$\psi^{(j)}$ $(j=1, 2, 3, 4, 5, 6)$. As in [5] we can show that $\psi^{(j)}$ are a complete system of linearly independent solutions of the equation (I_2^0). From this result it follows the completeness of the associate system $U_0^{(j)}$ $(j=1, 2, 3, 4, 5, 6)$. We can now formulate the following theorem.

Theorem 7.3. *The problem* (I_2) *can be solved for any* $G(z)=(T_1, T_2, T_3, H_1)^*$ *from* $H(\Sigma)$ *if and only if*

$$\int_\Sigma T_i d\sigma=0, \qquad \int_\Sigma \varepsilon_{ijk} x_k T_j d\sigma=0. \tag{7.6}$$

The solution is expressed as a single layer potential and is determined within an additive rigid displacement vector (6.2).

Proof. According to Fredholm's third theorem, the necessary and sufficient conditions for existence of the solution of the equation (I_2) are

$$\int_\Sigma G^*(y)U_0(y)d\sigma_y=0, \qquad (k=1, 2, 3, 4, 5, 6). \tag{7.7}$$

These conditions lead to (7.6).

We can prove also

Theorem 7.4. *The problem* (E_1) *can be solved for any* $G(z)\in H(\Sigma)$. *The solution is unique and can be expressed as a linear combination of a double layer potential with some single layer potentials.*

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THE MINIMUM RISK APPROACH TO MAX-MIN BILINEAR PROGRAMMING

BY

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1. **Introduction.** In this paper the minimum risk approach for the max-min bilinear programming problem is formulated. We show that this problem can be reduced, under certain hypotheses, to a max-min bilinear fractional problem. In the last section a parametrical algorithm for solving the bilinear fractional max-min problem is proposed.

2. **Problem formulation.** The following max-min problem is studied in the papers [1], [2], [4]. PM Problem. Find:

$$\max_{x\in X}\ \min_{y\in Y}\ [xAy+dx+ay], \tag{2.1}$$

where

$$X=\{x\in R^n \mid Bx\leqslant b,\ x\geqslant 0\}, \quad Y=\{y\in R^n \mid Ey\leqslant e,\ y\geqslant 0\}. \tag{2.2}$$

and x and y are variable vectors, whereas $A\in R^{n\times m}$, $B\in R^{p\times n}$, $E\in R^{q\times m}$, $d\in R^n$, $a\in R^m$, $b\in R^p$ and $e\in R^q$ are given matrices and vectors, respectively. As can be seen, the constraints are separate in problem (2.1)–(2.2).

Definition 2.1. *Given a function* $f: X\times Y\to R$, $X\times Y\in R^{n\times m}$, *we say that* $(x', y')\in X, Y$ *is an optimal solution for the max-min problem*:

$$\max_{x\in X}\ \min_{y\in Y} f(x, y),$$

if (x', y') *verifies the following conditions*:

(i) $f(x', y')=\min_{y\in Y} f(x', y)$, (ii) $f(x', y')\geqslant \min_{y\in Y} f(x, y)$, $x\in X$.

In what follows, we assume that the matrix A and vectors d and a are random:

$$A(\omega)=A^1+t(\omega)A^2,\quad d(\omega)=d^1+t(\omega)d^2,\quad a(\omega)=a^1+t(\omega)a^2, \tag{2.3}$$

where A^1, $A^2\in R^{n\times m}$ are given matrices, $d^i\in R^n$, $a^i\in R^m$ $(i=1, 2)$ are given constant vectors whereas $t(\omega)$ is a random variable on a probability space (Ω, K, P), with the continuous and strictly increasing distribution function $T(z)$.

The minimum risk problem corresponding to level z, associated to PM problem, consists in finding the optimal solution of the following problem:

PRM Problem. *Find*

$$v(z) = \max_{x\in X} P\{\omega \mid Z(x,\omega) > z\}, \tag{2.4}$$

where

$$Z(x,\omega) = \min_{y\in Y} [xA(\omega)\,y + d(\omega)x + a(\omega)\,y]. \tag{2.5}$$

Definition 2.2. *A pair $(x', y') \in X\times Y$ is called minimum risk solution for the PRM problem, if it verifies the conditions:*

$$P\{\omega \mid x'A(\omega)\,y' + d(\omega)\,x' + a(\omega)\,y' > z\} = P\{\omega \mid \min_{y\in Y} (x'A(\omega)\,y + d(\omega)\,x' + a(\omega)\,y) > z\}, \tag{2.6}$$

$$P\{\omega \mid x'A(\omega)\,y' + d(\omega)\,x' + a(\omega)\,y' > z\} \geqslant P\{\omega \mid \min_{y\in Y} \{xA(\omega)\,yd(\omega)\,x + a(\omega)\,y) > z\}, \quad \forall x \in X, \tag{2.7}$$

Let us denote:

$$Z_i(x,y) = xA^iy + d^ix + a^iy, \quad (i = 1,2), \quad \forall (x,y) \in X\times Y.$$

Further, we shall suppose that:

$$Z_2(x,y) \neq 0, \quad \forall (x,y) \in X\times Y, \tag{2.8}$$

(2.9) X and Y are nonempty and bounded sets.

By the continuity of the function Z_2 and by the assumptions (2.8), (2.9), it results that only one of the following cases holds:

(i) $Z_2(x,y) > 0$, $\forall (x,y) \in X\times Y$; or (ii) $Z_2(x,y) < 0$, $\forall (x,y) \in X\times Y$.

Thus, we will show that, under the assumptions (2.8), (2.9), the minimum risk PRM problem, can be solved by a deterministic max-min problem, which does not depend on the distribution function of the random variable $t(\omega)$. This fact generalizes a result obtained in [3] for linear programming.

Theorem 1. *If the conditions (2.8), (2.9) are met and if the distribution function T is continuous and strictly increasing, then the minimum-risk solution of the PRM problem does not depend on T and it can be obtained by solving one of the max-min bilinear fractional problems:*

PA1 Problem $\max_{x\in X} \min_{y\in Y} [Z_1(x,y) - z]/Z_2(x,y)$, if $Z_2(x,y) > 0$

or

PA2 Problem $\max_{x\in X} \min_{y\in Y} [z - Z_1(x,y)]/Z_2(x,y)$, if $Z_2(x,y) < 0$.

Proof: We get from (2.3) and (2.5):

$$F(x,z) = P\{\omega \mid \min_{y\in Y} [xA(\omega)\,y + d(\omega)\,x + a(\omega)\,y] > z\} = P\{\omega \mid \min_{y\in Y} [Z_1(x,y) + t(\omega)\,Z_2(x,y)] > z\} = P\{\omega \mid Z_1(x,y) + t(\omega)\,Z_2(x,y) > z, \quad \forall y \in Y\},$$

hence, according to (2.8), it results:

$$F(x,z) = \begin{cases} P\{\omega \mid t(\omega) > [z - Z_1(x,y)]/Z_2(x,y), & \forall y \in Y\}, \text{ if } Z_2(x,y) > 0 \\ P\{\omega \mid t(\omega) < [z - Z_1(x,y)]/Z_2(x,y), & \forall y \in Y\}, \text{ if } Z_2(x,y) < 0. \end{cases}$$

Let $g(x,y) = [z - Z_1(x,y)]/Z_2(x,y)$ be for any $(x,y) \in X\times Y$. Then, we have:

$$F(x,z) = \begin{cases} P\{\omega \mid t(\omega) > \max_{y\in Y} g(x,y)\}, & \text{if } Z_2(x,y) > 0 \\ P\{\omega \mid t(\omega) < \min_{y\in Y} g(x,y)\}, & \text{if } Z_2(x,y) < 0 \end{cases} = \begin{cases} 1 - T(\max_{y\in Y} g(x,y)), & \text{if } Z_2(x,y) > 0, \\ T(\min_{y\in Y} g(x,y)), & \text{if } Z_2(x,y) < 0. \end{cases}$$

Then, the PRM Problem is equivalent to:

$$\max_{x\in Y} F(x,z) = \begin{cases} \max_{x\in X} [1 - T(\max_{y\in Y} g(x,y))], & \text{if } Z_2(x,y) > 0, \\ \max_{x\in X} T(\min_{y\in Y} g(x,y)), & \text{if } Z_2(x,y) < 0, \end{cases}$$

hence, by the assumption that T is continuous and strictly increasing, we get:

$$v(z) = \max_{x\in} F(x,z) = \begin{cases} 1 - T(\min_{x\in X} \max_{y\in Y} g(x,y)), & \text{if } Z_2(x,y) > 0, \\ T(\max_{x\in X} \min_{y\in Y} g(x,y)), & \text{if } Z_2(x,y) < 0. \end{cases}$$

Therefore, a minimum risk solution for *PRM* problem can be obtained by solving *PA2* problem or

PA1' Problem. $\min_{x\in X} \max_{y\in Y} g(x,y)$.

But *PA1'* Problem is equivalent to *PA1* problem. This ends the proof.

Remarks 2.1. A similar approach can be applied to the minimum risk problem associated to a min-max bilinear problem having random coefficients of the (2.3) form, that is:

PRM' Problem: $v'(z) = \max_{x\in} P\{\omega \mid \max_{y\in Y} [xA(\omega)\,y + d(\omega)\,x + a(\omega)\,y] \leqslant z\}$.

2.2. The minimum risk Chebyshev problems studied in [5], [6], [8] can be considered as minimum risk problems associated to discrete min-max problems (that is, min-max problems, whose feasible set Y is a finite set).

2.3. If T is continuous and strictly increasing, the *PRM* problem can be formulated as a max-min problem:

$$v(z) = \max_{x\in X} \min_{y\in Y} P\{\omega \mid xA(\omega)\,y + d(\omega)\,x + a(\omega)\,y > z\}:$$

3. **Parametrical algorithm for solving the max-min bilinear-fractional problems.** We will propose here a parametrical algorithm for the following max-min bilinear-fractional problem:

PF Problem: Find

$$v=\max_{x\in X}\min_{y\in Y}\frac{xA^1y+d^1x+a^1y+w^1}{xA^2y+d^2x+a^2y+w^2}, \tag{3.1}$$

where $A^i\in R^{n\times m}$ $(i=1,2)$, $d^i\in R^n$, $a^i\in R^m$ and $w^i\in R$ $(i=1,2)$ are given matrices, vectors and real numbers, respectively.

The feasible sets X and Y are, as in section 2, polyhedral sets defined by (2.2). Obviously, the max-min $PA1$ and $PA2$ problems are particular cases of the PF problem. For every $x\in X$ and $y\in Y$, we denote by:

$$z_i(x,y)=xA^iy+d^ix+a^iy+w^i,\quad i=1,2 \tag{3.2}$$

$$h(x,y)=z_1(x,y)/z_2(x,y) \tag{3.3}$$

Throughout this section, we make the following hypotheses:

(H1) X and Y are nonempty and bounded sets;

(H2) $z_2(x,y)>0$, for all $(x,y)\in X\times Y$.

To solve the max-min bilinear fractional PF problem, we adapt a parametrical method given in [7], which can be applied to rather general classes of max-min non-linear fractional problems.

The following theorem gives a necessary and sufficient condition for a pair $(x^*,y^*)\in X\times Y$, to be an optimal solution of the PF problem.

Theorem 2. *Let $x^*\in X$, $y^*\in Y$ and $t^*\in R$ be, such that:*

$$t^*=h(x^*,y^*)=\min_{y\in Y}h(x^*,y).$$

Then (x^*,y^*) is an optimal solution for PF problem if and only if $F(t^*)=0$, where

$$F(t)=\max_{x\in X}\min_{y\in Y}(z_1(x,y)-tz_2(x,y)),\ \textit{for any}\ \ t\in R.$$

The proof is similar to that of Theorem 1 from Ref. [7].

The algorithm consists in the following steps:

Step 1 Choose $x_0\in X$ and take $k:0$.

Step 2 Find $y_k\in Y$, which is an optimal solution of the linear fractional problem:

$$\min_{y\in Y}h(x_k,y) \tag{3.4}$$

and take:

$$t_k=h(x_k,y_k). \tag{3.5}$$

Step 3 Solve the max-min bilinear problem:

$$F(t_k)=\max_{x\in X}\min_{y\in Y}(z_1(x,y)-t_kz_2(x,y)), \tag{3.6}$$

and let (x'_{k+1},y'_{k+1}) be an optimal solution of this problem.

Step 4 i) If $F(t_k)=0$, then by Theorem 2 (x_k,y_k) is an optimal solution for PF, problem and the algorithm stops. In this case $v=t_k$.

ii) If $F(t_k)>0$, then go to Step 5.

Step 5 Take $x_{k+1}:=x'_{k+1}$, $k:=k+1$ and go to Step 2.

This algorithm generaly requires an infinite number of iterations. It can be stopped after a finite number of iterations, by using in Step 4 an appropriate stop criterion as:

Step 4': i) If $F(t_k)\leqslant\varepsilon$ then stop.

ii) If $F(t_k)>\varepsilon$, then go to Step 5.

We mention that in Step 4', ε is a very small preassigned positive quantity. Its value depends on the degree of approximation that is required.

The algorithm convergence, as well as the finiteness of approximative variant is based on the following theorem:

Theorem 3. *Let $\varepsilon\geqslant0$ be a given real number. Under hypotheses H1 and H2, if $F(t_k)\leqslant\varepsilon$ and if $z_2(x,y)\geqslant\beta>0$, for every $(x,y)\in X\times Y$, then, for every optimal solution (x',y') of PF problem, we have:*

$$0\leqslant h(x',y')-t_k\leqslant F(t_k)/\beta\leqslant\varepsilon/\beta$$

The proof of this theorem is similar to that of theorem 3, ref. [7].

Remarks: 3.1. Step 2 requires to solve a linear-fractional programming problem at each iteration. As only objective function is changed at iteration k, at $k+1$ iteration the optimal solution y_k obtained at the previous iteration can be taken as the initial solution for problem (3.4).

3.2. The max-min bilinear problem (3.6) in Step 3, can be solved by two linear programming problems (see, e. g. Refs. [1], [4], [5]).

3.3. When A^1 and A^2 are zero matrices, that is, when in fact the PF problem is a max-min linear fractional problem, then the parametrical algorithm stops after a finite number of iterations (see Ref. [7]).

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