no cluster points, by $X^{\vee}$ the set of the maximal elements of $\mathscr{X}$ and by $i(x)$ the neighbourhood filter of $x \in X$. O_0 denotes the topology of $\mathrm{Sp}(X)$ generated by the sets $O(A)=\{\mathcal{F} \in \mathrm{Sp}(X)$: $A \in \mathcal{F}$, *A open subset of X*$\}$ and O_1 the topology generated by the sets $\{i(A)$: *A open subset* of $X\}$ and $\{i(A) \cup \{\mathcal{F}\}$: *$A$ open subset of X and $A \in \mathcal{F}$*$\}$. The same notations will be used for the topology induced on different subspaces of $\mathrm{Sp}(X)$. If $A \subset X$, then cA denotes $X \setminus A$.

We shall use extensively the following theorem, proved in [3]:

Theorem 1. *Y is an H-closed extension of X iff it is homeomorphic to a subspace $\mathfrak{X}$ of* $\mathrm{Sp}(X)$ *such that: $i(X) \subset \mathfrak{X}$, $\mathfrak{X} \setminus i(X) \subset \mathscr{X}$, any maximal element of $\mathscr{X}$ has exactly one subfilter in $\mathfrak{X}$, and the topology of $\mathfrak{X}$ is between O_0 and O_1.*

§ 1. **Characterizations of LHC spaces.** X is locally H-closed (LHC) if every point has an H-closed neighbourhood. If we requested a fundamental system of H-closed neighbourhoods, we should obtain only the class of locally compact spaces. We agreed that any space denoted by X is Hausdorff because there exist non-Hausdorff topological spaces in which every point has an H-closed neighbourhood (and even a fundamental system of compact neighbourhoods [2, I, §9, 7]) but such spaces cannot have H-closed extensions.

X is LHC *iff either $\mathscr{X}$ has a least element or $\mathscr{X}=\emptyset$.*

X is LHC *iff either the intersection of all the maximal elements of $\mathscr{X}$ is in $\mathscr{X}$ or $\mathscr{X}=\emptyset$.*

These characterizations are close to those obtained in [6].

The following result is a corollary to Theorem 1 from [3]: a nowhere solvable space is LHC iff $\mathscr{X}$ contains the intersection of all the non-convergent ultrafilters with open base.

In [3] we proved that X is not H-closed iff there is at least a cover of X such that: it has an open subcover, it contains $\overline{A \cup B}$ whenever it con-