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PROPERTIES OF SOLUTIONS OF HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS. I

BY

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1. **Introduction.** Motivated with the work of H a r t m a n [8] on linear difference equations, here we shall discuss asymptotic and oscillatory properties of solutions of nonlinear difference equation

$$\Delta^n x(t)+f(t, x(t), \Delta x(t), \ldots, \Delta^{n-1}x(t))=0, \quad t\in I \tag{1.1}$$

and its variant

$$\Delta^n x(t)+p(t)F(x(t), \Delta x(t), \ldots, \Delta^{n-1}x(t))=0, \quad t\in I, \tag{1.2}$$

where I is the discrete set $\{0, 1, \ldots\}$ and Δ is the difference operator $\Delta x(t)=x(t+1)-x(t)$. Throughout, we shall assume that f, p and F are defined in a given domain and for all initial values of the type

$$\Delta^i x(t)_1=A_i\,; \quad 0\leqslant i\leqslant n-1, \quad t_1\in I \tag{1.3}$$

solutions of (1.1), (1.3) and (1.2), (1.3) exist for all $t\geqslant t_1$, $t\in I$.

By a solution of (1.1) or (1.2) we shall always refer to a nontrivial solution. A solution $x(t)$ of (1.1) or (1.2) is called *nonoscillatory* if it is eventually of fixed sign; and it is called *oscillatory* if there is no end of t_1 and $t_2(t_1<t_2)$ in I such that if $x(t_1)>0(x(t_1)<0)$ then $x(t_2)<0(x(t_2)>0)$.

In section 2, we prove a series of lemmas which are discrete analoque of well known results in the continuous case. These lemmas are used in section 3, to obtain our main results which are similar to known results [3, 9—13, 15] for the continuous case and cover some results known in the discrete case for $n=2$, [4—7, 14].

In what follows, we shall denote the set $\{t_1, t_1+1, \ldots\}$ where $t_1\in I$ by I_{t_1}, and $(t)^{(m)}=t(t-1)\ldots(t-m+1)$, the usual factorial notation.

2. **Basic lemmas. Lemma** 2.1. *Let $u(t)$ be some function defined on I_{t_1}. Then, for $0\leqslant k\leqslant n-1$*

$$\Delta^k u(t)=\sum_{i=k}^{n-1}\frac{(t-t_1)^{(i-k)}}{(i-k)\,!}\,\Delta^i u(t_1)+\frac{1}{(n-k-1)\,!}\sum_{s=t_1}^{t-n+k}(t-s-1)^{(n-k-1)}\,\Delta^n u(s). \tag{2.1}$$

Proof. The proof is by finite induction. Since

$$\Delta^m u(t)=\Delta^m u(t_1)+\sum_{s=t_1}^{t-1}\Delta^{m+1}u(s), \quad 0\leqslant m\leqslant n-1, \tag{2.2}$$

the identity (2.1) is trivial for $k=n-1$. Next, assume (2.1) is true for $k=m+1$, then from (2.2)

$$\Delta^m u(t)=\Delta^m u(t_1)+\sum_{s=t_1}^{t-1}\left[\sum_{i=m+1}^{n-1}\frac{(s-t_1)^{(i-m-1)}}{(i-m-1)\,!}\Delta^i u(t_1)+\right.$$

$$\left.+\frac{1}{(n-m-2)!}\sum_{s_1=t_1}^{s-n+m+1}(s-s_1-1)^{(n-m-2)}\Delta^n u(s_1)\right]=\Delta^m u(t_1)+$$

$$+\sum_{i=m+1}^{n-1}\frac{(s-t_1)^{(i-m)}}{(i-m)!}\Bigg|_{s=t_1}^{t}\Delta^i u(t_1)+$$

$$+\frac{1}{(n-m-2!)}\sum_{j=0}^{t-n+m-t_1}\sum_{s=t_1+j}^{t-n+m}(t-1-s-1)^{(n-m-2)}\Delta^n u(t_1+j)=$$

$$\sum_{i=m}^{n-1}\frac{(t-t_1)^{(i-m)}}{(i-m)\,!}\Delta^i u(t_1)+\frac{1}{(n-m-1)!}\sum_{s=t_1}^{t-n+m}(t-s-1)^{(n-m-1)}\Delta^n u(s),$$

and hence (2.1) is also true for $k=m$.

Lemma 2.2. *Let $A\geqslant 0$, $f(t)\geqslant 0$ on I_{t_1} and $u(t)$ satisfies an inequality*

$$u(t)\leqslant A+\sum_{s=t_1}^{t-1}f(s)u(s). \tag{2.3}$$

Then, for $t\in I_{t_1}$

$$u(t)\leqslant A\prod_{s=t_1}^{t-1}(1+f(s)).$$

For the proof and several linear and nonlinear generalizations of (2.3) see [1, 2].

Lemma 2.3. *Let $u(t)$ be some function defined on I_{t_1}. Then*

$$\sum_{s=t_1}^{t-1}(s)^{(n-1)}\Delta^n u(s)=\sum_{k=1}^{n}(-1)^{k+1}\Delta^{k-1}(s)^{(n-1)}\Delta^{n-k}u(s+k-1)\Bigg|_{s=t_1}^{t}. \tag{2.4}$$

Proof. The identity (2.4) is a repeated application of

$$\sum_{s=t_1}^{t-1}u(s)\Delta u(s)=u(s)v(s)\Bigg|_{s=t_1}^{t}-\sum_{s=t_1}^{t-1}\Delta u(s)v(s+1).$$

Lemma 2.4. *Let $1\leqslant q\leqslant n-1$ and $u(t)$ be some function defined on I. Then,*

(a) $\liminf\limits_{t\to\infty}\Delta^q u(t)>0$ *implies* $\lim\limits_{t\to\infty}\Delta^i u(t)=\infty$, $\quad 0\leqslant i\leqslant q-1$

(b) $\limsup\limits_{t\to\infty}\Delta^q u(t)<0$ *implies* $\lim\limits_{t\to\infty}\Delta^i u(t)=-\infty$, $0\leqslant i\leqslant q-1$.

Proof. $\operatorname{Lim\,inf}\limits_{t\to\infty}\Delta^q u(t)>0$ implies that for all sufficiently large t say $t\geqslant t_1$, $\Delta^q u(t)\geqslant c_1>0$. Now, from (2.2), for $m=q-1$, we find $\Delta^{q-1}u(t)\geqslant\Delta^q u(t_1)+ +c_1(t-t_1)$ and hence $\lim\limits_{t\to\infty}\Delta^{q-1}u(t)=\infty$. Rest of the proof follows by induction. The case (b) is treated similarly.

Lemma 2.5. *Let $u(t)$ be some function defined on I and $u(t)>0$, $\Delta^n u(t)<0$ on I_{t_1}. Then,*

(1) *$\Delta^k u(t)$, $0\leqslant k\leqslant n-1$ has a definite limit as $t\to\infty$.*

(2) *If we denote $\Delta^k u(\infty)=\lim\limits_{t\to\infty}\Delta^k u(t)$, then $u(\infty)=\Delta u(\infty)=\ldots=\Delta^{k-1}u(\infty)=\infty$, $\Delta^k u(\infty)=c\geqslant 0$, $\Delta^{k+1}u(\infty)=\Delta^{k+2}u(\infty)=\ldots=\Delta^{n-1}u(\infty)=0$ where $0\leqslant k\leqslant n-1$.*

(3) *If $k=(n-i)$, $1\leqslant i\leqslant n-1$, and i is odd, then c is finite.*

(4) *If $\Delta^{n-i}u(\infty)=0$; $1\leqslant i\leqslant n-1$, then $(-1)^{i+1}\Delta^{n-i}u(t)>0$.*

(5) *If n is even and $k=0$ then, $u(\infty)=c>0$.*

(6) *If n is odd and $u(t_1)=0$ then, $\Delta^k u(\infty)=c>0$ for some $k\geqslant 1$.*

Proof. Let $\Delta^k u(\infty)$, $0\leqslant k\leqslant n-1$ exist and negative, then from lemma
2.4, $u(\infty)<0$ (in fact $u(\infty)=-\infty$, if $1\leqslant k\leqslant n-1$) wich is a contradiction
to our assumption that $u(t)>0$ on I_{t_1}. Thus, we need to consider $\Delta^k u(\infty)\geqslant 0$,
$0\leqslant k\leqslant n-1$ if it exists. Since, $\Delta^n u(t)<0$ on I_{t_1}, $\Delta^{n-1}u(t)$ is decreasing on I_{t_1}
and hence $\Delta^{n-1}u(\infty)=c_1$ (finite)$\geqslant 0$ exists, so (1) is true for $k=n-1$ and (3)
for $i=1$. If $c_1>0$ then, (2) follows from lemma 2.4. If $c_1=0$ then, $\Delta^{n-1}u(t)>0$
on I_{t_1} and so (4) is true for $i=1$, and $\Delta^{n-2}u(t)$ is increasing and $\Delta^{n-2}u(\infty)=$
$=c_2\geqslant 0$ exists and hence (1) for $k=n-2$. If $c_2=\infty$, (2) follows from lemma
2.4 for $k=n-1$, similarly if c_2 (finite)>0, (2) follows from lemma 2.4 for
$k=n-2$ also, if $c_2=0$ we must have $\Delta^{n-2}u(t)<0$ on I_{t_1} which is (4) for $i=2$
and so $\Delta^{n-3}u(t)$ is decreasing and $\Delta^{n-2}u(\infty)=c_3$ (finite)$\geqslant 0$ exists which is
(3) for $i=3$. Repeating similar arguments, we get (1)—(4) easily.

If n is even $\Delta u(\infty)=c_{n-1}>0$ then, (5) is immediate, also if $\Delta u(\infty)=0$
then, from (4), $\Delta u(t)>0$ on I_{t_1} which implies (5).

If n is odd and $\Delta u(\infty)=0$ then, from (4), $\Delta u(t)<0$ on I_{t_1} and $u(t)$ is
decreasing. Since $u(t_1)=0$, we find $u(t)<0$ on $I_{t_1}-\{t_1\}$ which is a contra-
diction to $u(t)>0$ on I_{t_1}, and hence (6) is true.

Lemma 2.6. *Let in lemma 2.5, $u(t)$ be bounded also. Then,*

(a) $\lim\limits_{t\to\infty}\Delta^k u(t)=0$, $1\leqslant 1k\leqslant n-1$,

(b) $(-1)^{k+1}\Delta^{n-k}u(t)>0$, $1\leqslant k\leqslant n-1$.

Proof. The proof is contained in lemmas 2.4 and 2.5.

Lemma 2.7. *Let $u(t)\geqslant 0$, $v(t)>0$, $\Delta u(t)\geqslant 0$ and $\Delta v(t)>0$ for all sufficiently large t in I. Then, if $v(t)\to\infty$,*

$$\lim_{t\to\infty}\frac{\Delta u(t)}{\Delta v(t)}=r \quad \textit{implies} \quad \lim_{t\to\infty}\frac{u(t)}{v(t)}=r.$$

Proof. $\operatorname{Lim}\limits_{t\to\infty}\Delta u(t)/\Delta v(t)=r$ implies that for all sufficiently large $t\geqslant t_1$, in I, $(r-\varepsilon)\Delta v(t)\leqslant\Delta u(t)\leqslant(r+\varepsilon)\Delta v(t)$. Summing the above inequality from $t=t_1$ to $T-1$, we find

$$(r-\varepsilon)[v(T)-v(t_1)]\leqslant u(T)-u(t_1)\leqslant(r+\varepsilon)[v(T)-v(t_1)]$$

or,

$$(r-\varepsilon)\left[1-\frac{v(t_1)}{v(T)}\right]\leqslant\frac{u(T)}{v(T)}-\frac{u(t_1)}{v(T)}\leqslant(r+\varepsilon)\left[1-\frac{v(t_1)}{v(T)}\right]$$

and in the above inequality as $T\to\infty$, the result $\lim\limits_{t\to\infty}u(t)/v(t)=r$ follows.

Lemma 2.8. *Let $u(t)$ be as in lemma 2.5. Then,*

$$\lim_{t\to\infty}\frac{\Delta^i u(t)}{u(t)}=0,\quad 1\leqslant i\leqslant n-1 \tag{2.5}$$

provided $u(t_1)=0$ *in case* n *is odd.*

Proof. Let in lemma 2.5, $k=0$ which is from (6) is possible only when n is even, then from (5), $u(\infty)=c>0$ and (2.5) follows from (2). Next, let $k\geq 1$, then from (2), (2.5) is obvious for $k+1\leq i\leq n-1$, and for $k\leq i\leq n-1$ if $c=0$. Next, let $c\neq 0$ then, from lemma 2.7, we have

$$0=\lim_{t\to\infty}\frac{\Delta^{k+1}u(t)}{\Delta^{k}u(t)}=\lim_{t\to\infty}\frac{\Delta^{k}u(t)}{\Delta^{k-1}u(t)}=\ldots=\lim_{t\to\infty}\frac{\Delta^{2}u(t)}{\Delta u(t)}=\lim_{t\to\infty}\frac{\Delta u(t)}{u(t)}$$

i.e.

$$\lim_{t\to\infty}\frac{\Delta^{i}u(t)}{\Delta^{i-1}u(t)}=0,\quad 1\leq i\leq k. \tag{2.6}$$

Next, from (2.6)

$$\lim_{t\to\infty}\frac{\Delta^{i}u(t)}{u(t)}=\lim_{t\to\infty}\frac{\Delta^{i}u(t)}{\Delta^{i-1}u(t)}\lim_{t\to\infty}\frac{\Delta^{i-1}u(t)}{\Delta^{i-2}u(t)}\ldots\lim_{t\to\infty}\frac{\Delta u(t)}{u(t)}=0,\quad 1\leq i\leq k,$$

and hence (2.5) is true for $1\leq i\leq k$ also.

3. Main results. Theorem 3.1. *Assume that the function* $f(t, x_0, x_1, \ldots, x_{n-1})$ *for* $t\in I$, $(x_0, x_1, \ldots, x_{n-1})\in R^n$ *satisfies the condition*

$$|f(t, x_0, x_1, \ldots, x_{n-1})|\leq\sum_{i=0}^{n-1}p_i(t)|x_i| \tag{3.1}$$

where $p_i(t)$, $0\leq i\leq n-1$ *are nonnegative functions, defined on* I *and*

$$\prod^{\infty}\left[1+\sum_{i=0}^{n-1}(s)^{(n-i-1)}p_i(s)\right]<\infty. \tag{3.2}$$

Then, the equation (1.1) *has solutions which are asymptotic to* $\sum_{i=0}^{n-1}a_i(t)^{(i)}$ *as* $t\to\infty$, *where* a_i, $1\leq i\leq n-1$ *are real constants and* $a_{n-1}\neq 0$.

Proof. Let $x(t)$ be any solution of (1.1), then from lemma 2.1 for any $t_1\in I$, it follows that

$$\Delta^k x(t)=\sum_{i=k}^{n-1}\frac{(t-t_1)^{(i-k)}}{(i-k)!}\Delta^i x(t_1)-\frac{1}{(n-k-1)!}\sum_{s=t_1}^{t-n+k}(t-s-1)^{(n-k-1)}\cdot \tag{3.3}$$
$$f(s, x(s), \Delta x(s), \ldots, \Delta^{n-1}x(s))\ ;\ t\in I_{t_1},\ 0\leq k\leq n-1.$$

Using (3.1) in (3.3), we obtain the estimate

$$|\Delta^k x(t)|\leq(t)^{(n-1-k)}A_k+(t)^{(n-k-1)}B_k\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1}p_i(s)|\Delta^i x(s)|, \tag{3.4}$$

where $A_k=(t)^{-(n-1-k)}\sum_{i=k}^{n-1}(t)^{(i-k)}/(i-k)!\,|\Delta^i x(t_1)|$, and $B_k=1/(n-k-1)!$. Define, $A=\max\limits_{\substack{0\leq k\leq n-1\\ t\in I_{t_1}}}A_k$, then since $B_k\leq 1$, inequality (3.4) implies

$$|\Delta^k x(t)|\leq(t)^{(n-1-k)}F(t),\ 0\leq k\leq n-1, \tag{3.5}$$

where

$$F(t)=A+\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1}p_i(s)|\Delta^i x(s)|. \tag{3.6}$$

Using (3.5) in (3.6), we find

$$F(t)\leq A+\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1}(s)^{(n-1-i)}p_i(s)F(s).$$

An application of lemma 2.2 provides that

$$F(t)\leq A\prod_{s=t_1}^{t-n+k}\left[1+\sum_{i=0}^{n-1}(s)^{(n-1-i)}p_i(s)\right]$$

and hence from (3.2), $F(t)\leq C$, where C is some positive constant. Thus, inequality (3.5) implies that

$$|\Delta^k x(t)|\leq C(t)^{(n-1-k)},\quad 0\leq k\leq n-1. \tag{3.7}$$

Next, from (3.3), we have

$$\Delta^{n-1}x(t)=\Delta^{n-1}x(t_1)-\sum_{s=t_1}^{t-1}f(s, x(s), \Delta x(s), \ldots, \Delta^{n-1}x(s)). \tag{3.8}$$

Since, condition (3.2) implies that $\sum^{\infty}\sum_{i=0}^{n-1}(s)^{(n-i-1)}p_i(s)<\infty$, we find from (3.1) and (3.8) that the sum in (3.8) converges as $t\to\infty$ and therefore $\lim\limits_{t\to\infty}\Delta^{n-1}x(t)$ exists and is a finite nomber. To, ensure that this limit is not zero, we choose t_1 so large that $1-C\sum_{t_1}^{\infty}\sum_{i=0}^{n-1}(s)^{(n-i-1)}p_i(s)>0$ and impose the condition $\Delta^{n-1}x(t_1)=1$ on the solution of (1.1). This solution has the desired asymptotic property.

Corollary 3.2. *Under the hypothesis of theorem* 3.1, (1.1) *has nonoscillatory solutions.*

Next, we shall consider equation (1.2) and need some of these conditions :

(i) $p(t)$ is positive on I_{t_1} and $\sum_{s=t_1}^{\infty}p(s)=\infty$

(ii) $F(x_0, x_1, \ldots, x_{n-1})$ is continuous on R^n and $x_0 f(x_0, x_1, \ldots, x_{n-1})>0$ for $x_0\neq 0$

(iii) $F(\lambda x_0, \lambda x_1, \ldots, \lambda x_{n-1})=\lambda^{2\alpha+1}F(x_0, x_1, \ldots, x_{n-1})$ for all $(x_0, x_1, \ldots, x_{n-1})\in R^n$ and $\lambda\in R$ where α is some nonnegative integer

(iv) for any $(x_0, x_1, \ldots, x_{n-1})$, $c\geq 0$, $1\leq k\leq n-1$ the function $F(x_0, x_1, \ldots, x_{n-1})$ has a definite limit $\mathring{F}(k, c)$ which is positive or ∞ as $x_0\to\infty, \ldots, x_{k-1}\to\infty$, $x_k\to c$, $x_{k+1}\to 0, \ldots, x_{n-1}\to 0$.

Theorem 3.3. *Let in* (1.2) p *and* F *satisfy* (i) — (iii) *Then,*

(a) *for* n *even, every solution of* (1.2) *is oscillatory on* I,

(b) *for* n *odd, any solution of* (1.2) *is oscillatory on* I, *whenever it has a zero.*

Proof. Let $x(t)$ be nonoscillatory solution of (1.2), which must then eventually be of one sign. Without loss of generality, we assume that

$x(t)>0$ for $t\geq t_1\geq 0$. Thus, from (i) and (ii), we find $\Delta^n x(t)<0$ and lemma 2.8 provides that $\lim_{t\to\infty}\Delta^i x(t)/x(t)=0$, $1\leq i\leq n-1$. Since, F is continuous, for any $\varepsilon>0$, there exists $t_2\geq t_1$ such that

$$\left|F\left(1,\frac{\Delta x(t)}{x(t)},\dots,\frac{\Delta^{n-1}x(t)}{x(t)}\right)-F(1,0,\dots,0)\right|<\varepsilon,\quad t\geq t_2.$$

From (ii), $F(1,0,\dots,0)>0$ and we may assume $0<\varepsilon<F(1,0,\dots,0)$. From lemma 2.5, we have $\Delta x(t)>0$, $\Delta^{n-1}x(t)>0$, also as a consequence $x(t)\geq c_n>0$ for all $t\in I_{t_2}$. Define, $y(t)=\Delta^{n-1}x(t)/x(t)$, then we have

$$\Delta y(t)=\frac{\Delta^{n-1}x(t+1)}{x(t+1)}-\frac{\Delta^{n-1}x(t)}{x(t)}\leq\frac{\Delta^{n-1}x(t+1)}{x(t)}-\frac{\Delta^{n-1}x(t)}{x(t)}=\frac{\Delta^n x(t)}{x(t)}.$$

Thus, from (1.2)

$$\sum_{s=t_2}^{t-1}\Delta y(s)\leq-\sum_{s=t_2}^{t-1}p(s)\frac{F(x(s),\Delta x(s),\dots,\Delta^{n-1}x(s))}{x(s)}$$

and hence,

$$y(t)-y(t_2)=-\sum_{s=t_2}^{t-1}p(s)x^{2\alpha}(s)F\left(1,\frac{\Delta x(s)}{x(s)},\dots,\frac{\Delta^{n-1}x(s)}{x(s)}\right)\leq$$

$$\leq-[F(1,0,\dots,0)-\varepsilon]c_n^{2\alpha}\sum_{s=t_2}^{t-1}p(s).$$

In the above inequality right side tends to $-\infty$ as $t\to\infty$ whereas left side remains bounded. This contradiction proves our result.

Remark. In theorem 3.3, it is enough to assume $(\text{iii})_{\lambda\in R_+}$ and α any nonnegative number.

Theorem 3.4. *Let in* (1.2) *p and F satisfy* (i), (ii), $(\text{iii})_{-1}$, (iv). *Then, the conclusion of theorem* 3.3 *holds.*

Proof. Assume $x(t)>0$ on I_{t_1} as in the proof of theorem 3.3. Then, $\Delta x(t)>0$, $\Delta^{n-1}x(t)>0$ on I_{t_1} and $x(\infty)=c>0$. From, (2.2) we find

$$-\Delta^{n-1}x(t)+\Delta^{n-1}x(t_1)=\sum_{s=t_1}^{t-1}p(s)F(x(s),\Delta x(s),\dots,\Delta^{n-1}x(s))$$

and hence

(3.9) $$\Delta^{n-1}x(t_1)>\sum_{s=t_1}^{t-1}p(s)F(x(s),\Delta x(s),\dots,\Delta^{n-1}x(s)).$$

Now, let $k\geq 1$ in the conclusion of lemma 2.5 (which is always the case when n is odd) then, from (iv), $\lim_{t\to\infty}F(x(t),\Delta x(t),\dots,\Delta^{n-1}x(t))=\dot{F}(k,c)>\varepsilon>0$. Thus, there exists $t_2\geq t_1$, such that $F(x(t),\Delta x(t),\dots,\Delta^{n-1}x(t))>\varepsilon$ for all $t\geq t_2$. Replacing t_1 by t_2 in (3.9), we find $\Delta^{n-1}x(t_2)>\varepsilon\sum_{s=t_2}^{t-1}p(s)$ and this leads to a contradiction.

Next, let $k=0$ (which is possible only when n is even). Then, for every $0<\varepsilon<F(c,0,\dots,0)$ there exists $t_2\geq t_1$ such that $F(c,0,\dots,0)-\varepsilon<F(x(t),\Delta x(t),\dots,\Delta^{n-1}x(t))$ for all $t\geq t_2$ and we get from (3.9)

$$\Delta^{n-1}x(t_2)>[F(c,0,\dots,0)-\varepsilon]\sum_{s=t_2}^{t-1}p(s)$$

which is again a contradiction.

Theorem 3.5. *Let in* (1.2) *F satisfy* (ii) *and p be positive on I_a, also let*

(3.10) $$\sum^{\infty}(s)^{(n-1)}p(s)=\infty.$$

Then, (a) *for n even, every bounded solution of* (1.2) *is oscillatory on I*

(b) *for n odd, any bounded solution of* (1.2) *is oscillatory on I, whenever it has a zero*

(c) *for n odd, every bounded solution of* (1.2) *is either oscillatory or tend to zero monotonically.*

Proof. Let as in the proof of theorem 3.3 $x(t)>0$ on I_{t_1} be a bounded solution of (1.2). Then, $\Delta^n x(t)<0$ and lemma 2.6 is applicable. In cases (a) and (b) from lemma 2.5 and the boundedness of $x(t)$, we find $x(\infty)=c>0$ also, in case (c) since $\Delta x(t)\leq 0$ either $x(\infty)=0$ or $x(\infty)=c>0$ exists. Thus, to complete the proof, we need to consider only when $x(\infty)=c>0$.

Using lemma 2.6, we find, $\lim_{t\to\infty}F(x(t),\Delta x(t),\dots,\Delta^{n-1}x(t))=F(c,0,\dots,0)>\varepsilon>0$. Hence, there exists $t_2\geq t_1$ such that $F(c,0,\dots,0)-\varepsilon<F(x(t),\Delta x(t),\dots,\Delta^{n-1}x(t))$ for all $t\geq t_2$, ank from (1.2), we find

(3.11) $$\Delta^n x(t)+[F(c,0,\dots,0)-\varepsilon]p(t)<0,\quad t\geq t_2.$$

Multiplying (3.11) by $(t)^{(n-1)}$ and using lemma 2.3, to obtain

(3.12) $$[F(c,0,\dots,0)-\varepsilon]\sum_{s=t_2}^{t-1}(s)^{(n-1)}p(s)+\sum_{k=1}^{n}(-1)^{k+1}\Delta^{k-1}(s)^{(n-1)}\Delta^{n-k}x(s+k-1)\Big|_{s=t_2}^{t}<0.$$

Using lemma 2.6 in (3.12), we get

(3.13) $$[F(c,0,\dots,0)-\varepsilon]\sum_{s=t_2}^{t-1}(s)^{(n-1)}p(s)<K+(-1)^{n+2}(n-1)!\,x(t+n-1)$$

where K is some finite constant. Inequality (3.13) together with (3.10) gives a contradiction to our assumption that $x(t)$ is bounded.

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LE PROBLÈME DE DARBOUX POUR UNE ÉQUATION AUX DÉRIVÉES PARTIELLES MULTIVOQUE

PAR

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Le but de cette Note est de donner un résultat d'existence locale pour le problème de Darboux défini pour une équation hyperbolique multivoque.

Supposons satisfaites les hypothèses suivantes :

(H_1) $F : D\times\Omega\to 2^{R^n}$ est une fonction multivoque à valeurs compactes, convexes, non vides dans R^n, $D=[0, a]\times[0, b]$, Ω un ensemble ouvert dans R^n,

(H_2) Pour tout $(x, y)\in D$, l'application $z\to F(x, y, z)$ est semi-continue supérieurement sur Ω,

(H_3) Pour tout $z\in\Omega$, l'application $(x, y)\to F(x, y, z)$ est mesurable Lebesgue dans D,

(H_4) Il existe une fonction $g : D\to R_+$, $g\in\mathcal{L}^1(D\,;R_+)$ telle que $\|u\|\leq g(x, y)$, $\forall u\in F(x, y, z)$, $\forall(x, y)\in D$, $\forall z\in\Omega$.

(H_5) Il existe un ensemble convexe, compact, $M\subset\Omega$ et un point $(x_0, y_0)\in\,]0, a]\times]0, b]$, tel que

$$\int_0^{x_0}\int_0^{y_0} g(\xi, \eta)\, d\xi\, d\eta\leq d(M, C_\Omega),$$

où $d(M, C_\Omega)$ est la distance de M à $C_\Omega=R^n-\Omega$.

(H_6) Les fonctions $\sigma\in AC([0, a]\,;R^n)$, $\tau\in AC([0, b]\,;R^n)$, où $AC([\alpha_1, \alpha_2]\,;R^n)$ est l'espace des fonctions absolument continues $f : [\alpha_1, \alpha_2]\subset R\to R^n$, avec la norme

$$|f|=\sup_{t\in[\alpha_1,\alpha_2]}\|f(t)\|+\int_{\alpha_1}^{\alpha_2}\|f'(t)\|\,dt,$$

satisfont à la condition $\sigma(0)=\tau(0)$, et la fonction $\alpha : D\to R^n$, définie par

$$\alpha(x, y)=\sigma(x)+\tau(y)-\sigma(0) \tag{1}$$

a les valeurs dans M pour $(x, y)\in[0, x_0]\times[0, y_0]$.

Il résulte que la fonction α est absolument continue au sens de Carathéodory sur D : $\alpha\in C^*(D\,;R^n)$, ([1], p. 651).

Définition 1. *On appelle problème de Darboux pour l'équation hyperbolique multivoque*