For $\varepsilon \to 0$ we have :

$$(*) \qquad \frac{L(x_0, r)}{l(x_0, r)} \leq \frac{1}{\theta^{-1}(1)} .$$

In the case $l(x_0, r) = L(x_0, r)$, (**) is true. Hence, the inequality (*) is true for every $r \in (0, 1)$ so that $\overline{\lim}\, L(x_0, r)/l(x_0, r) \leq 1/\theta^{-1}(1) < \infty$ and, since x_0 is an arbitrary point of D, the theorem is proved.

4. Mappings with bounded triangular dilatation. Definition. 4. *The mapping $f : D \subset V_n \to V_n$ is said to have bounded triangular dilatation (b.t.d.) if there exists a real number C, $C \geq 1$, such that for every $x, y, z \in D$ with $d(x, y) \leq \leq d(y, z)$ we have $d(f(x), f(y)) \leq C d(f(y), f(z))$.*

Definition 5. *The mapping $f : D \subset V_n \to V_n$ is said to have locally b.t.d. if, for every $x \in D$, there exists $r > 0$ such that $d(f(y), f(x)) \leq C d(f(z), f(x))$ for every $y, z \in D$ with $d(y, x) \leq d(z, x) \leq r$, $C \geq 1$, $C \in R$.*

Theorem 4. *Let $f : D \subset V_n \to V_n$ be a homeomorphism, D a domain of V_n : If f has locally b.t.d. on D (with the constant C) then : $\delta(x) \leq C$ for every $x \in D$.*

Proof. For any $x \in D$, let $y, z \in D$ be such that $d(y, x) = d(z, x) = r > 0$. Since f has locally b.t.d. it follows $d(f(y), f(x)) \leq C d(f(z), f(x))$ hence:

$$\sup_{y}\{d(f(y), f(x)),\ d(y, x) = r\} \leq C \inf_{z}\{d(f(z), f(x)),\ d(z, x) = r\}$$

and then $L(x, r)/l(x, r) \leq C$ for every r. It follows that $\delta(x) = \overline{\lim_{r \to 0}}\, L(x, r)/l(x, r) \leq C$.

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INVERSES OF SINGULAR INTEGRAL OPERATORS AND PROBLEMS ON DEFICIENCY SUBSPACES

BY

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1. **Introduction.** In the present note we give some examples of singular integral operators of the following nature. First example is about constructing a bounded self-adjoint singular integral operator from the resolvents of unbounded self-adjoint singular integral operators considered by Pincus and Rovnyak [6]. Second example goes as follows. Let S be a bounded self-adjoint singular integral operator (Pincus [3], [4], Clancey [1], [2]). Let λ be a real number which does not belong to the spectrum of S. Then we give a singular integral operator representation of bounded self-adjoint operator $(S-\lambda)^{-1}$. Further we invoke a technique of Pincus [6] to determine the finite dimensional deficiency subspaces of an unbounded symmetric inverse of a bounded self-adjoint singular integral operator. A similar treatment is also hinted for symmetric Toeplitz operators without details.

2. **Example 2.1.** Let $A(x)$ be a real valued measurable function on $(-\infty, \infty)$, and $k(x)$ be a complex valued measurable function on $(-\infty, \infty)$. We consider the following singular integral operator :

$$(2.1) \qquad S : f(x) \to A(x) f(x) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{k(x)\bar{k}(t)}{t - x} f(t)\,dt$$

on $L^2(-\infty, \infty)$. The essential facts about the operator S are given as follows (see [5]). Let

$$(2.2) \qquad \begin{aligned} A_a(x) &= A(x) \text{ if } |A(x)| \leq a \\ &= 0 \text{ otherwise.} \end{aligned} \qquad \begin{aligned} k_a(x) &= k(x) \text{ if } |k(x)| \leq a \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then we define the family of truncated operators S_a with the coefficients $A_a(x)$ and $k_a(x)$. The sufficient conditions are given in [5] for the normal convergence $\lim_{a \to \infty} S_a = S$ (means that there exists an unbounded self-adjoint operator S such that $\lim_{a \to \infty} (S_a - l)^{-1} = (S - l)^{-1}$ in the strong operator topology for $I_m(l) \neq 0$).

If we define a sectionally holomorphic function $E(l, z)$ $(\mathrm{Im}(l) \neq 0,$ $\mathrm{Im}(z) \neq 0)$ in the following manner :

$$E(l, z)=\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(l, x)\,\frac{dx}{x-l}\;\frac{1+tz}{t-z}\;\frac{dt}{1+t^2}\right), \tag{2.3}$$

where $g(t, x)$ is the characteristic function of a set $\Gamma=\{(t, x): A(t)-|k(t)|^2<x<A(t)+|k(t)|^2\}$, then under the assumption that $\int_{-\infty}^{\infty}|k(t)|^2(1+t^2)^{-1}dt<\infty$, and $k(t)\neq 0$ *a.e.* in $(-\infty, \infty)$, we find that the double improper integral in the definition of $E(l, z)$ exists. If $E_a(l, z)$ are the corresponding functions with $A_a(t)$ and $k_a(t)$ in its definitions via the characteristic function $g_a(t, x)$, then $\lim_{a\to\infty} E_a(l, z)=E(l, z)$. The representation of $(S_a-l)^{-1}$ and $(S-l)^{-1}$ are discussed in [5]. What we need here is the following formula:

$$(S-l)^{-1}\frac{k(t)}{t-w}=\frac{1}{2}k(t)^{-1}(t-w)^{-1}E(l, w)\bar{H}(l, t) \quad \text{where} \tag{2.4}$$

$$\bar{H}(\bar{l}, t)=\bar{E}(l, t+i0)-\bar{E}(\bar{l}, t-i0). \tag{2.5}$$

The lemmas 2.1 and 2.2 given next will establish that $(S-i)^{-1}+(S+i)^{-1}$ is a bounded self-adjoint singular integral operator.

Lemma 2.1. *If an operator S possibly bounded or unbounded and self-adjoint satisfies the following commutation relation:*

$$SU-US=\frac{1}{\pi i}C, \tag{2.6}$$

where U is a bounded self-adjoint operator and C is of trace class then

(i) $$(S-i)^{-1}U-U(S-i)^{-1}=-\frac{1}{\pi i}(S-i)^{-1}C(S-i)^{-1},$$

(ii) $$(S+i)^{-1}U-U(S+i)^{-1}=-\frac{1}{\pi i}(S+i)^{-1}C(S+i)^{-1}.$$

Proof: To prove (i) from (2.6) we see that

$$(S-i)U-U(S-i)=\frac{1}{\pi i}C. \tag{2.7}$$

Applying $(S-i)^{-1}$ from the left hand side of (2.7) we get

$$U-(S-i)^{-1}U(S-i)=\frac{1}{\pi i}(S-i)^{-1}C. \tag{2.8}$$

Next we apply $(S-i)^{-1}$ from the right hand side of (2.8): we get

$$U(S-i)^{-1}-(S-i)^{-1}U=\frac{1}{\pi i}(S-i)^{-1}C(S-i)^{-1}$$

from which (i) follows. Similarly we can prove (ii). In case $C=(\cdot, k)k$ where $k(t)\in L^2(-\infty, \infty)$, it is easy to show that the operators on the right hand sides of (i) and (ii) are of the form:

(i') $$-\frac{1}{\pi i}\left(\cdot, \frac{1}{2}\frac{1}{k(t)}H(-i, t)\right)\frac{1}{2}k(t)^{-1}\bar{H}(i, t),$$

(ii') $$-\frac{1}{\pi i}\left(\cdot, \frac{1}{2}\frac{1}{k(t)}H(i, t)\right)\frac{1}{2}k(t)^{-1}\bar{H}(-i, t).$$

In relations (i') and (ii') above, the operators are rank one operators of the form $(\cdot, f)g$. We need to show that f and g are essentially bounded functions. To do that we quote some results of Pincus [6] as follows.

$$E(i, z)\bar{E}(i, \bar{z})=1+\frac{1}{\pi}\int\frac{d\mu_i(t)}{t-z}, \tag{2.9}$$

$$E(-i, z)\bar{E}(-i, \bar{z})=1-\frac{1}{\pi}\int\frac{d\mu_{-i}(t)}{t-z}, \tag{2.10}$$

where $d\mu_i$ and $d\mu_{-i}$ are some positive Borel measures on the real line.

(2.11) S is an unbounded self-adjoint singular integral operator in case $d\mu_{-i}$ and $d\mu_i$ both are absolutely continuous with respect to the Lebesgue measure

(2.12) $E(i, z)$ and $E(-i, z)$ both are devoid of any real zeros as functions of z in case $d\mu_{-i}$ and $d\mu_i$ are absolutely continuous with respect to the Lebesgue measure.

$$\frac{E(i, t+i0)}{E(i, t-i0)}=\frac{A(t)-i+|k(t)|^2}{A(t)-i-|k(t)|^2}, \tag{2.13}$$

$$\frac{E(-i, t+i0)}{E(-i, t-i0)}=\frac{A(t)+i+|k(t)|^2}{A(t)-i-|k(t)|^2}. \tag{2.14}$$

Lemma 2.2 *$1/2k(t)H(i, t)$ and $(1/2)k(t)H(-i, t)$ are essentially bounded functions on $(-\infty, \infty)$.*

Proof. We show that $(1/2)k(t)^{-1}H(i, t)$ is essentially bounded. We have from (2.11)

$$\begin{aligned}&\frac{1}{2}k(t)^{-1}H(i, t)=\frac{1}{2}k(t)^{-1}[E(i, t+i0)-E(i, t-i0)]=\\ &\frac{1}{2}k(t)^{-1}\frac{2|k(t)|^2}{A(t)-i-|k(t)|^2}E(i, t-i0)=\frac{k(t)}{A(t)-i-|k(t)|^2}E(i, t-i0).\end{aligned} \tag{2.15}$$

Since $E(-i, z)$ is devoid of real zeros as a function of z, it is easy to see that the right hand side of (2.15) is an essentially bounded function on $(-\infty, \infty)$.

3. **Example** 3.1. We now consider operators of the form:

$$Sf(\lambda)=A(\lambda)f(\lambda)+\frac{1}{\pi i}k(\lambda)\int_E\frac{k(\mu)f(\mu)}{\mu-\lambda}d\mu,$$

where $E\subset R$ is a measurable set and $A(t)$, $k(t)\in L^\infty(E)$. Assume that $A(\lambda)$ is real valued and $\operatorname{ess\,inf}_{\lambda\in E}|k(\lambda)|>0$. It is a well known fact that S is a bounded self-adjoint singular integral operator on $L^2(E)$ (P i n c u s [3], [4], C l a n c e y [1], [2]). Also if $M = \operatorname{ess\,sup}_{t\in E} [A(t)+|k(t)|^2]$ and $m = \operatorname{ess\,inf}_{t\in E} [A(t)-|k(t)|^2]$, then $\sigma(S) = [m, M]$. In case $\lambda \notin [m, M]$ Clancey has shown that $(S-\lambda)^{-1}$ is a bounded self-adjoint operator on $L^2(E)$. The following lemmas show that $(S-\lambda)^{-1}$ is a singular integral operator.

Lemma 3.1.

$$(S-\lambda)^{-1}U - U(S-\lambda)^{-1} = -\frac{1}{\pi i}(S-\lambda)^{-1}C(S-\lambda)^{-1}$$

where C is the one dimensional operator $(\ \cdot\ , k)k$.

Proof: Proof is similar to Lemma 2.1.

Lemma 3.2. $(S-\lambda)^{-1}k(t) \in L^\infty(E)$.

Proof: Define a function $\Phi(\lambda, z)$ by the following exponential formula which is well defined because $\lambda \notin [m, M]$

$$\Phi(\lambda, z) = \exp\left(\frac{1}{2\pi i}\int_E \log \frac{A(t)+|k(t)|^2-\lambda}{A(t)-|k(t)|^2-\lambda}\,\frac{dt}{t-z}\right) \quad (\operatorname{Im}(z)\neq 0) \tag{3.1}$$

Then from an analysis applied in solving Riemann-Hilbert problems [1] we can deduce that

$$(S-\lambda)^{-1}k(t) = (1/2)\bar{k}(t)^{-1}\{\Phi(\lambda, t+i0) - \Phi(\lambda, t-i0)\} \tag{3.2}$$

Also from (3.1) we can get

$$\frac{\Phi(\lambda, t+i0)}{\Phi(\lambda, t-i0)} = \frac{A(t)+|k(t)|^2-\lambda}{A(t)-|k(t)|^2-\lambda}, \tag{3.3}$$

from which it follows that the right hand side of (3.1)

$$=\frac{1}{2}\bar{k}(t)^{-1}\frac{2|k(t)|^2}{A(t)-|k(t)|^2-\lambda}\Phi(\lambda, t-i0) = \frac{k(t)}{A(t)-|k(t)|^2-\lambda}\Phi(\lambda, t-i0) \tag{3.4}$$

Since $\lambda \notin [m, M]$, the denominator of (3.4) is bounded from below and from an earlier analysis in ([1], [2]) it follows that $\Phi(\lambda, t-i0)$ and $k(t)$ are essentially bounded functions. Therefore the right hand side of (3.4) is essentially bounded.

Remark. The example 2.1 which we have given for bounded self-adjoint singular integral operators constructed from the resolvents of unbounded self-adjoint singular integral operators is important in the sense that it is analogous to the situation when an unbounded self-adjoint ordinary differential operator encountered in Sturm-Liouville Problem has a compact resolvent, which is an integral operator with square-summable Kernel.

4. In [6] P i n c u s characterizes the finite dimensional deficiency subspaces of an unbounded symmetric singular integral operator. We adopt his techniques to characterize the finite dimensional deficiency subspaces of an unbounded symmetric singular integral operator $(S-\lambda)^{-1}$ where $\lambda \in \sigma(S)$, and S is a bounded symmetric singular integral operator. We

adopt usual notations for operator theoretic definitions for the matter which follow next.

Definition. *For a closed, unbounded, symmetric operator T defined on a Hilbert space H the deficiency subspaces are $\overline{R(T+i)}^{\perp}$ and $\overline{R(T-i)}^{\perp}$.*

Let S be the bounded symmetric singular integral operator of section 3, where $A(t), k(t) \in L^\infty(E)$. Then by adopting techniques of section 3, we can see that $L = (S-\lambda)^{-1}$ for $\lambda \in \sigma(S)$ (where $\sigma(S) = [m, M]$) has a singular integral representation of the type

$$Lf(x) = a(x)f(x) + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\psi(x)\bar{\psi}(t)}{t-x}f(t)\,dt.$$

The above operator is defined on $L^2(-\infty, \infty)$, $a(t) \in L^\infty(-\infty, \infty)$, $\psi(t) \in L^2(-\infty, \infty)$. This operator has a natural characterization as an unbounded symmetric singular integral operator on a dense domain in $L^2(-\infty, \infty)$.

Remark. In fact one can easily see that

$$\psi(t) = \frac{k(t)}{A(t)-|k(t)|^2-\lambda}\Phi(\lambda, t-i0),$$

where Φ is defined in section 3. By a simplified analysis (assuming some mild smoothness conditions on $k(t)$) one can see further that $\psi(t) \in L^2(-\infty, \infty)$.

Now we define a new $\hat{E}(l, z)$ function for the above operator $(S-\lambda)^{-1}$ following expression (2.2) of section 2 with $A(t)$ and $k(t)$ replaced by $a(t)$ and $\psi(t)$.

As in [6] we have a function

$$\theta_l(t) = \frac{1}{\pi}\tan^{-1}(a(t)+|\psi(t)|^2) - \frac{1}{\pi}\tan^{-1}(a(t)-|\psi(t)|^2),$$

with $0 \leq \theta_l(t) \leq 1$. There exists positive Borel measures on the real line m_l so that

$$\exp\int\frac{\theta_l(t)}{t-z}\,dt = 1 + \frac{1}{\pi}\int\frac{dm_l(t)}{t-z}.$$

Let $m_{l,s}$ and $m_{-l,s}$ be the singular part of m_l and m_{-l} entering into the Lebesgue decomposition of these measures. Let $m = \dim L_2(m_{l,s})$ and $n = \dim L_2(m_{-l,s})$. Then we have the following results.

Theorem 4.1. *The deficiency indices of the symmetric operator $L = (S-\lambda)^{-1}$ described above are (m, n). If the measures above are not atomic then the corresponding deficiency indices are infinite.*

For the $\hat{E}(l, z)$ function define $\hat{H}(l, \lambda) = \hat{E}(l, \lambda+i0) - \hat{F}(l, \lambda-i0)$. Consider the case when the two singular measures defined above are purely atomic. Suppose that the atoms of $m_{-l,s}$ are $\{\lambda_1^-, \lambda_2^-, \ldots, \lambda_n^-\}$ while those of $m_{-l,s}$ are $\{\lambda_1^+, \lambda_2^+, \ldots, \lambda_m^+\}$.

Theorem 4.2. *The functions*

$$W_j^-(\lambda)=\frac{\hat{H}(i,\lambda)}{\psi(\lambda)}\ \frac{1}{\lambda-\lambda_j^-}\qquad j=1,2,\ldots,n.$$

$$W_r^+(\lambda)=\frac{\hat{H}(-i,\lambda)}{\psi(\lambda)}\ \frac{1}{\lambda-\lambda_r^+}\qquad r=1,2,\ldots,m.$$

are in $L_2(-\infty,\infty)$. They satisfy the formal eigenvalue equations $(L+i)W_j^-(\lambda)=0$ and $(L-i)W_r^+(\lambda)=0$ and span the deficiency spaces of the operator L.

The proofs of theorems 4.1 and 4.2 are analogous to the corresponding results of Pincus [6].

Symetric Toeplitz Operators : Let $\alpha=P_+W$, where $W(0)$ is real and periodic of period 2π and is in $L_2(0,\pi)$ and P_+ denotes the projection of $L_2(0,\pi)$ onto the Hardy space L^2. Define $L_n=P_+W_n$ by

$$W_n(0)=\begin{cases} W(0), & \text{if } |W(0)|\leqslant n\\ 0 & \text{otherwise.}\end{cases}$$

Then L_n is a bounded self-adjoint operator on L_+^2, and the „limit" of the sequence $\{L_n\}$ in the sense of section 2 was studied in [6]. The limiting operator is an unbounded symmetric Toeplitz operator whose finite dimensional deficiency subspaces have been characterized in [6]. We observe that for a bounded symmetric Toeplitz operator S and $\lambda\in\sigma(S)$ we can characterize the deficiency subspaces of the unbounded symmetric Toeplitz operator (constructed in the above manner) for the operator $(S-\lambda)^{-1}$ by a treatment analogous to the case of singular integral operator developed earlier in this section.

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LOCALIZATION OF SOME NON-NILPOTENT GROUPS AND SPACES

BY

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In this paper we continue the study of the category $\mathfrak{Gr}_l$ and of the l- spaces defined by us in [7], for l- a set of primes. Then we investigate the localization at l of the l-spaces.

1. **The category $\mathfrak{Gr}_l$.** 1.1. Let P be the set of all primes, $l\subseteq P$ a subset (possibly empty) of primes and $\bar{l}$ be the complementary set of l. Define $S(l)=\{a\in Z/a>0,\ \forall p\in l\Rightarrow p$ does not divide $a\}$ and let Z_l be the integers localized at l. We shall associate to l a (full) subcategory $\mathfrak{Gr}_l$ of the category $\mathfrak{Gr}$ of groups as follows :

2.1. **Definition.** [7]. *For $l\subseteq P$, we say that a group G belongs to the category $\mathfrak{Gr}_l$ iff the following conditions are satisfied :*

(i) *for each $x,y\in G$ and every $s\in S(l)$, there exists $t\in S(l)$, such that s divides t (we write $s\mid t$) and*

$$(x^sy^s)^{\frac{t}{s}}=(xy)^t, \tag{1}$$

(ii) *for each $x,y,z\in G$ and every $s,s',s''\in S(l)$, there exists $t\in S(l)$ such that $ss's''\mid t$ and*

$$[(x^{s'}y^{s})^{s''}z^{ss'}]^{\frac{t}{ss's''}}=[x^{s's''}(y^{s''}z^{s'})^{s}]^{\frac{t}{ss's''}}. \tag{2}$$

3.1. If $G\in\mathrm{Ob}\,\mathfrak{Gr}_l$, then for each $x,y\in G$ and every $s,s',s''\in S(l)$, such that $s\mid s''$ and $s'\mid s''$, there exists $t\in S(l)$, such that $ss'\mid t$ and

$$(x^{\frac{s''}{s}}y^{\frac{s''}{s'}})^t=(x^{s'}y^{s})^{s}]^{\frac{ts''}{s's}}. \tag{3}$$

4.1. For the statement of the theorem on localization of the category $\mathfrak{Gr}_l$, we consider $S(l)$ as a directed set as follows : if $s,s'\in S(l)$, then we put $s\leqslant s'$ iff s divides s'.

If G is a group, then we can construct a directed system of sets $\{G(s),\alpha_{ss'}\}_{s,s'\in S(l)}$, with $G(s)=G$, $\forall s\in S(l)$ and for $s,s'\in S(l)$, with $s\leqslant s'$, $\alpha_{ss'}:G(s)=G\longrightarrow G(s')=G$ is the following mapping of sets :

$$\alpha_{ss'}(x)=x^{\frac{s'}{s}},\quad \forall x\in G. \tag{4}$$

Firstly, we observe that the maps $\alpha_{ss'}$ are not necessary homomorphisms of groups. But, we can consider the limit G_l of the directed system of sets $\{G(s),\alpha_{ss'}\}$ and let $q_s:G(s)\to G_l$ the canonical injection, corresponding to $s\in S(l)$.