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EXISTENCE THEOREMS IN ELASTOSTATICS OF MATERIALS WITH VOIDS

BY

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1. Introduction. In [1], J. W. N u n z i a t o and S. C. C o w i n gave a nonlinear theory of elastic materials with voids. The intended applications of this theory are to geological materials like rocks and soils and to manufactured porous materials. The linear theory of the elastic materials with voids was established by S. C. C o w i n and J. W. N u n z i a t o in [2]. Some basic theorems for statical and dynamical cases were obtained by D. I e ş a n ([3]). In this paper we consider some fundamental three-dimensional boundary value problems of equilibrium for isotropic and homogeneous elastic materials with voids. In Section 3 the fundamental matrix is given and its properties are established. A. S o m i g l i a n a's type representation of the solution is given in Section 4. In Sections 5 and 6, using the method of potentials ([4], [5]) we reduce the boundary value problems to two-dimensional singular integral equations for which Fredholm's basic theorems hold ([6]). In Section 7, existence theorems for interior and exterior problems are established.

2. Basic Equations. Throughout this paper a rectangular coordinate system $Ox_k (k=1, 2, 3)$ is employed. We shall use summation and differentiation conventions: Latin subscripts — unless otherwise specified — are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and a subscript preceded by a comma denotes partial differentiation with respect to the corresponding cartesian coordinate.

Let us assume that the body occupies a finite region B bounded by a closed Liapunov surface Σ.

The basic equations of the linear theory of homogeneous and isotropic materials with voids are ([2])

— the equations of equilibrium

$$t_{ji,j}+f_i=0, \quad h_{i,i}+g+l=0, \tag{2.1}$$

— the constitutive equations

$$\begin{aligned} t_{ij}&=\lambda e_{rr}\delta_{ij}+2\mu e_{ij}+\beta\varphi\delta_{ij}, \\ h_i&=\alpha\varphi_{,i} \\ g&=-\xi\varphi-\beta e_{rr}, \end{aligned} \tag{2.2}$$

— the kinematic equations

(2.3) $$2e_{ij} = u_{i,j} + u_{j,i}.$$

Here we have used the following notations: u_i — the components of the displacement vector; φ — the change in volum fraction from the reference volume fraction; e_{ij} — the components of the strain tensor; t_{ij} — the components of the stress tensor; h_i — the components of the equilibrated stress vector; g — the intrinsic equilibrated body force; f_i — the components of the body force vector; l — the extrinsic equilibrated body force; $\lambda, \mu, \alpha, \beta, \xi$ — the characteristic coefficients of the material satisfying the conditions ([2])

(2.4) $$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad 3\lambda + 2\mu > 0, \quad (3\lambda + 2\mu)\xi > 3\beta^2.$$

If the surface tractions are prescribed on the boundary, then

(2.5) $$t_{ij} n_j = \tilde{t}_i, \quad h_i n_i = \tilde{h},$$

where $n_j = \cos(n_x, x_j)$ are the components of the outward unit normal to Σ in the point $x = (x_j)$ and $\tilde{t}_i, \tilde{h}$ are prescribed functions defined on Σ.

From (2.1), (2.2), (2.3) we obtain the field equations of the static theory of elastic materials with voids in terms of u_i and φ:

(2.6) $$\mathcal{L}_i U \equiv \mu \Delta u_i + (\lambda + \mu) u_{j,ji} + \beta \varphi_{,i} = f_i,$$
$$\mathcal{L}_4 U \equiv \alpha \Delta \varphi - \xi \varphi - \beta u_{j,j} = l,$$

where Δ is the three-dimensional Laplace operator and $U = (u_1, u_2, u_3, \varphi)^*$; v^* denotes the transposed of the matrix v.

Using (2.2) and (2.3), the boundary conditions (2.5) become

(2.7) $$\mathcal{T}_i U \equiv (\lambda \operatorname{div} \mathbf{u} + \beta\varphi) n_i + \mu \frac{\partial u_i}{\partial n_x} + \mu \frac{\partial u_j}{\partial n_x} + \mu \frac{\partial u_j}{\partial x_i} n_j = \tilde{t}_i, \quad \mathcal{T}_4 U \equiv \alpha \frac{\partial \varphi}{\partial n_x} = \tilde{h}.$$

If we introduce the operators

(2.8) $$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)^*, \quad \mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4)^*,$$

then the system (2.6) and the surface conditions (2.7) become

(2.9) $$\mathcal{L} U = -\mathcal{F},$$

(2.10) $$\mathcal{T} U = T,$$

where

$$\mathcal{F} = (f_1, f_2, f_3, l)^*, \quad T = (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \tilde{h})^*.$$

3. Fundamental Solutions of the Equilibrium Equations. The force $\mathcal{F}$ in the right-hand side of the equations (2.9) are in general different from zero only in a some finite range of B. It is of particular interest to study the case where the force f_i or l is concentrated at a point.

First we give a Galerkin type representation of the solution.

Theorem 3.1. *If*

(3.1) $$u_i = \alpha(\lambda + 2\mu)\Delta(\Delta - \varkappa^2)\Phi_i - [(\lambda + \mu)(\alpha\Delta - \xi) + \beta^2]\Phi_{j,ji} - \beta\Upsilon_{,i},$$
$$\varphi = \mu\beta\Delta\Phi_{j,j} + (\lambda + 2\mu)\Delta\Upsilon,$$

where the fields Φ_i of class C^6 and Υ of class C^4 on B satisfy the equations

(3.2) $$\Delta\Delta(\Delta - \varkappa^2)\Phi_i = -a f_i,$$
$$\Delta(\Delta - \varkappa^2)\Upsilon = -\mu a l,$$

(3.3) $$\varkappa^2 = \frac{\xi(\lambda + 2\mu) - \beta^2}{\alpha(\lambda + 2\mu)}, \quad a = \frac{1}{\mu\alpha(\lambda + 2\mu)},$$

then u_i and φ satisfy the equations (2.6).

Proof. Taking into account (3.2), (3.3) and substituting (3.1) in (2.6) we have

$$\mu\Delta u_i + (\lambda + \mu) u_{j,ji} + \beta\varphi_{,i} + f_i = \mu\alpha(\lambda + 2\mu)\Delta\Delta(\Delta - \varkappa^2)\Phi_i +$$
$$+ (\lambda + \mu)[(\lambda + 2\mu)(\alpha\varkappa^2 - \xi) + \beta^2]\Delta\Phi_{j,ji} + f_i = 0,$$
$$\alpha\Delta\varphi - \xi\varphi - \beta u_{j,j} + l = \beta[\beta^2 + (\lambda + 2\mu)(\alpha\varkappa^2 - \xi)]\Delta\Phi_{j,j} +$$
$$+ \alpha(\lambda + 2\mu)\Delta(\Delta - \varkappa^2)\Upsilon + l = 0,$$

and the proof is complete.

Let us assume that a concentrated force acts at a point $y = (y_i)$ along the axis Ox_j,

$$f_i = \delta_{ij}\delta(x - y), \quad l = 0,$$

where $\delta(\cdot)$ is Dirac delta. In this case, Φ_j satisfies

(3.4) $$\Delta\Delta(\Delta - \varkappa^2) q = -a\delta(x - y),$$

and $\Phi_i = \Upsilon = 0$ $(i \neq j)$.

The solution obtained from (3.1) for $\Phi_j = q$, $\Phi_i = \Upsilon = 0$ $(i \neq j)$ will be denoted by $U^{(j)}$.

If a extrinsic equilibrated force acts in the point $y = (y_i)$

$$f_i = 0, \quad l = \delta(x - y),$$

then $\Phi_i = 0$ and Υ satisfies

(3.5) $$\Delta(\Delta - \varkappa^2)\tau = -a\mu\delta(x - y).$$

We denote by $U^{(4)}$ the solution obtained from (3.1) when $\Phi_i = 0$, $\Upsilon = \tau$.

If q_1, q_2, q_3 satisfy respectively the equations

$$\Delta\Delta q_1 = s, \quad \Delta q_2 = s, \quad \Delta(\Delta - \varkappa^2) q_3 = s,$$

then

$$q = \frac{-1}{\varkappa^4}(\varkappa^2 q_1 + q_2 - q_3),$$

satisfies the equation

$$\Delta\Delta(\Delta - \varkappa^2) q = s.$$

It is easy to verify that for $s = -a\delta(x - y)$, we have

(3.6) $$q = -\frac{a}{8\pi\varkappa^4}\left(\varkappa^2 r + \frac{2}{r} - \frac{2}{r} e^{-\varkappa r}\right)$$

where $r=r(x,y)=|x-y|$ is the distance between the points $x=(x_i)$ and $y=(y_i)$.

Similarly, the solution of the equation (3.5) is given by

$$\tau=\left(\frac{-a\mu}{4\pi\varkappa^2 r}\right)(1-e^{-\varkappa r}). \tag{3.7}$$

Formulae (3.1), (3.6) and (3.7) yield the following expressions for the components of the solutions

$$\begin{aligned} u_i^{(j)}&=\alpha(\lambda+2\mu)\delta_{ij}\Delta(\Delta-\varkappa^2)q-[(\lambda+\mu)(\alpha\Delta-\xi)+\beta^2]q_{,ij},\\ \varphi^{(j)}&=\mu\beta\Delta q_{,j},\quad u_i^4=-\beta\tau_{,i},\quad \varphi^{(4)}=(\lambda+2\mu)\Delta\tau. \end{aligned}\tag{3.8}$$

We introduce the notations

$$\Gamma^{(j)}=(u_1^{(j)},\ u_2^{(j)},\ u_3^{(j)},\ \varphi^{(j)})^*,\ (j=1,2,3,4). \tag{3.9}$$

The matrix whose columns are the vectors $\Gamma^{(k)}(x,y)$, $(k=1,2,3,4)$ will be designated by $\Gamma(x,y)$.

It is easy to prove that

$$\Gamma(x,y)=\Gamma^*(y,x). \tag{3.10}$$

and that each column of the matrix $\Gamma(x,y)$ satisfies at x the homogeneous system

$$\mathcal{L}\Gamma^{(k)}(x,y)=0,\ (k=1,2,3,4). \tag{3.11}$$

The extensions in series of powers of the functions $q(x,y)$ and $\tau(x,y)$ are

$$q(x,y)=\frac{a}{96\pi}r^3-\frac{a}{8\pi\varkappa^4}\left(2\varkappa+\frac{\varkappa^3r^2}{3}+\frac{2\varkappa^5r^4}{5!}+\dots\right),$$

$$\tau(x,y)=\frac{a\mu}{8\pi}r-\frac{a\mu}{4\pi\varkappa^2}\left(\varkappa+\frac{\varkappa^3r^2}{3!}-\frac{\varkappa^4r^3}{4!}+\dots\right).$$

Let us note that

$$\bar q(x,y)=\frac{1}{96\pi}r^3,\qquad \bar\tau(x,y)=\frac{1}{8\pi}r.$$

Satisfy the equations

$$\Delta\Delta\Delta\bar q=-\delta(x-y),\qquad \Delta\Delta\bar\tau=-\delta(x-y)$$

respectively.

The functions

$$\begin{aligned} U_j^{(k)}&=\alpha(\lambda+2\mu)\delta_{jk}\Delta\Delta(a\bar q)-\alpha(\lambda+\mu)\Delta(a\bar q)_{,jk}=\\ &=\frac{1}{8\pi\mu(\lambda+2\mu)r}[(\lambda+3\mu)\delta_{jk}+(\lambda+\mu)r_{,j}r_{,k}],\\ U_j^{(4)}&=\chi^{(k)}=0,\quad \chi^{(4)}=(\lambda+2\mu)\Delta(a\mu\bar\tau)=\frac{1}{4\pi ar}, \end{aligned}\tag{3.12}$$

represent the main part of the components of the matrix $\Gamma(x,y)$ that is, the order of the singularities in the remaining terms are smaller that in $U_j^{(k)}$, $\chi^{(k)}$ $(k=1,2,3,4)$. Moreover, $U_j^{(k)}$ represent the fundamental solutions in the statical case in the theory of classical elasticity.

In what follows, we shall denote by $\Pi(x,y)$ the matrix whose columns are $\Pi^{(k)}(x,y)$, where

$$\Pi^{(k)}(x,y)=(U_1^{(k)},\ U_2^{(k)},\ U_3^{(k)},\ \chi^{(k)})^*,\ (k=1,2,3,4). \tag{3.13}$$

Obviously

$$\Pi(x,y)=\Pi^*(y,x),\quad \Pi(x,y)=\Pi(y,x). \tag{3.14}$$

It will be noticed that the difference

$$G(x,y)=\Gamma(x,y)-\Pi(x,y), \tag{3.15}$$

has all its components bounded when $x=y$ and their first derivatives have in $x=y$ a pole of the first order or are bounded.

We introduce the notations

$$P^{(k)}(x,y)=\mathcal{T}\Gamma^{(k)}(x,y),\ R^{(k)}(x,y)=\mathcal{T}\Pi^{(k)}(x,y)\quad (k=1,2,3,4) \tag{3.16}$$

The square matrices having the columns $P^{(k)}(x,y)$, respectively $R^{(k)}(x,y)$ will be denoted by $P(x,y)$ and $R(x,y)$.

From (2.7), (3.12), (3.16) we obtain the components of the matrix $R(x,y)$

$$\begin{aligned} R_j^{(k)}(x,y)&=\frac{1}{4\pi}\left[\delta_{jk}+\frac{3(\lambda+\mu)}{(\lambda+2\mu)}r_{,j}r_{,k}\right]+\frac{\mu}{4\pi(\lambda+2\mu)}\left[n_k\left(\frac{1}{r}\right)_{,j}-n_j\left(\frac{1}{r}\right)_{,k}\right],\\ R_4^{(k)}(x,y)&=R_k^{(4)}(x,y)=0,\quad R_4^{(4)}(x,y)=\frac{1}{4\pi}\frac{\partial}{\partial n_x}\left(\frac{1}{r}\right). \end{aligned}\tag{3.17}$$

In the classical elasticity was introduced the generalized stress vector ([4])

$$P^{(x)}\mathbf{u}=(\alpha'+\mu')\frac{\partial\mathbf{u}}{\partial n_x}+\beta'\mathbf{n}_x\operatorname{div}\mathbf{u}+\alpha'\mathbf{n}_x\times\operatorname{rot}\mathbf{u}, \tag{3.18}$$

where α', β' *are two real numbers restricted by the condition*

$$\alpha'+\beta'=\lambda'+\mu',$$

λ' and μ' being the Lamé constants. If $\alpha'=\mu'$ and $\beta'=\lambda'$, the generalized stress vector becomes the stress vector in the classical elasticity. Due to the fact that $U_j^{(k)}(x,y)$ are the fundamental solutions in the statical theory of classical elasticity for an elastic isotropic and homogeneous body having the elastic constants λ and μ, taking $\alpha=\mu$ and $\beta=\lambda$, from (3.18) we have

$$P^{(x)}\Pi^{(k)}(x,y)=R^{(k)}(x,y)\quad (k=1,2,3,4). \tag{3.19}$$

From (3.19), we obtain ([4])

$$\int_\Sigma P^{(x)}\Pi^{(k)}(x,y)d\sigma_y=\int_\Sigma R^{(k)}(x,y)d\sigma_y=-\delta_{ik}\eta_i(x),\quad (k=1,2,3,4), \tag{3.20}$$

where

$$\eta(x)=\begin{cases}1 & \text{if } x\in B,\\ 1/2 & \text{if } x\in\Sigma,\\ 0 & \text{if } x\notin B\cup\Sigma.\end{cases}\tag{3.21}$$

From (3.12), (3.17), using a well-known relation in the theory of potentials ([7]), we have

$$\int_{\Sigma} R_i^{(4)}(x, y)d\sigma_y=-\eta(x).\tag{3.22}$$

Taking into account the above properties of $\Gamma(x, y)$ and $\Pi(x, y)$ we see that the difference

$$L(x, y)=P(x, y)-R(x, y),\tag{3.23}$$

has all its components $L_i^{(j)}(x, y)$ with a pole of the first order in $x=y$. $(i, j=1, 2, 3, 4)$.

The transposed of the matrix $P(x, y)$ is designated by $\Lambda(x, y)$ and the transposed of the matrix $R(x, y)$ is denoted by $\Omega(x, y)$.

Obviously, the difference

$$N(x, y)=\Lambda(x, y)-\Omega(x, y),\tag{3.24}$$

has also its components with a pole of the first order in $x=y$.

From (3.21), (3.22) we get

$$\int_{\Sigma} \Omega_j^{(k)}(x, y)d\sigma_y=-\eta(x)\delta_{jk},\quad (j, k=1, 2, 3, 4).\tag{3.25}$$

Theorem 3.2. *Each column of the matrix $\Lambda(x, y)$ considered as a function of x satisfies the equation*

$$\mathcal{L}\Lambda^{(k)}(x, y)=0,\quad (k=1, 2, 3, 4)$$

for $x\neq y$.

Proof. To prove the above result we use the property (3.11) of the matrix $\Gamma(x, y)$ and the definitions of $P(x, y)$ and $\Lambda(x, y)$.

4. Somigliana Relation. The reciprocity relation in the linear theory of elastic materials with voids can be written in the form ([3])

$$\int_{B} (U\mathcal{L}V-V\mathcal{L}U)dv=\int_{\Sigma} (U\mathcal{T}V-V\mathcal{T}U)d\sigma.\tag{4.1}$$

As in classical theory, from (4.1) we obtain

$$U(x)=\int_{\Sigma}\left[\Gamma(x, y)\mathcal{T}\left(\frac{\partial}{\partial y}, n_y\right)U(y)-\Lambda(x, y)U(y)\right]d\sigma_y,\quad x\in B.\tag{4.2}$$

Following [4] we obtain for $x\notin B\cup\Sigma$ a representation analoguous to (4.2), if U satisfies the relations.

$$u_i\in O(r^{-1}),\quad u_{i,k}\in O(r^{-2}),\quad \varphi\in O(r^{-2}),\quad \varphi_{,k}\in O(r^{-3}).\tag{4.3}$$

5. The Elastic Potentials. Let $H(\Sigma)$ be the class of functions satisfying Hölder condition at Σ. For $\psi(x)\in H(\Sigma)$ we introduce the potential of a single layer

$$V(x, \psi)=\int_{\Sigma} \Gamma(x, y)\psi(y)d\sigma_y,\tag{5.1}$$

the potential of a double layer

$$W(x, \psi)=\int_{\Sigma} \Lambda(x, y)\psi(y)d\sigma_y,\tag{5.2}$$

and the volume potential

$$U(x, \psi)=\int_{B} \Gamma(x, y)\psi(y)dv_y.\tag{5.3}$$

The behaviour of the surface potentials $V(x, \psi)$ and $W(x, \psi)$ near the boundary is the same as that for the harmonic potentials. Therefore we shall not give extensive proofs of the following theorems.

Theorem 5.1. *The potential of a single layer is continuous throughout.*

Theorem 5.2. *The potential of a double layer has finite limits when the point x tends to $z\in\Sigma$ both from within and from without, and these limits are respectively equal to*

$$\begin{aligned}W_i(z, \psi)&=-\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda(z, y)\psi(y)d\sigma_y,\\ W_e(z, \psi)&=\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda(z, y)\psi(y)d\sigma_y,\end{aligned}\tag{5.4}$$

the integrals in the right-hand sides being conceived of as principal values.

Proof. We use the Hölder's condition satisfied by $\psi(z)$ and the above properties of $\Lambda(x, y)$, $\Gamma(x, y)$. The double layer potential (5.2) can be presented in the form

$$\begin{aligned}W(x, \psi)=\int_{\Sigma} \Lambda(x, y)[\psi(y)-\psi(z)]d\sigma_y+\int_{\Sigma} [\Lambda(x, y)-\Omega(x, y)]\psi(z)d\sigma_y+\\ +\int_{\Sigma} \Omega(x, y)\psi(z)d\sigma_y.\end{aligned}\tag{5.5}$$

Using (3.24), (3.25) we obtain

$$W_i(z, \psi)=\psi(z)=W(z, \psi)+(1/2)\psi(z)=W_e(z, \psi),$$

from which it follows the desired result.

Theorem 5.3. *The $\mathcal{T}$ operator of the single layer potential (5.1) has finite limits as the point x tends to $z\in\Sigma$ both from within and from without and these limits are respectively equal to*

$$\begin{aligned}(\mathcal{T}V)_i&=\frac{1}{2}\psi(z)+\int_{\Sigma}\mathcal{T}\left(\frac{\partial}{\partial z}, n_z\right)\Gamma(z, y)\psi(y)d\sigma_y,\\ (\mathcal{T}V)_e&=-\frac{1}{2}\psi(z)+\int_{\Sigma}\mathcal{T}\left(\frac{\partial}{\partial z}, n_z\right)\Gamma(z, y)\psi(y)d\sigma_y.\end{aligned}\tag{5.6}$$

Proof. The proof is based on the possibility to represent $\mathcal{T}V(x)$ for x non located on Σ in the form

$$\mathcal{T}V(x)=\int_{\Sigma}\left[\mathcal{T}\left(\frac{\partial}{\partial x}, n_x\right)\Gamma(x, y)+\mathcal{T}\left(\frac{\partial}{\partial y}, n_y\right)\Gamma(y, x)\right]\psi(y)d\sigma_y-$$
$$-\int_{\Sigma}\mathcal{T}\left(\frac{\partial}{\partial y}, n_y\right)\Gamma(y, x)\psi(y)d\sigma_y.$$

The last term in the right-hand side is a double layer potential and its behaviour near the boundary was established by the Theorem 5.2. The first term becomes

$$\int_{\Sigma}\left[\mathcal{T}\left(\frac{\partial}{\partial x}, n_x\right)\Gamma(x, y)+\mathcal{T}\left(\frac{\partial}{\partial y}, n_y\right)\Gamma(y, x)\right]\psi(y)d\sigma_y=$$
$$=\int_{\Sigma}[R(x, y)+R(y, x)+J(x, y)]\psi(y)d\sigma_y,$$

where $J(x, y)$ has no singularities on Σ. Using the properties of the Liapunov surface it follows that

$$\int_{\Sigma}[R(x, y)+R(y, x)]\psi(y)d\sigma_y,$$

is continuous, and then, the relations (5.6) are established.

We can prove that

$$\mathcal{L}U(x, \psi)=-\psi(x), \qquad x\in B,$$
$$\mathcal{L}V(x, \psi)=0, \quad \mathcal{L}W(x, \psi)=0 \qquad x\notin\Sigma. \tag{5.7}$$

6. **Reduction to Integral Equations.** We consider the system of differential equations

$$\mathcal{L}\left(\frac{\partial}{\partial x}\right)U=0 \tag{6.1}$$

and the following boundary value problems:

Interior problems. To find the regular solution in B of the system (6.1) satisfying one of the conditions

(I_1) $\qquad \lim_{x\to z} U(x)=G(z).$

(I_2) $\qquad \lim_{x\to z}\mathcal{T}\left(\frac{\partial}{\partial x}, n_z\right)U(x)=G(z), \quad x\in B, \quad z\in\Sigma.$

where $G(z)$ is a given function from $H(\Sigma)$.

Exterior problems. To find the regular solution in B_e of the system (6.1) satisfying the conditions (4.3) and one of the conditions

(E_1) $\qquad \lim_{x\to z} U(x)=G(z).$

(E_2) $\qquad \lim_{x\to z}\mathcal{T}\left(\frac{\partial}{\partial x}, n_z\right)U(x)=G(z), \quad x\in B_e, \quad z\in\Sigma,$

where B_e is the complementary of $B\cup\Sigma$ to Euclidean three-dimensional space.

The homogeneous problems corresponding to (I_1), (I_2), (E_1), (E_2) for $G(z)=0$ will be denoted by (I_1^0), (I_2^0), (E_1^0), (E_2^0).

In [3] the following theorems on uniqueness are established.

Theorem 6.1. *The regular solution in B of the problem (I_1^0) is identically zero.*

Theorem 6.2. *The regular solution in B of the problem (I_2^0) is*

$$U_0(x)=(u_1^0, u_2^0, u_3^0, 0)^*, \tag{6.2}$$

where

$$u_k^0=a_k+\varepsilon_{kij}b_ix_j,$$

a_k, b_k being arbitrary constants and ε_{ijk} the alternating symbol.

Theorem 6.3. *The regular solution in B_e of the problems (E_1^0), (E_2^0) satisfying the conditions (4.3) are identically zero.*

We look after the solutions of the problems (I_1), (E_1) in the form of a double layer potential and the solutions of the problems (I_2), (E_2) in the form of a single layer potential. On the basis of the Theorem 5.2 and Theorem 5.3 we obtain the following integral equations for determining the density $\psi(z)$

(I_1) $$-\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda(z, y)\psi(y)d\sigma_y=G(z),$$

(I_2) $$\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda^*(z, y)\psi(y)d\sigma_y=G(z),$$

(E_1) $$\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda(z, y)\psi(y)d\sigma_y=G(z),$$

(E_2) $$-\frac{1}{2}\psi(z)+\int_{\Sigma}\Lambda^*(z, y)\psi(y)d\sigma_y=G(z).$$

The homogeneous equations coresponding to the problems (I_1^0), (I_2^0), (E_1^0), (E_2^0) will be denoted by the same symbols.

These equations are two-dimensional singular integral equations. The equations (I_1^0), (E_2^0) and (I_2^0), (E_1^0) are pair-wise mutually associate equations. We shall prove that Fredholm's theory ([6]) is applicable.

The integral operator of the problem (I_1) can be written in the form

$$K\psi=-\psi(z)+\int_{\Sigma}k(z, y)\psi(y)d\sigma_y, \tag{6.3}$$

where

$$k(z, y)=2\Lambda(z, y). \tag{6.4}$$

According to (3.17) we have $\quad k(z, y)=k_1(z, y)+k_2(z, y)$

with

$$k_{ij}^{(1)}=\frac{1}{2\pi r^3}\,\frac{\mu}{(\lambda+2\mu)}\,[\cos(n_y, z_i)(z_j-y_j)-\cos(n_y, z_j)(z_i-y_i)],$$

$$k_{4j}^{(1)}=k_{j4}^{(1)}=0,\quad k_{44}^{(1)}=0,$$

$$k_{ij}^{(2)}=\frac{1}{2\pi}\left[\delta_{ij}+\frac{3(\lambda+\mu)}{\lambda+2\mu}\,\frac{1}{r^3}(z_j-y_j)(z_i-y_i)\right]\frac{\partial}{\partial n_y}\left(\frac{1}{r}\right)+2L_{ji}(y, z), \tag{6.5}$$

$$k_{4j}^{(2)}=2L_{j4}(y, z),\quad k_{i4}^{(2)}=2L_{4i}(y, z).\qquad k_{44}^{(2)}=\frac{1}{2\pi}\frac{\partial}{\partial n_y}\left(\frac{1}{r}\right)+2L_{44}(y, z),$$

As in [4] we determine the characteristic matrix of the operator K by considering a local system of coordinates $\xi_1\xi_2\xi_3$ with its origin at $z\in\Sigma$ and the axis ξ_3 directed along $\mathbf{n}_z$. Following [4] we write

$$y_j-z_j=a_{jk}(z)n_k,\quad a_{si}a_{sj}=\delta_{ij},$$

and we obtain the characteristic matrix $\chi(z, 0)$ with components $\chi_{ij}(z, 0)$ $(i, j=1, 2, 3, 4)$,

$$\left.\begin{aligned}
&\chi_{ii}=-1\ \text{(no sum)}\ (i=1, 2, 3, 4),\\
&\chi_{12}=-\chi_{21}=D[n_2(a_{11}\cos 0+a_{12}\sin 0)-n_1(a_{21}\cos 0+a_{22}\sin 0)],\\
&\chi_{13}=-\chi_{31}=D[n_3(a_{11}\cos 0+a_{12}\sin 0)-n_1(a_{31}\cos 0+a_{32}\sin 0)],\\
&\chi_{23}=-\chi_{32}=D[n_3(a_{21}\cos 0+a_{22}\sin 0)-n_2(a_{31}\cos 0+a_{32}\sin 0)],\\
&\chi_{i4}=\chi_{4j}=0,\ (i, j=1, 2, 3, 4).
\end{aligned}\right\} \tag{6.6}$$

The symbolic matrix of the operator K, $\sigma(z, 0)$ has the components $\sigma_{ij}(z, 0)$ $(i, j=1, 2, 3, 4)$ given by ([4])

$$\begin{aligned}
&\sigma_{ii}=-1\ \text{(no sum)}\ (i=1, 2, 3, 4),\\
&\sigma_{ih}=2\pi i\chi_{ih},\quad (i, h=1, 2, 3, 4).
\end{aligned} \tag{6.7}$$

We obtain $\det\sigma(z, 0)=1-D^2/(4\pi^2)$, where $D=\mu/(\lambda+2\mu)$.
This result proves that the operators of equations (I_1), (E_2) are of normal type. In a similar way we can prove that the operators of the remaining equations are also of normal type.

If ρ is a complex parameter, let us consider the operator

$$K_\rho\psi=-\psi(z)+\rho\int_\Sigma k(z, y)\psi(y)d\sigma_y. \tag{6.8}$$

We obtain $\det\sigma_\rho(z, 0)=1-\rho^2D^2/4\pi^2$,
where $\sigma_\rho(z, 0)$ is the symbolic matrix of the operator K_ρ. It is known ([6]) that if K_ρ is of the normal type and the symbolic matrix is antisymmetric, then the index of the operator is zero and the Fredholm theory is valid. For $\rho=1$, the operator K_ρ is the operator of Fredholm type in the space $L_p^{(4)}(\Sigma)$ ([5]). In a similar way the operators of the equations (I_2), (E_1) are Fredholm operators in the space $L_p^{(4)}(\Sigma)$. As in [5] we can show that the integral

operators of the equations (I_1), (I_2), (E_1), (E_2) are Fredholm operators in the space of functions satisfying Hölder's condition.

7. Existence Theorems. Let us consider the problems (I_1), (E_2). The homogeneous equations (I_1^0), (E_2^0) have only trivial solutions. Indeed, assuming the opposite, let ψ_0 be a non-trivial solution of (E_2^0). The single layer potential

$$V_0(x, \psi_0)=\int_\Sigma\Gamma(x, y)\psi_0(y)d\sigma_y, \tag{7.1}$$

satisfies the relations (4.3) and

$$\mathcal{L}\left(\frac{\partial}{\partial x}\right)V_0=0,\quad\left[\mathcal{T}\left(\frac{\partial}{\partial x}, n_x\right)V_0\right]_e=0. \tag{7.2}$$

According to the Theorem 6.3

$$V_0(x, \psi_0)=0\ \text{for all}\ x\in B_e. \tag{7.3}$$

Taking into account the Theorem 5.1, $V_0(x, \psi_0)$ is continuous, hence

$$[V_0(z, \psi_0)]_e=[V_0(z, \psi_0)]_i=0\quad z\in\Sigma.$$

The Theorem 6.1 shows that

$$V_0(x, \psi_0)=0,\ \text{for all}\ x\in B. \tag{7.4}$$

From (5.6), (7.3) and (7.4),

$$\psi_0(z)=\left[\mathcal{T}\left(\frac{\partial}{\partial z}, n_z\right)V_0(z, \psi_0)\right]_i-\left[\mathcal{T}\left(\frac{\partial}{\partial z}, n_z\right)V_0(z, \psi_0)\right]_e=0,\quad z\in\Sigma,$$

which is in contradiction with the initial assumption and proves our assertion

Applying the first Fredholm's theorem, we have

Theorem 7.1. *The problem (I_1) has solution for any $G(z)\in H(\Sigma)$. This solution is unique and can be expressed as a double layer potential.*

Theorem 7.2. *The problem (E_2) has solution for any $G(z)\in H(\Sigma)$. This solution is unique and can be expressed as a double layer potential.*

Let us consider the equations (I_2^0), (E_1^0). The vector U_0 given by (6.2) satisfies

$$\mathcal{L}\left(\frac{\partial}{\partial x}\right)U_0=0\qquad\mathcal{T}\left(\frac{\partial}{\partial x}, n_x\right)U_0=0,\quad x\in B. \tag{7.5}$$

From the representation formula (4.2), we get

$$U_0(x)=-\int_\Sigma\Lambda(x, y)U_0(y)d\sigma_y.$$

It is not difficult to prove that $U_0(x)$ is a solution of the equation (E_1^0). Taking succesively in (6.2) one of the constants a_i, b_i different from zero and the others zero, we have six linearly independent solutions $U_0^{(j)}$ $(j=1, 2, 3, 4, 5, 6)$ of the equation (E_1^0). According to the second Fredholm's theorem ([8]), the associate equation (I_2^0) has at least six linearly independent solutions

$\psi^{(j)}$ $(j=1, 2, 3, 4, 5, 6)$. As in [5] we can show that $\psi^{(j)}$ are a complete system of linearly independent solutions of the equation (I_2^0). From this result it follows the completeness of the associate system $U_0^{(j)}$ $(j=1, 2, 3, 4, 5, 6)$. We can now formulate the following theorem.

Theorem 7.3. *The problem* (I_2) *can be solved for any* $G(z)=(T_1, T_2, T_3, H_1)^*$ *from* $H(\Sigma)$ *if and only if*

$$\int_\Sigma T_i d\sigma=0, \qquad \int_\Sigma \varepsilon_{ijk}x_k T_j d\sigma=0. \tag{7.6}$$

The solution is expressed as a single layer potential and is determined within an additive rigid displacement vector (6.2).

Proof. According to Fredholm's third theorem, the necessary and sufficient conditions for existence of the solution of the equation (I_2) are

$$\int_\Sigma G^*(y)U_0(y)d\sigma_y=0, \quad (k=1, 2, 3, 4, 5, 6). \tag{7.7}$$

These conditions lead to (7.6).

We can prove also

Theorem 7.4. *The problem* (E_1) *can be solved for any* $G(z)\in H(\Sigma)$. *The solution is unique and can be expressed as a linear combination of a double layer potential with some single layer potentials.*

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THE MINIMUM RISK APPROACH TO MAX-MIN BILINEAR PROGRAMMING

BY

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1. **Introduction.** In this paper the minimum risk approach for the max-min bilinear programming problem is formulated. We show that this problem can be reduced, under certain hypotheses, to a max-min bilinear fractional problem. In the last section a parametrical algorithm for solving the bilinear fractional max-min problem is proposed.

2. **Problem formulation.** The following max-min problem is studied in the papers [1], [2], [4]. P M Problem. Find:

$$\max_{x\in X}\ \min_{y\in Y}\ [xAy+dx+ay], \tag{2.1}$$

where:

$$X=\{x\in R^n|\ Bx\leqslant b,\ x\geqslant 0\}, \quad Y=\{y\in R^n|Ey\leqslant e,\ y\geqslant 0\}. \tag{2.2}$$

and x and y are variable vectors, whereas $A\in R^{n\times m}$, $B\in R^{p\times n}$, $E\in R^{q\times m}$, $d\in R^n$, $a\in R^m$, $b\in R^p$ and $e\in R^q$ are given matrices and vectors, respectively. As can be seen, the constraints are separate in problem (2.1)—(2.2).

Definition 2.1. *Given a function* $f: X\times Y\to R$, $X\times Y\in R^{n\times m}$, *we say that* $(x', y')\in X, Y$ *is an optimal solution for the max-min problem*:

$$\max_{x\in X}\ \min_{y\in Y} f(x, y),$$

if (x', y') *verifies the following conditions*:

(i) $f(x', y')=\min_{y\in Y} f(x', y)$, (ii) $f(x', y')\geqslant \min_{y\in Y} f(x, y)$, $x\in X$.

In what follows, we assume that the matrix A and vectors d and a are random:

$$A(\omega)=A^1+t(\omega)\,A^2,\quad d(\omega)=d^1+t(\omega)\,d^2,\quad a(\omega)=a^1+t(\omega)\,a^2, \tag{2.3}$$

where A^1, $A^2\in R^{n\times m}$ are given matrices, $d^i\in R^n$, $a^i\in R^m$ $(i=1, 2)$ are given constant vectors whereas $t(\omega)$ is a random variable on a probability space (Ω, K, P), with the continuous and strictly increasing distribution function $T(z)$.

The minimum risk problem corresponding to level z, associated to *PM* problem, consists in finding the optimal solution of the following problem: