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ON QUASICONFORMAL MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS OF DIMENSION n

BY

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1. **Introduction.** It is well-known that each 2-dimensional Riemann surface is locally conformally equivalent to the plane; in the 2-dimensional case the local properties of quasiconformal mappings between Riemann surfaces can be studied in the plane. The class of n-dimensional Riemann surfaces analogous to the bidimensional one (from above) is very poor. That is why, K. Suominen [6] considered instead of them, the class of n-dimensional Riemannian manifolds. This author defines the quasiconformal mappings between Riemannian manifolds of dimension n, by means of the module of a curve family. The purpose of this paper is to study these mappings starting from a definition with metric character.

In the sequel we denote by V_n a complete Riemann manifold of dimension n, by $T(V_n)$ its tangent bundle, by $T_x(V_n)$ its tangent space at x and by exp_x the exponential map. It is well known (see [4]) that exp_x is a diffeomorphism of a neighbourhood $U \subset T(V_n)$ onto a neighbourhood of x in V_n.

If x, y are two points of V_n, the distance $d(x, y)$ is defined by:

$$d(x, y)=\inf_{\gamma\in C} l(\gamma),$$

where C is the family of piecewise differentiable curves $\gamma : [a, b] \to V_n$ joining x and y and $l(\gamma)$ is the length of γ.

2. **Quasiconformal mappings.** Let us consider a map $f : D \subset V_n \to D^* = f(D) \subset V_n$, where D is a domain in V_n. We define:

$$L(x, r)=\sup_{y}\{d(f(x), f(y)) : d(x, y)=r\},$$

$$l(x, y)=\inf_{y}\{d(f(x), f(y)) : d(x, y)=r\},$$

The *linear dilatation* of f at $x \in D$ is, by definition:

$$\delta(x)=\overline{\lim_{r\to 0}}\frac{L(x, r)}{l(x, r)}.$$

Definition 1. *A homeomorphism $f : D \subset V_n \to D^* = f(D) \subset V_n$ is called K-qc (K-quasiconformal) by the metric definition in Gehring's sense $(1 \leq K < \infty)$ if $\delta(x) \leq K$, for every point x of D.*

A regular family of parameter k, $k=k(x_0)$, relatively to a point x_0, is a family of hipersurfaces (submanifolds of dimension $n-1$) $\{\sigma_\alpha\}$ $\alpha\in(0, 1)$, of class C^l ($l\geq 0$), where $\sigma_\alpha=\varphi(S(O, \alpha))$, $S(O, \alpha)$ the sphere of center $O\in R^n$ and radius α, while φ is a homeomorphism of the unit ball onto a neighbourhood of the point $x_0=\varphi(O)$, such that :

$$\overline{\lim_{\alpha\to 0}}\frac{\sup_{x\in\sigma_\alpha} d(x, x_0)}{\inf_{x\in\sigma_\alpha} d(x, x_0)}=k(x_0)<\infty.$$

A homeomorphism f is called regular at $x_0\in D$, if for at least one regular family $\{\sigma_\alpha\}$ of parameter $k(x_0)$ relatively to x_0, the family $\{f(\sigma_\alpha)\}$ is a regular one of parameter $k^*(x_0)$ relatively to $f(x_0)$.

A homeomorphism is said to be regular in a domain D if it is regular at each of its points. If f is a regular homeomorphism then :

$$q(x_0)=\inf_{\{\sigma_\alpha\}} k(x_0).k^*(x_0),$$

where the infimum is taken over all regular families relatively to x_0 — is called the *characteristic* of f at x_0.

Remark. It is easy to see that if $f: D\to D^*$ is a regular homeomorphism such that its characteristic $q(x)\leq K<\infty$, then f^{-1} is a regular homeomorphism such that its characteristic $q_1(x)\leq K<\infty$. If, in addition, $g: D^*\to D^{**}$ is a regular homeomorphism such that its characteristic $q^*(f(x))\leq K^*<\infty$, then $g\circ f: D\to D^{**}$ is a regular homeomorphism and its characteristic satisfies $q^{**}(x)\leq K.K^*$.

Theorem 1. *If $f: D\to D^*$ is a K-qc mapping in D, then it is a regular homeomorphism and its characteristic $q(x)\leq K$ for every point $x\in D$.*

Proof. Let us consider $f: D\to D^*$ a K-qc homeomorphism. Let $B(O_x, 1)$ be the unit ball and $B(O_x, a)$ the ball of radius a in the tangent space $T_x(V_n)$ with $x\in D$ (where O_x is the null vector in $T_x(V_n)$). Let the diffeomorphism $h: B(O_x, 1)\to B(O_x, a)$ be defined by $h(X_x)=a.X_x$ for every $X_x\in B(O_x, 1)$. Let a be fixed real and sufficiently small such that the map $\exp_x: B(O_x, a)\to V_n$ be well defined. We denote by $g=\exp_x\circ h: B(O_x, 1)\to V_n$. For every $\alpha\in(0, 1)$, let us consider the hipersurfaces :

$$\sigma_\alpha=g[S(O_x, \alpha)]=\exp_x[h(S(O_x, \alpha))]=\{y: d(x, y)=a\alpha\}.$$

It follows that :

$$\overline{\lim_{\alpha\to 0}}\frac{\sup_{y\in\sigma_\alpha} d(x, y)}{\inf_{y\in\sigma_\alpha} d(x, y)}=1;$$

therefore the family $\{\sigma_\alpha\}$ is regular at x.

Let us consider $\tau_\alpha=f(\sigma_\alpha)=(f\circ\exp_x\circ h)(S(O_x, \alpha))$, then

$$k^*(x)=\overline{\lim_{\alpha\to 0}}\frac{\sup_{f(y)\in\tau_\alpha} d(f(x), f(y))}{\inf_{f(y)\in\tau_\alpha} d(f(x), f(y))}=\overline{\lim_{\alpha\to 0}}\frac{\sup_{d(x, y)=a} d(f(x), f(y))}{\inf_{d(x, y)=a} d(f(x), f(y))}=\delta(x)\leq K<\infty.$$

Therefore $q(x)=\inf k(x).k^*(X)=\inf\delta(x)\leq K<\infty$.

3. **0-homeomorphisms. Definition 3.** *A homeomorphism f of D into D^* (D, D^* domains of V_n) is said to be a θ-mapping, if there exists a function θ, which is continuous and strictly increasing in $[0, 1]$ with $\theta(0)=0$, such that, if $x, y\in\Delta$ with $d(x, y)<d(x, \partial\Delta)$, then :*

$$\frac{d(f(x), f(y))}{d(f(x), \partial\Delta^*)}\leq\theta\left[\frac{d(x, y)}{d(x, \partial\Delta)}\right] \textit{for every subdomain } \Delta\subset D,\ \Delta^*=f(\Delta).$$

Theorem 2. *A θ-mapping $f: D\to D^*$ is a K-qc homeomorphism with $K=\theta(1)$.*

Proof. Let us consider $x_0\in D$ and C such that $0<C<d(x_0, \partial\Delta)$ and $d(f(x_0), f(x))<d(f(x_0), \partial\Delta^*)$ whenever $d(x_0, x)<C$. For every $r\in(0, C)$ let the points $x_1, x_2\in D$ be such that $d(x_1, x_0)=d(x_2, x_0)=r$ and: $d(f(x_1), f(x_0))>L(x_0, r)-\varepsilon$; $d(f(x_2), f(x_0))<l(x_0, r)+\varepsilon$. Let us consider also $\Delta=D-\{x_2\}$ and $\Delta^*=D-\{f(x_2)\}$; then :

$$\frac{L(x_0, r)-\varepsilon}{l(x_0, r)+\varepsilon}<\frac{d(f(x_1), f(x_0))}{d(f(x_2), f(x_0))}=\frac{d(f(x_1), f(x_0))}{d(f(x_0), \partial\Delta^*)}\leq\theta\left[\frac{d(x_1, x_0)}{d(x_0, \partial\Delta)}\right]=$$

$$=\theta\left[\frac{d(x_1, x_0)}{d(x_2, x_0)}\right]=\theta(1)=K,$$

for every $\varepsilon>0$. Then, for $\varepsilon\to 0$ we obtain : $L(x_0, r)/l(x_0, r)\leq\theta(1)=k$, for every $r\in[0, C]$.

Hence :

$$\overline{\lim_{r\to 0}}\frac{L(x_0, r)}{l(x_0, r)}\leq K.$$

Theorem 3. *A homeomorphism $f: D\to D^*$ is K-qc if f^{-1} is a θ-mapping The constant $K=1/\theta^{-1}(1)$, where θ^{-1} is the inverse of θ.*

Proof. Let us consider $x_0\in D$ and $0<C<d(x_0, \partial D)$ such that : $d(f(x), f(x_0))<d(f(x_0), \partial D^*)$ if $d(x, x_0)<C$. For every $0<r<C$ and $\varepsilon'>0$, we choose x_1 and x_2 such that $d(x_1, x_0)=d(x_2, x_0)=r$ and

$$L(x_0, r)=\sup_{d(x, x_0)=r}\{d(f(x), f(x_0))\}<d(f(x_1), f(x_0))+\varepsilon',$$

$$l(x_0, r)=\inf_{d(x, x_0)=r}\{d(f(x), f(x_0))\}>d(f(x_2), f(x_0))-\varepsilon'.$$

Let us consider first the case in which $l(x_0, r)<L(x_0, r)$. Then, for an $r>0$, $l(x_0, r)+\varepsilon<L(x_0, r)-\varepsilon$ for any ε with $0<\varepsilon<\varepsilon'$, ε' suitable. Let $\Delta=D-\{x_1\}$ and $\Delta^*=f(\Delta)$, then $\Delta^*=D^*-\{f(x_1)\}$ and :

$$d(f(x_2), f(x_0))<l(x_0, r)+\varepsilon<L(x_0, r)-\varepsilon<d(f(x_1), f(x_0))=d(f(x_0), \partial\Delta^*)$$

Since f^{-1} is a θ-mapping (where θ is increasing) we have :

$$l=\frac{d(f^{-1}(f(x_2)), f^{-1}(f(x_0)))}{d(f^{-1}(f(x_0)), \partial\Delta)}\leq\theta\left[\frac{d(f(x_2), f(x_0))}{d(f(x_0), \partial\Delta^*)}\right]\leq\theta\left[\frac{l(x_0, r)+\varepsilon}{L(x_0, r)-\varepsilon}\right],$$

hence $[L(x_0, r)-\varepsilon]/[l(x_0, r)+\varepsilon]<1/\theta^{-1}(1)$.

For $\varepsilon \to 0$ we have:

$$(^*)\qquad \frac{L(x_0, r)}{l(x_0, r)} \leq \frac{1}{\theta^{-1}(1)}$$

In the case $l(x_0, r) = L(x_0, r)$, $(*)$ is true. Hence, the inequality $(^*)$ is true for every $r \in (0, 1)$ so that $\overline{\lim}\, L(x_0, r)/l(x_0, r) \leq 1/\theta^{-1}(1) < \infty$ and, since x_0 is an arbitrary point of D, the theorem is proved.

4. Mappings with bounded triangular dilatation. Definition. 4. *The mapping $f: D \subset V_n \to V_n$ is said to have bounded triangular dilatation (b.t.d.) if there exists a real number C, $C \geq 1$, such that for every $x, y, z \in D$ with $d(x, y) \leq \leq d(y, z)$ we have $d(f(x), f(y)) \leq C d(f(y), f(z))$.*

Definition 5. *The mapping $f: D \subset V_n \to V_n$ is said to have locally b.t.d. if, for every $x \in D$, there exists $r > 0$ such that $d(f(y), (f(x)) \leq C d(f(z), f(x))$ for every $y, z \in D$ with $d(y, x) \leq d(z, x) \leq r$, $C \geq 1$, $C \in R$.*

Theorem 4. *Let $f: D \subset V_n \to V_n$ be a homeomorphism, D a domain of V_n: If f has locally b.t.d. on D (with the constant C) then: $\delta(x) \leq C$ for every $x \in D$.*

Proof. For any $x \in D$, let $y, z \in D$ be such that $d(y, z) = d(z, x) = r > 0$. Since f has locally b.t.d. it follows $d(f(y), f(x)) \leq C d(f(z), f(x))$ hence:

$$\sup_y \{d(f(y), f(x)), d(y, x) = r\} \leq C \inf_z \{d(f(z), f(x)), d(z, x) = r\}$$

and then $L(x, r)/l(x, r) \leq C$ for every r. It follows that $\delta(x) = \overline{\lim_{r\to 0}}\, L(x, r)/ /l(x, r) \leq C$.

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INVERSES OF SINGULAR INTEGRAL OPERATORS AND PROBLEMS ON DEFICIENCY SUBSPACES

BY

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1. Introduction. In the present note we give some examples of singular integral operators of the following nature. First example is about constructing a bounded self-adjoint singular integral operator from the resolvents of unbounded self-adjoint singular integral operators considered by Pincus and Rovnyak [6]. Second example goes as follows. Let S be a bounded self-adjoint singular integral operator (Pincus [3], [4], Clancey [1], [2]). Let λ be a real number which does not belong to the spectrum of S. Then we give a singular integral operator representation of bounded self-adjoint operator $(S-\lambda)^{-1}$. Further we invoke a technique of Pincus [6] to determine the finite dimensional deficiency subspaces of an unbounded symmetric inverse of a bounded self-adjoint singular integral operator. A similar treatment is also hinted for symmetric Toeplitz operators without details.

2. Example 2.1. Let $A(x)$ be a real valued measurable function on $(-\infty, \infty)$, and $k(x)$ be a complex valued measurable function on $(-\infty, \infty)$. We consider the following singular integral operator:

$$(2.1)\qquad S: f(x) \to A(x) f(x) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{k(x)\bar{k}(t)}{t-x} f(t)\mathrm{d}t$$

on $L^2(-\infty, \infty)$. The essential facts about the operator S are given as follows (see [5]). Let

$$(2.2)\qquad \begin{aligned} A_a(x) &= A(x) \text{ if } |A(x)| \leq a \\ &= 0 \text{ otherwise.} \end{aligned} \qquad \begin{aligned} k_a(x) &= k(x) \text{ if } |k(x)| \leq a \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then we define the family of truncated operators S_a with the coefficients $A_a(x)$ and $k_a(x)$. The sufficient conditions are given in [5] for the normal convergence $\lim_{a\to\infty} S_a = S$ (means that there exists an unbounded self-adjoint operator S such that $\lim_{a\to\infty}(S_a - l)^{-1} = (S-l)^{-1}$ in the strong operator topology for $\mathrm{I}_m(l) \neq 0$).

If we define a sectionally holomorphic function $E(l, z)$ $(\mathrm{Im}(l) \neq 0,$ $\mathrm{Im}(z) \neq 0)$ in the following manner: