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SHOCK WAVES IN MICROPOLAR ELASTIC MATERIALS WITH VOIDS

BY

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1. **Introduction.** In [1], Nunziato and Cowin have presented a general theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. The linear theory of elastic materials with voids has been established by Cowin and Nunziato in [2]. Some applications of this theory were presented in [2—4].

The basic feature of the theory of materials with voids is the concept of a material, for which the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation of the bulk density of the material introduces an additional degree of kinematic freedom and was employed previously by Goodman and Cowin [5] to develop a continuum theory for flowing granular materials.

The intended applications of the theory of elastic materials with voids are to geological materials like rock and soils and to manufactured porous materials.

In this paper we develop a linear theory of micropolar elastic solids with voids. Micropolar materials, roughly speaking, are continuous media with extra independent degrees of freedom for the local intrinsic rotation. The micropolar continuum mechanics seems to be an adequate tool to describe the behavior of the granular materials (see e.g. [6—8]). It appears that the micropolar continuum posseses promise for entering the Earth sciences because of the fact that most materials of interest are made of granules. Some applications of micropolar mechanics to earthquake problems were presented in [9—11].

In the first part of this paper, we establish the basic equations of the linear theory of micropolar elastic materials with voids. The theory differs from the classical theory of micropolar elasticity [6, 8] in that the volume fraction corresponding to the void volume is taken as an independent kinematical variable. In the second part of the paper we study the shock waves in isotropic micropolar elastic materials with voids. The coupling between the discontinuities in the macroscopic and microscopic fields is studied.

2. **Basic equations.** We refer the motion of the continuum to a fixed system of rectangular Cartesian axes $Ox_i (i=1, 2, 3)$. We shall employ the

usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the reference coordinates.

In what follows we present a linear theory for the behavior of porous solid in which the skeletal is a micropolar elastic material and the interstices are void of material. The concept of a distributed body asserts that the mass density ρ^*, at time t, has the decomposition

$$\rho^* = \nu\gamma, \tag{2.1}$$

where γ is the density of the matrix material and ν is the volume fraction field [1, 2].

We denote by u_i the components of the displacement vector and by φ_i the components of the microrotation vector [6, 8]. We assume that the initial body is free from stress and couple stress.

We consider an arbitrary material volume V in the continuum, bounded by a surface $\mathcal{A}$ at time t, and we suppose that Ω is the corresponding volume in the initial undeformed state of the continuum, bounded by a surface S. Let the outward unit normal at S be n_i, referred to the rectangular frame of reference. The fundamental system of balance laws of a micropolar solid with voids are:

(i) the law of balance of linear momentum

$$\frac{d}{dt}\int_\Omega \rho v_i dv = \int_\Omega \rho f_i \, dv + \int_S t_i \, da, \tag{2.2}$$

(ii) the law of balance of moment of momentum

$$\frac{d}{dt}\int_\Omega \rho(e_{ijk}x_j v_k + J_{ij}\omega_j)dv = \int_\Omega \rho(e_{ijk}x_j f_k + l_i)dv + \int_S (e_{ijk}x_j t_k + m_i)da, \tag{2.3}$$

(iii) the law of balance of equilibrated force [1,5]

$$\frac{d}{dt}\int_\Omega \rho k\dot{\nu} \, dv = \int_\Omega (\rho l + g)dv + \int_S h \, da, \tag{2.4}$$

(iv) the law of balance of energy

$$\frac{d}{dt}\int_\Omega \rho\left[\frac{1}{2}(v_i v_i + J_{ij}\omega_i\omega_j + k\dot{\nu}^2) + \varepsilon\right]dv = \int_\Omega \rho(f_i v_i + l_i\omega_i + l\dot{\nu})dv + \int_S (t_i v_i + m_i\omega_i + h\dot{\nu})da. \tag{2.5}$$

Here ρ is the density in the reference configuration, $J_{ij}(=J_{ji})$ are coefficients of inertia, k is the equilibrated inertia, ε is the internal energy per unit mass, f_i is the body force, l_i is the body couple, l is the extrinsic equilibrated body force, t_i is the stress vector associated with the surface $\mathcal{A}$ but measured per unit area of the surface S, m_i is the couple stress vector, h is the equilibrated stress, g is the intrinsic equilibrated body force, e_{ijk} is the alternating tensor, $v_i = \dot{u}_i$, $\omega_i = \dot{\varphi}_i$, and a dot denotes the material derivative with respect to the time.

From (2.2) we obtain the well-known relations

$$t_i = t_{ji}n_j, \tag{2.6}$$

and

$$t_{ji,j} + \rho f_i = \rho\ddot{u}_i. \tag{2.7}$$

Taking into account (2.6) and (2.7), the equation (2.3) reduces to

$$\int_\Omega I_{ij}\ddot{\varphi}_j dv = \int_\Omega (\rho l_i + e_{irs}t_{rs})dv + \int_S m_i da, \tag{2.8}$$

where $I_{ij} = \rho J_{ij}$. With an argument similar to that used in obtaining (2.6), we have

$$m_i = m_{ji}n_j, \tag{2.9}$$

where m_{ij} is the couple stress tensor. With the help of (2.9), the equation (2.8) reduces to

$$m_{ji,j} + e_{irs}t_{rs} + \rho l_i = I_{ij}\ddot{\varphi}_j. \tag{2.10}$$

From (2.4) we obtain

$$h = h_i n_i, \tag{2.11}$$

and

$$h_{i,i} + \rho l + g = \rho k\ddot{\nu}, \tag{2.12}$$

where h_i is the equilibrated stress vector.

With the help of (2.6), (2.7), (2.9)–(2.12), the equation (2.5) reduces to

$$\rho\dot{\varepsilon} = t_{ij}\dot{e}_{ij} + m_{ij}\dot{\varkappa}_{ij} - g\dot{\nu} + h_i\dot{\nu}_{,i}, \tag{2.13}$$

where we have used the notations

$$e_{ij} = u_{j,i} + e_{jik}\varphi_k, \quad \varkappa_{ij} = \varphi_{j,i}. \tag{2.14}$$

Let us introduce the notation

$$\psi = \nu - \nu_0, \tag{2.15}$$

where ν_0 is the volume distribution function for the reference configuration. We assume that ν_0 is a given constant.

We restrict attention only to the linear theory of elastic materials where

$$\varepsilon = \varepsilon(e_{rs}, \varkappa_{rs}, \psi, \psi_{,k}), \quad t_{ij} = t_{ij}(e_{rs}, \varkappa_{rs}, \psi, \psi_{,k}), \quad m_{ij} = m_{ij}(e_{rs}, \varkappa_{rs}, \psi, \psi_{,k}),$$
$$g = g(e_{rs}, \varkappa_{rs}, \psi, \psi_{,k}), \quad h_i = h_i(e_{rs}, \varkappa_{rs}, \psi, \psi_{,k}), \tag{2.16}$$

the functionals in (2.16) being consistent with the assumption of the linear theory. From (2.13) and (2.16) it follows

$$t_{ij}=\frac{\partial\sigma}{\partial e_{ij}},\quad m_{ij}=\frac{\partial\sigma}{\partial \varkappa_{ij}},\quad g=-\frac{\partial\sigma}{\partial\psi},\quad h_i=\frac{\partial\sigma}{\partial\psi_{,i}},\tag{2.17}$$

where $\sigma=\rho\varepsilon$.

In the linear theory, we have

$$\sigma=(1/2)A_{ijrs}e_{ij}e_{rs}+B_{ijrs}e_{ij}\varkappa_{rs}+(1/2)\,C_{ijrs}\varkappa_{ij}\varkappa_{rs}+F_{ij}e_{ij}\psi+$$
$$+G_{ij}\varkappa_{ij}\psi+H_{ijk}e_{ij}\psi_{,k}+K_{ijr}\varkappa_{ij}\psi_{,r}-L_i\psi\psi_{,i}+(1/2)M_{ij}\,\psi_{,i}\psi_{,j}+(1/2)a\psi^2.\tag{2.18}$$

Using (2.17) and (2:18) we obtain the following constitutive equations

$$\begin{aligned}
t_{ij}&=A_{ijrs}e_{rs}+B_{ijrs}\varkappa_{rs}+F_{ij}\psi+H_{ijk}\psi_{,k},\\
m_{ij}&=B_{rsij}e_{rs}+C_{ijrs}\varkappa_{rs}+G_{ij}\psi+K_{ijr}\psi_{,r},\\
g&=-F_{ij}e_{ij}-G_{ij}\varkappa_{ij}-L_i\psi_{,i}-a\psi,\\
h_i&=H_{rsi}e_{rs}+K_{rsi}\varkappa_{rs}+L_i\psi+M_{ij}\psi_{,j}.
\end{aligned}\tag{2.19}$$

The coefficients in (2.19) have the following symmetries

$$A_{ijrs}=A_{rsij},\quad C_{ijrs}=C_{rsij},\quad M_{ij}=M_{ji}.\tag{2.20}$$

In the case of a centrosymmetric isotropic material the relations (2.19) become

$$\begin{aligned}
t_{ij}&=\lambda e_{rr}\delta_{ij}+(\mu+\alpha)e_{ij}+(\mu-\alpha)e_{ji}+b\psi\delta_{ij},\\
m_{ij}&=\beta\varkappa_{rr}\delta_{ij}+(\gamma+\varepsilon)\varkappa_{ij}+(\gamma-\varepsilon)\varkappa_{ji},\\
g&=-be_{rr}-a\psi,\quad h_i=d\psi_{,i}
\end{aligned}\tag{2.21}$$

We assume that the body occupies a bounded region B with Lipschitz boundary ∂B which consists of a finite number of smooth surfaces. Let us consider the subsets $\Sigma_s(s=1,2,\ldots,6)$ of ∂B so that $\bar{\Sigma}_1\cup\Sigma_2=\bar{\Sigma}_3\cup\Sigma_4=\bar{\Sigma}_5\cup\Sigma_6=\partial B$, $\Sigma_1\cap\Sigma_2=\Sigma_3\cap\Sigma_4=\Sigma_5\cap\Sigma_6=0$. We can consider the boundary conditions

$$\begin{aligned}
u_i&=\tilde{u}_i \ \text{on}\ \bar{\Sigma}_1\times[0,t_0), & t_i&=\tilde{t}_i \ \text{on}\ \Sigma_2\times[0,t_0),\\
\varphi_i&=\tilde{\varphi}_i \ \text{on}\ \bar{\Sigma}_3\times[0,t_0), & m_i&=\tilde{m}_i \ \text{on}\ \Sigma_4\times[0,t_0),\\
\psi&=\tilde{\psi} \ \text{on}\ \bar{\Sigma}_5\times[0,t_0), & h&=\tilde{h} \ \text{on}\ \Sigma_6\times[0,t_0),
\end{aligned}\tag{2.22}$$

where $\tilde{u}_i, \tilde{t}_i, \tilde{\varphi}_i, \tilde{m}_i, \tilde{\psi}, \tilde{h}$ are prescribed functions and t_0 is some instant that may be infinite.

The basic equations of the theory are : the equations of motion (2.7), (2.10) : the balance of equilibrated forces (2.12) ; the constitutive equations (2.19) ; the geometrical equations (2.14). To the system of field equations we adjoin the boundary conditions (2.22) and the initial conditions

$$\begin{aligned}
&u_i(x,0)=u_i^0(x),\quad \dot{u}_i(x,0)=v_i^0(x),\quad \varphi_i(x,0)=\varphi_i^0(x),\\
&\dot{\varphi}_i(x,0)=\omega_i^0(x),\quad \psi(x,0)=\psi^0(x),\quad \dot{\psi}(x,0)=\gamma^0(x),\quad x\in\bar{B},
\end{aligned}\tag{2.23}$$

where $u_i^0, v_i^0, \varphi_i^0, \omega_i^0, \psi^0, \gamma^0$ are prescribed functions.

From (2.7), (2.10), (2.12), (2.14), (2.21) we obtain the field equations in terms of the displacement, microrotation and volume fraction for centrosymmetric homogeneous and isotropic bodies

$$\begin{aligned}
&(\mu+\alpha)\Delta u_i+(\lambda+\mu-\alpha)u_{r,ri}+2\alpha e_{ijk}\varphi_{k,j}+b\psi_{,i}+\rho f_i=\rho\ddot{u}_i,\\
&(\gamma+\varepsilon)\Delta\varphi_i+(\beta+\gamma-\varepsilon)\varphi_{r,ri}+2\alpha e_{ijk}u_{k,j}-4\alpha\varphi_i+\rho l_i=I\ddot{\varphi}_i\\
&d\Delta\psi-bu_{r,r}-a\psi+\rho l=\rho k\ddot{\psi}.
\end{aligned}\tag{3.24}$$

If we introduce the notations

$$\begin{aligned}
&\rho c_1^2=\lambda+2\mu,\quad \rho c_2^2=\mu+\alpha,\quad Ic_3^2=\beta+2\gamma,\quad Ic_4^2=\gamma+\varepsilon,\\
&kc_5^2=d,\quad I\omega_0=4\alpha,\quad p\rho=2\alpha,\quad \rho kq=b,\quad \rho k\omega_1^2=a,
\end{aligned}\tag{2.25}$$

then the equations (2.24) can be written in the form

$$\begin{aligned}
&c_2^2\Delta u_i+(c_1^2-c_2^2)u_{r,ri}+pe_{ijk}\varphi_{k,j}+kq\psi_{,i}+f_i=\ddot{u}_i,\\
&c_4^2\Delta\varphi_i+(c_3^2-c_4^2)\varphi_{r,ri}+\frac{1}{2}\omega_0^2e_{ijk}u_{k,j}-\omega_0^2\varphi_i+\frac{\rho}{I}l_i=\ddot{\varphi}_i,\\
&c_5^2\Delta\psi-qu_{r,r}-\omega_1^2\psi+\frac{1}{k}l=\ddot{\psi}.
\end{aligned}\tag{2.26}$$

3. **Shock waves.** Let Σ be a moving surface defined by the equations

$$x_i=x_i(\theta^1,\theta^2,t),\tag{3.1}$$

where θ^1, θ^2 are curvilinear coordinates on the surface. We assume that the functions (3.1) are continuously differentiable with respect to their arguments and that Σ is smooth in the sense that the matrix $(\partial x_i/\partial\theta^\alpha)$ has rank two. The metric tensor of the surface is given by $a_{\alpha\beta}=x_{i,\alpha}\,x_{i,\beta}$. In what follows we will denote by n_i the unit normal to Σ. We note that [12]

$$n_in_i=1,\quad n_ix_{i;\alpha}=0,\quad x_{i;\alpha\beta}=b_{\alpha\beta}\,n_i,\quad n_{i;\alpha}=-a^{\lambda\varphi}b_{\rho\alpha}x_{i;\lambda},\tag{3.2}$$

where indices folowed by a semicolon represents covariant partial differentiation based on the metric of Σ, $b_{\alpha\beta}$ is the second fundamental form of the surface and $a^{\alpha\beta}$ are the elements of the inverse of matrix $(a_{\alpha\beta})$. We also note that

$$a^{\alpha\beta}x_{i;\alpha}x_{j;\beta}=\delta_{ij}-n_in_j,\quad H=\frac{1}{2}a^{\alpha\beta}b_{\alpha\beta},\tag{3.3}$$

where H is the mean curvature of the surface.

If f is a continuously differentiable function of x_i and t on each side of the moving surface Σ which suffers a jump discontinuity across Σ, then we denote the discontinuity in f across the surface by $[f]$. The discontinuities in the first and second derivatives of f satisfy the following relations [13]

$$[f_{,i}]=a^{\alpha\beta}A_{;\alpha}x_{i;\beta}+Bn_i,\quad [\dot f]=\frac{\delta A}{\delta t}-VB,$$

$$[f_{,ij}]=a^{\alpha\beta}(B_{;\alpha}+a^{\lambda\rho}b_{\alpha\lambda}A_{;\rho})(n_ix_{j;\beta}+n_jx_{i;\beta})+$$
(3.4)
$$+a^{\alpha\beta}a^{\nu\rho}(A_{;\alpha\nu}-b_{\alpha\nu}B)x_{i;\beta}x_{j;\rho}+Cn_in_j,$$
$$[\dot f_{,i}]=a^{\alpha\beta}\left(\frac{\delta A}{\delta t}-VB\right)_{;\alpha}x_{i;\beta}+\left(\frac{\delta B}{\delta t}+a^{\alpha\beta}A_{;\alpha}V_{;\beta}-CV\right)n_i,$$
$$[\ddot f]=\frac{\delta}{\delta t}\left(\frac{\delta A}{\delta t}-VB\right)-V\left(\frac{\delta B}{\delta t}+a^{\alpha\beta}A_{;\alpha}V_{;\beta}-CV\right),$$

where

$$\frac{\delta}{\delta t}=\frac{\partial}{\partial t}+Vn_i\frac{\partial}{\partial x_i}\,.$$

is the convected derivative for an observer moving with the surface, V is the speed of propagation of the surface, and

(3.5) $$A=[f],\quad B=[f_{,i}n_i],\quad C=[f_{,ij}n_in_j].$$

By a wave of order N, where $N\geq 1$ is a fixed integer, we mean a solution (u_i,φ_i,ψ) of the equations (2.7), (2.10), (2.12), (2.19), (2.14) such that

i) the functions u_i, φ_i, ψ together with their first $N-1$ derivatives with respect to their arguments, are continuous functions of x and t;

ii) the Nth order derivatives of the vector $U=(u_i,\varphi_i,\psi)$ have jump discontinuities across the surface Σ, but are continuous functions of x and t everywhere else;

iii) the body force, the body couple and the extrinsic equilibrated body force are continuous and if $N\geq 2$, $N-1$ times continuously differentiable for all x and t.

When $N=1$, we call the surface Σ a shock wave.

The integral forms of the equations of balance are special cases of the equation

$$\frac{d}{dt}\int_\Omega\rho F\,dv=\int_S G_in_i\,da+\int_\Omega\rho Q\,dv,$$

which, at a singular surface Σ, is equivalent to the condition [13]

(3.6) $$-V[\rho F]=[G_i]n_i.$$

When (3.6) is applied to equations (2.2)—(2.4), we have the jump conditions

(3.7) $$\rho V[\dot u_i]+[t_{ji}]n_j=0,\quad IV[\dot\varphi_i]+[m_{ji}]n_j=0,\quad \rho kV[\dot\psi]+[h_i]n_i=0.$$

In the case of shock waves, from (3.4) it follows that

(3.8) $$[u_{i,j}]=\xi_in_j,\quad [\varphi_{i,j}]=\gamma_in_j,\quad [\psi_{,i}]=\zeta n_i,$$
$$[\dot u_i]=-V\xi_i,\quad [\dot\varphi_i]=-V\gamma_i,\quad [\dot\psi]=-V\zeta,$$

where

$$\xi_i=\left[\frac{\partial u_i}{\partial n}\right],\quad \gamma_i=\left[\frac{\partial\varphi_i}{\partial n}\right],\quad \zeta=\left[\frac{\partial\psi}{\partial n}\right].$$

By taking the jumps in (2.21), and making use of (2.14), we have

(3.9) $$[t_{ij}]=\xi_rn_r\delta_{ij}\lambda+(\mu+\alpha)\xi_jn_i+(\mu-\alpha)\xi_in_j,$$
$$[m_{ij}]=\beta\gamma_rn_r\delta_{ij}+(\gamma+\varepsilon)\gamma_jn_i+(\gamma-\varepsilon)\gamma_in_j,\quad [g]=-b\xi_rn_r,\quad [h_i]=d\zeta n_i.$$

Taking into account (3.8) and (3.9), the equations (3.7) become

(3.10) $$\{(c_1^2-c_2^2)n_in_j+c_2^2\delta_{ij}-V^2\delta_{ij}\}\xi_j=0,$$

(3.11) $$\{(c_3^2-c_4^2)n_in_j+c_4^2\delta_{ij}-V^2\delta_{ij}\}\gamma_j=0,$$

(3.12) $$(c_5^2-V^2)\zeta=0.$$

If the vector ξ_i does not vanish, the wave is a macro-shock wave. Equation (3.10) admits a non trivial solution for ξ_s if and only if $(c_2^2-V^2)^2(c_1^2-V^2)=0$. It is an immediate consequence of equation (3.10) that longitudinal macro-shock waves (for which $\xi_i=\xi n_i$) propagate with speed $V=c_1$. The speed of propagation of transverse macro-shocks waves (for which $\xi_in_i=0$) is $V=c_2$.

If $\gamma_i\neq 0$, the wave is a micro-shock wave. The possible speeds of propagation of micro-shock waves are determined by $(c_4^2-V^2)^2(c_3^2-V^2)=0$. Longitudinal macro-shock waves (for which $\gamma_i=\gamma n_i$) propagate with speed $V=c_3$, and the transverse micro-shock waves with the speed of propagation $V=c_4$.

If $\zeta\neq 0$, the wave is a shock wave of compaction or distension. The possible speed of propagation of this wave is $V=c_5$.

Let us study now the growth of the waves. Taking into account that V is constant for all waves, from (3.4) we obtain

$$[\ddot u_i]=-2V\frac{\delta\xi_i}{\delta t}+V^2\mu_i,\quad [u_{s,ij}]=a^{\alpha\beta}\xi_{s;\alpha}(n_ix_{j;\beta}+n_jx_{i;\beta})-$$
(3.13)
$$-a^{\alpha\beta}a^{\nu\rho}b_{\alpha\nu}\xi_sx_{i;\beta}x_{j;\rho}+\mu_sn_in_j,$$
$$[u_{r,ri}]=a^{\alpha\beta}\xi_{r;\alpha}(n_ix_{r;\beta}+n_rx_{i;\beta})-a^{\alpha\beta}a^{\nu\rho}b_{\alpha\nu}\xi_rx_{i;\alpha}x_{r;\rho}+\mu_rn_rn_i,$$
$$[u_{i,jj}]=-2H\xi_i+\mu_i,$$

where $\mu_i=[u_{i,rs}n_rn_s]$. We note that

(3.14) $$a^{\alpha\beta}\xi_{r;\alpha}x_{r;\beta}=a^{\alpha\beta}(\xi_rx_{r;\beta})_{;\alpha}-2H\xi,\quad \xi_{r;\alpha}n_r-\xi_ra^{\nu\rho}b_{\alpha\nu}x_{r;\rho}=\xi_{;\alpha},$$

where

(3.15) $$\xi=\xi_in_i\,.$$

Using (3.14) we can write

(3.16) $$[u_{r,ri}]=a^{\alpha\beta}(\xi_rx_{r;\beta})_{;\alpha}n_i-2H\xi n_i+a^{\alpha\beta}x_{i;\beta}\xi_{;\alpha}+\mu_rn_rn_i\,.$$

From (2.26) we obtain

$$(V^2-c_2^2)\mu_i-(c_1^2-c_2^2)\{\mu_rn_rn_i+a^{\alpha\beta}(\xi_rx_{r;\beta})_{;\alpha}n_i+a^{\alpha\beta}x_{i;\beta}\xi_{;\alpha}\}-$$
(3.17)
$$-2V\frac{\delta\xi_i}{\delta t}+2H\{(c_1^2-c_2^2)\xi n_i+c_2^2\xi_i\}-pc_{ijk}\gamma_kn_j-kq\zeta n_i=0.$$

$$(3.18)\quad \begin{aligned}&(V^2-c_4^2)\gamma_i-(c_3^2-c_4^2)\{\gamma_j n_j n_i+a^{\alpha\beta}(\gamma_r x_{r;\beta})_{;\alpha}n_i+a^{\alpha\beta}x_{i;\beta}\gamma_{;\alpha}\}-\\&-2V\frac{\delta\gamma_i}{\delta t}+2H\{(c_3^2-c_4^2)\gamma n_i+c_4^2\gamma_i\}-\frac{1}{2}\omega_0^2\varepsilon_{ijk}\xi_k n_j=0,\end{aligned}$$

$$(3.19)\quad (V^2-c_5^2)\tau+2Hc_5^2\zeta+q\xi-2V\frac{\delta\zeta}{\delta t}=0,$$

where

$$(3.20)\quad \gamma_i=[\varphi_{i,rs}n_r n_s],\quad \tau=[\phi_{,ij}n_i n_j],\quad \gamma=\gamma_i n_i .$$

If we multiply (3.17) by n_i and sum on i, we obtain

$$(3.21)\quad (V^2-c_1^2)\mu_i n_i-(c_1^2-c_2^2)a^{\alpha\beta}(\xi_r x_{r;\beta})_{;\alpha}-2V\frac{\delta\xi}{\delta t}+2Hc_1^2\xi-kq\zeta=0.$$

We have used the fact that $\delta n_i/\delta t=0$.

In the case of longitudinal macro-shock waves $V=c_1$, $\xi_i=\xi n_i$, and assuming that $c_1\neq c_4$, (3.21) yields the growth equation

$$(3.22)\quad \frac{1}{c_1}\frac{\delta\xi}{\delta t}=\frac{d\xi}{dn}=H\xi,$$

where n is the distance measured along the normal to the wave surface. In this case the equation (3.19) reduces to

$$(3.23)\quad \tau=q\xi/(c_5^2-c_1^2).$$

If, at some time $t=t_0$, the mean and Gaussian curvatures of surface are H_0 and K_0, respectively, then, at a subsequent time t, we have [12] $H=(H_0-nK_0)/(1-2nH_0+n^2K_0)$, and the transport equation (3.22) can be integrated to yield

$$(3.24)\quad \xi=\xi_0(1-2nH_0+n^2K_0)^{-1/2},$$

where ξ_0 is the strength of the wave at time $t=t_0$. From (2.23) we obtain $\tau=q\xi_0(1-2nH_0+n^2K_0)^{-1/2}/(c_5^2-c_1^2)$, so that a disturbance in the porosity which is of second order, is induced by a longitudinal macro-shock wave. This discontinuity propagates with the same speed as the inducing longitudinal macro-shock wave.

If $c_1\neq c_3$, $c_1\neq c_4$, then $\eta_i=0$ and the equation (3.28) implies $\gamma_i=0$. Thus, the longitudinal macro-shock waves induce no micro-acceleration discontinuities.

The speed of propagation of irrotational macro-shock waves is $V=c_2$. If we assume $c_2\neq c_5$ we have $\zeta=0$ and the equation (3.21) reduces to

$$a^{\alpha\beta}(\xi_r x_{r;\beta})_{;\alpha}=-n_j\mu_j .$$

Using this result in (3.17), we have

$$\frac{1}{c_2}\frac{\delta\xi_i}{\delta t}=\frac{d\xi_i}{dn}=H\xi_i.$$

As before, we have $\xi_i=\xi_i^0(1-2nH_0+n^2K_0)^{-1/2}$, where $\xi_i=\xi_i^0$ when $t=t_0$. In this case $\xi=0$, $\tau=0$.

If $c_2\neq c_3$, $c_2\neq c_4$ then $\eta_i=0$ and the equations (3.18) reduce to

$$\{(c_2^2-c_4^2)\delta_{ij}-(c_3^2-c_4^2)n_i n_j\}\gamma_j=\frac{1}{2}\omega_0^2\varepsilon_{ijk}\xi_k n_j .$$

Since, by hypothesis, the speed of propagation of the possible irrotational macro-shock wave is not equal to c_3 or c_4, these equations have a nontrivial solution for γ_i. This means that a transverse macro-shock wave is accompanied by a discontinuity in $\ddot{\varphi}_i$.

In the case of micro-shock waves ($\xi_i=0$) when (3.18) is multiplied by n_i and summation convention is applied, we find that the growth equation for longitudinal micro-shocks implies $\gamma=\gamma_0(1-2nH_0+n^2K_0)^{-1/2}$, where γ_0 is the strength of the wave at time t_0.

If $c_3\neq c_1$, $c_3\neq c_2$, $c_3\neq c_5$ then $\xi_i=0$, $\zeta=0$ and the equations (3.17) and (3.18) imply that $\mu_i=0$, $\tau=0$. Thus, the longitudinal micro-shock wave is a pure micro-wave.

The growth equation for transverse micro-shock wave is obtained similarly. If $c_4\neq c_1$, $c_4\neq c_2$, $c_4\neq c_5$ then $\xi_i=0$, $\zeta=0$. From (3.19) we obtain $\tau=0$. The equations (3.17) reduce to

$$\{(c_4^2-c_2^2)\delta_{ij}-(c_1^2-c_2^2)n_i n_j\}\mu_j=p\varepsilon_{ijk}\gamma_k n_j .$$

By hypothesis these equations have a nontrivial solution for μ_i. This means that a macro-acceleration discontinuity is induced by the transverse micro-shock waves.

If $V=c_5$, $c_5\neq c_1$, $c_5\neq c_2$, then $\xi_i=0$ and from (3.19) we obtain the growth equation

$$\frac{1}{c_5}\frac{\delta\zeta}{\delta t}=\frac{d\zeta}{dn}=H\zeta.$$

Thus, we have

$$\zeta=\zeta_0(1-2nH_0+n^2K_0)^{-1/2},$$

where $\zeta=\zeta_0$ when $t=t_0$. Moreover, if $c_5\neq c_3$, $c_5\neq c_4$ we have $\eta_i=0$. The equations (3.17) reduce to

$$\{(c_5^2-c_2^2)\delta_{ij}-(c_1^2-c_2^2)n_i n_j\}\mu_j=kqn_i ,$$

so that

$$\mu_i=-kq\zeta n_i(c_1^2-c_5^2)^{-1}.$$

Thus, a macro-acceleration discontinuity is induced by a shock wave of compaction or distension.

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SOME THEOREMS IN A GRADE CONSISTENT MICROPOLAR THEORY OF THERMOELASTICITY

BY

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1. **Introduction.** Brulin and Hjalmars [1, 2] have derived a theory of micropolar continua, where the second-order displacement gradient is added to the classic set of independent constitutive variables. The grade consistent micropolar theory of thermoelasticity was studied in [2, 3].

The present paper is concerned with the linear theory of thermoelastic solids. The purpose of this paper is to prove some basic theorems concerning the uniqueness of solution, the reciprocity relations and the variational characterization of the solution.

2. **Basic equations.** We assume that the body occupies a bouded region B with Lipschitz boundary ∂B which consists of a finite number of smooth surfaces. Denote γ_α intersections of two adjoined smooth surfaces and $\Gamma = \bigcup_\alpha \gamma_\alpha$. We denote by u_i the components of the displacement vector and by φ_i the components of the microrotation vector.

The basic equations of the linear theory are (see [1], [2]),

— the equations of motion

$$\tau_{ji,j} - \mu_{kji,kj} + F_i = \rho \ddot{u}_i, \quad m_{ji,j} + \varepsilon_{irs}\tau_{rs} + M_i = I_{ij}\ddot{\varphi}_j, \tag{2.1}$$

— the equation of entropy

$$\rho T_0 \dot{\eta} - q_{i,i} = S, \tag{2.2}$$

— the constitutive equations

$$\begin{aligned} \tau_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}\varkappa_{rs} + D_{ijpqr}\varkappa_{pqr} + a_{ij}\theta, \\ m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\varkappa_{rs} + E_{ijpqr}\varkappa_{pqr} + c_{ij}\theta, \\ \mu_{ijk} &= D_{rsijk}e_{rs} + E_{rsijk}\varkappa_{rs} + F_{ijkpqr}\varkappa_{pqr} + g_{ijk}\theta, \\ \rho\eta &= d\theta - a_{ij}e_{ij} - c_{ij}\varkappa_{ij} - g_{ijk}\varkappa_{ijk}, \\ q_i &= k_{ij}\theta_{,j}, \end{aligned} \tag{2.3}$$

— the geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \varkappa_{ij} = \varphi_{j,i}, \quad \varkappa_{ijk} = u_{k,ij}. \tag{2.4}$$