$$W_j^-(\lambda)=\frac{\hat{H}(i,\lambda)}{\psi(\lambda)}\ \frac{1}{\lambda-\lambda_j^-}\qquad j=1,2,\ldots,n.$$

$$W_r^+(\lambda)=\frac{\hat{H}(-i,\lambda)}{\psi(\lambda)}\ \frac{1}{\lambda-\lambda_r^+}\qquad r=1,2,\ldots,m.$$

are in $L_2(-\infty,\infty)$. They satisfy the formal eigenvalue equations $(L+i)W_j^-(\lambda)=0$ and $(L-i)W_r^+(\lambda)=0$ and span the deficiency spaces of the operator L.

The proofs of theorems 4.1 and 4.2 are analogous to the corresponding results of Pincus [6].

Symetric Toeplitz Operators : Let $\alpha=P_+W$, where $W(\theta)$ is real and periodic of period 2π and is in $L_2(0,\pi)$ and P_+ denotes the projection of $L_2(0,\pi)$ onto the Hardy space L^2. Define $L_n=P_+W_n$ by

$$W_n(\theta)=\begin{cases} W(\theta), & \text{if } |W(\theta)|\leqslant n\\ 0 & \text{otherwise.}\end{cases}$$

Then L_n is a bounded self-adjoint operator on L_+^2, and the „limit" of the sequence $\{L_n\}$ in the sense of section 2 was studied in [6]. The limiting operator is an unbounded symmetric Toeplitz operator whose finite dimensional deficiency subspaces have been characterized in [6]. We observe that for a bounded symmetric Toeplitz operator S and $\lambda\in\sigma(S)$ we can characterize the deficiency subspaces of the unbounded symmetric Toeplitz operator (constructed in the above manner) for the operator $(S-\lambda)^{-1}$ by a treatment analogous to the case of singular integral operator developed earlier in this section.

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LOCALIZATION OF SOME NON-NILPOTENT GROUPS AND SPACES

BY

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In this paper we continue the study of the category $\mathfrak{Gr}_l$ and of the l- spaces defined by us in [7], for l- a set of primes. Then we investigate the localization at l of the l-spaces.

1. **The category $\mathfrak{Gr}_l$.** 1.1. Let P be the set of all primes, $l\subseteq P$ a subset (possibly empty) of primes and $\bar{l}$ be the complementary set of l. Define $S(l)=\{a\in Z/a>0,\ \forall p\in\bar{l}\Rightarrow p$ does not divide $a\}$ and let Z_l be the integers localized at l. We shall associate to l a (full) subcategory $\mathfrak{Gr}_l$ of the category $\mathfrak{Gr}$ of groups as follows :

2.1. **Definition.** [7]. *For $l\subseteq P$, we say that a group G belongs to the category $\mathfrak{Gr}_l$ iff the following conditions are satisfied :*

(i) *for each $x, y\in G$ and every $s\in S(l)$, there exists $t\in S(l)$, such that s divides t (we write $s\mid t$) and*

$$(x^sy^s)^{\frac{t}{s}}=(xy)^t, \tag{1}$$

(ii) *for each $x, y, z\in G$ and every $s, s', s''\in S(l)$, there exists $t\in S(l)$ such that $ss's''\mid t$ and*

$$[(x^{s'}y^{s})^{s''}z^{ss'}]^{\frac{t}{ss's''}}=[x^{s's''}(y^{s''}z^{s'})^{\frac{t}{ss's''}}. \tag{2}$$

3.1. If $G\in\mathrm{Ob}\,\mathfrak{Gr}_l$, then for each $x, y\in G$ and every $s, s', s''\in S(l)$, such that $s\mid s''$ and $s'\mid s''$, there exists $t\in S(l)$, such that $ss'\mid t$ and

$$(x^{\frac{s''}{s}}y^{\frac{s''}{s'}})^t=(x^{s'}y^{s})^{s}]^{\frac{ts''}{s's}}. \tag{3}$$

4.1. For the statement of the theorem on localization of the category $\mathfrak{Gr}_l$, we consider $S(l)$ as a directed set as follows : if $s, s'\in S(l)$, then we put $s\leqslant s'$ iff s divides s'.

If G is a group, then we can construct a directed system of sets $\{G(s), \alpha_{ss'}\}_{s,s'\in S(l)}$ with $G(s)=G$, $\forall s\in S(l)$ and for $s, s'\in S(l)$, with $s\leqslant s'$, $\alpha_{ss'}: G(s)=G\longrightarrow G(s')=G$ is the following mapping of sets :

$$\alpha_{ss'}(x)=x^{\frac{s'}{s}},\quad \forall x\in G. \tag{4}$$

Firstly, we observe that the maps $\alpha_{ss'}$ are not necessary homomorphisms of groups. But, we can consider the limit G_l of the directed system of sets $\{G(s), \alpha_{ss'}\}$ and let $q_s: G(s)\rightarrow G_l$ the canonical injection, corresponding to $s\in S(l)$.

5.1. **Theorem.** *Let G be a group of the category $\mathfrak{Gr}_l$. Then:*

(i) G_l has a natural structure of group.

(ii) The maps $q_s : G(s)\to G_l$ are homomorphisms of groups. Moreover, if H is an arbitrary group and $f_s : G(s)\to H$, $\forall s\in S(l)$, are homomorphisms of groups satisfying the conditions $f_{s'}\circ\alpha_{ss'}=f_s$, for any $s, s'\in S(l)$, with $s\leqslant s'$, then there exists a unique homomorphism of groups $f : G_l\to H$, such that $f\circ q_s=f_s$.

(iii) If $f : G\to H$ is a morphism in the category $\mathfrak{Gr}_l$, one can define, in a natural way, a homomorphism of groups $f_l : G_l\to H_l$.

(iv) The correspondences $G\to G_l$, $f\to f_l$, for $G\in \mathrm{Ob}\,\mathfrak{Gr}_l$ and f a morphism in the category $\mathfrak{Gr}_l$, define a covariant functor $E : \mathfrak{Gr}_l\to\mathfrak{Gr}_l$.

(v) The functor E together with the homomorphism $q_1 : G\to G_l$ define a monad (localization) on the category $\mathfrak{Gr}_l$.

The proof of this theorem is given in [7], excepting the second part of (ii). For this part, if H is an arbitrary group and $f_s : G(s)\to H$ are homomorphisms of groups satisfying the conditions $f_{s'}\circ\alpha_{ss'}=f_s$ for any $s, s'\in S(l)$, with $s\leqslant s'$, then by the universal property of the direct limit, there exists a unique map of sets $f : G_l\to H$, such that $f\circ q_s=f_s$, $\forall s\in S(l)$. Therefore, $\forall x\in G$, we have $f(q_s(x))=f_s(x)$ and in particular, $f(q_s(x)q_{s'}(x'))=f(q_{ss'}(x^{s'}x'^{s}))=f_{ss'}(x^{s'}x'^{s})=f_{ss'}(x^{s'})f_{ss'}(x'^{s})=f(q_{ss'}(x^{s'}))f((q_{ss'}(x'^{s}))=f(q_s(x))f(q_{s'}(x'))$, which proves that f is in fact a homomorphism of groups.

2. New properties of the functor E. 1.2. **Theorem.** *The covariant functor $E : \mathfrak{Gr}_l\to\mathfrak{Gr}_l$ is exact and preserves the central extensions.*

Proof. Let

$$G'\xrightarrow{f} G\xrightarrow{g} G'' \tag{5}$$

be an exact sequence in the category $\mathfrak{Gr}_l$ and let

$$G'_l\xrightarrow{f_l} G_l\xrightarrow{g_l} G''_l \tag{6}$$

be the localized sequence. From $g\circ f=e''$, it follows that $g_l\circ f_l=e''_l$ and therefore $\mathrm{Im}\, f_l\subseteq \mathrm{Ker}\, g_l$. Then, if $g_l(q_s(y))=e''_l$, for $y\in G$, i.e. $q_s(g(y))=e''_l$, which implies $g(y^{t})=e''$, with $t\in S(l)$, $t\geqslant s$. By this, it follows $y^{t/s}=f(x)$, $x\in G'$, which implies $q_s(y)=q_t(f(x))$, i.e. $q_s(y)=f_l(q_t(x))$. So, we proved that the sequence (6) is exact. Now we suppose that (5) is a central extension, that is, f is a monomorphism, g is epimorphism and $G'\subseteq Z(G)$-the centre of the group G. First, because E is an exact functor, f_l is a monomorphism and g_l is an epimorphism. Then, if $q_s(x)\in G'_l$, $q_t(y)\in G_l$, then $f_l(q_s(x))\cdot q_t(y)=q_s(f(x))q_t(y)=q_{st}((f(x))^t y^s)=$ $=q_{st}(f(x^t)y^s)$ and, since $f(x^t)y^s=y^s f(x^t)$, we deduce that $f_l(q_s(x))q_t(y)=$ $=q_{st}(y^s(f(x))^t)=q_t(y)\cdot q_s(f(x))=q_t(y)\cdot f_l(q_s(x))$, hence $G'_l\subseteq Z(G_l)$ and this proves that (6) is a central extension.

2.2. **Theorem.** *If G is an abelian group, then his localization at $l\subseteq P$ defined by $Z_l\otimes G$ (for example in [2]) is a group isomorphic with the group G_l above considered.*

Prof. If G is an abelian group, then the system $\{G(s), \alpha_{ss'}\}$ is a directed system of abelian groups, indexed on a directed set and therefore the direct limit exists and (by 2, Proposition 2.6) it is a group isomorphic to $Z_l\otimes G$. On the other hand, from 5.1 (ii), with the hypothesis that $\alpha_{ss'}$ are homomorphisms of groups, which holds if G is an abelian group, the group G_l is isomorphic to the direct limit of the above directed system of groups.

3.2. Naturally, it is interesting also to compare the localization G_l of a group G with its nilpotent localization, supposing that G is simultaneously in the category $\mathfrak{Gr}_l$ and in the category $\mathfrak{N}_c$ (the category of all nilpotent groups G with nil $(G)\leqslant c$). If $G\in \mathrm{Ob}\ \mathfrak{N}_c$ then its localization L_cG is constructed by induction on $c=$ nil (G) For $c=1$, G is an abelian group and therefore $L_1G=G\otimes Z_l$, which, according with 2.2. is isomorphic to the group G_l. We can prove the general result:

4.2. **Theorem.** *If $G\in \mathrm{Ob}\,\mathfrak{Gr}_l$ and $G\in \mathrm{Ob}\,\mathfrak{N}_c$, then the groups G_l and L_cG are isomorphic.*

Proof. The proof proceeds by induction on $c=$ nil (G). We suppose that if $G\in \mathrm{Ob}\,\mathfrak{N}_{c-1}$ and $G\in \mathrm{Ob}\,\mathfrak{Gr}_l$, then there exists an isomorphism $\varphi_{c-1} : G_l\to L_{c-1}G$ for which the diagram

$$\begin{tikzcd} G \arrow[r, "q_1"] \arrow[dr, "\varepsilon_1"'] & G_l \arrow[d, "\varphi_{c-1}"] \\ & L_{c-1}G \end{tikzcd}$$

is commutative.

Now, if we suppose that $G\in \mathrm{Ob}\ \mathfrak{N}_c$, we consider the central extension

$$\Gamma \rightarrowtail^{u} G \xrightarrow{v} G/\Gamma \tag{7}$$

with $\Gamma=\Gamma^c(G)$, nil $(\Gamma)=1$ and nil $(G/\Gamma)\leqslant c-1$. Then the construction of the group L_cG [4, p. 15] is the following: by Theorem E i l e n b e r g - M a c L a n e, (7) corresponds to a cohomology class $\eta\in H^2(G/\Gamma ; \Gamma)$, where Γ is considered as G/Γ-module with the trivial action (see for example [8, p. 206]). Then, considering the homomorphism $e : \Gamma\to L_1\Gamma$ one obtains $e_*\eta\in H^2$ $(G/\Gamma ; L_1\Gamma)$ and hence, [4, Corollary 1.14, p. 14], there is an element $\eta_l\in H^2$ $(L_{c-1}(G/\Gamma) ; L_1\Gamma)$ such that $e_*\eta_l=e_*\eta$, where e_* is the induced by the homomorphism $e : G/\Gamma\to L_{c-1}(G/\Gamma)$. Then one considers the central extension corresponding to the cohomology class η_l:

$$L_1\Gamma \xrightarrow{u'} L_cG \xrightarrow{v'} L_{c-1}(G/\Gamma). \tag{8}$$

In virtue of the induction hypothesis, we have $L_{c-1}(G/\Gamma)\cong(G/\Gamma)_l$ and $L_l\Gamma\cong\Gamma_1$. Thus η_l corresponds to an element $\eta'_l\in H^2((G/\Gamma)_l ; \Gamma_l)$ and this cohomology class corresponds to a central extension which can be identified with (8). On the other hand, from Theorem 1.2 the sequence

$$\Gamma_l \xrightarrow{u_l} G_l \xrightarrow{v_l} (G_l/\Gamma)_l \tag{9}$$

is also a central extension of the same abstract kernel $((G/\Gamma)_l, \Gamma_l, \psi)$ as easily can be verified. It follows that there exists an isomorphism $\varphi_c : G \to L_c G$ such that the following diagram:

$$\begin{tikzcd}
\Gamma_l \arrow[r, tail, dashed, "u_l"] \arrow[d, "\varphi_1"] & G_l \arrow[r, two heads] \arrow[d, "\varphi_c"] & (G/\Gamma)_l \arrow[d, "\varphi_{c-1}"] \\
L_1 \arrow[r, tail, "u'"] & L_c G \arrow[r, two heads, "u_1"] & L_{c-1}(G/\Gamma)
\end{tikzcd}$$

is commutative.

5.2. **Definition.** *A group G is called $S(l)$-torsion group if for every $x \in G$, there exists $t \in S(l)$ such that $x^t = e$ (the unity of the group).*

6.2. *Remark.* If e_l is the unity of the group G_l, then $q_s(x) = e_l$ iff there exists $t \in S(l)$ with $t \geqslant s$ and such that $x^{\frac{t}{s}} = e$.

7.2. **Corollary.** *If G is a $S(l)$-torsion group, then $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and its localization G is the trivial group.*

8.2. *Remark.* For any group $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$, the kernel of the homomorphim $q_1 : G \to G_l$ is a $S(l)$-torsion group and, for every $q_s(x) \in G_l$, we have $(q_s(x))^s = q_1(x) \in \mathrm{Im}\, q_1$.

9.2. **Proposition.** *If $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and G has not $S(l)$-torsion elements, then, if $x^s = y^s$, with $x, y \in G$, $s \in S(l)$, it follows $x = y$.*

Proof. From $x^s = y^s$ we deduce $x^s(y^{-1})^s = e_l$. Applying 2.1, there exists $t \in S(l)$, with $s \mid t$ and such that $e = (x^s(y^{-1})^s)^{\frac{t}{s}}$ and since G has not $S(l)$-torsion elements, it follows $x = y$.

3. **l-local groups and l-equivalences. 1.3. Definition.** [7]. *A group $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$ is called l-local if the homomorphism $q_1 : G \to G_l$ is an isomorphism.*

2.3. **Proposition.** *If $G' \xrightarrow{\mu} G \xrightarrow{\varepsilon} G''$ is a central extension in the category $\mathfrak{Gr}_l$ and G', G'' are l-local groups, then G is also l-local.*

Proof. Let $s \in S(l)$. We prove that $\alpha_s : G \to G$, with $\alpha_s(x) = x^s$ is a bijection and then we can apply Lemma 1. of [7] First, if $x \in G$, then $\varepsilon(x) \in G''$ and because $\alpha_s : G'' \to G''$ is a bijection, we have $\varepsilon(x) = y''^s$ for a $y'' \in G''$. Then $y'' = \varepsilon(y)$, with $y \in G$ and therefore $y''^s = \varepsilon(y^s)$. We deduce that $(y^s)^{-1}x \in \mathrm{Ker}\,\varepsilon$ and by this $(y^s)^{-1}x = \mu(y')$, with $y' \in G'$. But, since $y' = x'^s$, with $x' \in G'$ and $\mu(G') \subseteq Z(G)$, we deduce that $x = y^s\mu(y') = y^s(\mu(x'))^s = (y\mu(x'))^s$. Consequently, α_s is a surjection. Suppose now that $x^s = y^s$, for $x, y \in G$ and $s \in S(l)$. Then, $\varepsilon(x^s) = \varepsilon(y^s)$, and this implies $(\varepsilon(x))^s = (\varepsilon(y))^s$, therefore $\varepsilon(x) = \varepsilon(y)$ and by this, $x = y\mu(x')$, with $x' \in G'$. Thus, $x^s = y^s\mu(x'^s)$ and therefore $\mu(x'^s) = 1$ and then $x'^s = 1$. Since G' is l-local group, we obtain $x' = 1$ and therefore $x = y\mu(1)$, that is $x = y$.

3.3. **Theorem.** *A homomorphism $f : G \to H$ in the category $\mathfrak{Gr}_l$ is l-equivalence iff f is l-bijective.*

The proof is given in [7, Lemma 2].

From 3.3 and by Remarks 6.2, 8.2, we obtain .

4.3. **Corollary.** *If $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$, then the homomorphism $q_1 : G \to G_l$ is an l-equivalence.*

5.3. **Theorem.** *If $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$, then the homomorphism $(q_1)_* : \tilde{H}_n(G) \to \tilde{H}_n(G_l)$ is an l-equivalence, for every n.*

Proof. According to 5.1, the homomorphism $q_1 : G \to G_l$ is an l-equivalence. Then, using the property of the homology of groups to commute with direct limits and by 5.1 and 2.2 we have:

$$(\tilde{H}_n(G))_l = \varinjlim_{s \in S(l)} \tilde{H}_n(G) = H_n(\varinjlim_{s \in S(l)} G) = \tilde{H}_n(G_l) = (q_1')_* \tilde{H}_n((G_l)_l) = \tilde{H}_n(\varinjlim_{s \in S(l)} G_l) =$$

$$= \varinjlim_{s \in S(l)} \tilde{H}_n(G_l) = (\tilde{H}_n(G_l))_l$$ and this composition of homomorphisms is $((q_1)_*)_l$.

6.3. **Corollary.** *If $G \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and A is an l-local abelian group which is a trivial G-module, then $q_1^* : H^n(G_l ; A) \cong H^n(G ; A)$. is an isomorphism*

Proof. The homomorphism $q_1 : G \to G_l$ induces the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}\,(H_{n-1}(G_l), A) \arrow[r, tail] \arrow[d, "q_1'^*"] & H^n(G_l ; A) \arrow[r, two heads] \arrow[d, "q_1^*"] & \mathrm{Hom}\,(H_n(G_l), A) \arrow[d, "q_1''^*"] \\
\mathrm{Ext}\,(H_{n-1}(G), A) \arrow[r] & H^n(G ; A) \arrow[r, two heads] & \mathrm{Hom}\,(H_n(G), A)
\end{tikzcd}$$

Using [7, Lemmas 3, 4] and 5.3, it follows that $q_1'^*$ and $q_1''^*$ are isomorphisms which implies that q_1^* is also an isomorphism.

4. **l-n-simple spaces.** 1.4. Let X be a topological space, which we suppose path-connected, and let $x_0 \in X$. If $a \in \pi_1(X, x_0)$, then denote by $h_a : \pi_n(X, x_0) \to \pi_n(X, x_0)$, $n \geqslant 1$, the induced isomorphism by the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ (for example see [9, p. 491]). For $n \geqslant 2$, we consider the homomorphism:

$$\varepsilon(x_0, a) = 1_{\pi_n(X, x_0)} - h_a : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0).$$

Then, $\mathrm{Im}\,\varepsilon(x_0, a)$ is a subgroup of the group $\pi_n(X, x_0)$.

2.4. **Definition.** [7]. *Let X be a path-connected space. If for a point $x_0 \in X$, we have $\pi_n(X, x_0) \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and the group $\pi_1(X, x_0)$ operates trivially on the group $\pi_n(X, x_0)$, then X is called an l-n-simple space.*

3.4. **Proposition.** (see also [7]). *A path-connected space X is l-1-simple iff $\pi_1(X, x_0) \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and the group $\pi_1(X, x_0)_l$ is abelian for a point $x_0 \in X$.*

Proof. We suppose that $\pi_1(X, x_0) \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and its action on the group $\pi_1(X, x_0)$ is trivial. This means that $\forall s \in S(l)$, $\forall [\alpha], [\beta] \in \pi_1(X, x_0)$, we have $q_s([\alpha]\,[\beta]) = q_s([\beta]\,[\alpha])$. It follows that if $q_s([\alpha]), q_{s'}([\beta]) \in \pi_1(X, x_0)$, then $q_s([\alpha]) \cdot q_{s'}([\beta]) = q_{ss'}([\alpha]^{s'}[\beta]^s) = q_{ss'}([\beta]^s[\alpha]^{s'}) = q_{s'}([\beta])q_s([\alpha])$, that is, $\pi_1(X, x_0)_l$ is abelian. Conversely, if $\pi_1(X, x_0)_l$, there exists and this group is abelian, then, from $q_s([\alpha])q_s([\beta]) = q_s([\alpha]\,[\beta]) = q_s([\beta])q_s([\alpha])$ we deduce that we have $q_s([\alpha]\,[\beta]\,[\alpha]^{-1}) = q_s([\beta])$, such that $q_s(h_{[\alpha]}\,[\beta]) = q_s([\beta])$.

4.4. **Definition.** [7]. *A continuous map $f : X \to Y$ between path connected spaces is called l-n-equivalence, $n \geqslant 1$, if the homomorphisms $f_{\#q} : \pi_q(X, x) \to \pi_q(Y, f(x))$ are l-injective for $1 \leqslant q < n$ and l-surjective for $1 \leqslant q \leqslant n$.*

5. **l-simple and l-local spaces. 1.5. Definition.** *A path-connected space X is called l-simple if this is l-n-simple, for any $n \geq 1$. X is called l-local if this is l-simple and all its homotopy groups $\pi_n(X, x_0)$, in a point $x_0 \in X$, are l-local.*

2.5. **Definition.** *A continuous map $f : X \to Y$, between path-connected spaces, is said to l-localize homotopy if the following conditions are satisfied :*

a) Y is a *l-local* space ; b) $\pi_q(X, x) \in \mathrm{Ob}\,\mathfrak{Gr}_l$, $\forall q \geq 1$, $\forall x \in X$, c) The homomorphism $f_{\#q} : \pi_q(X, x) \to \pi_q(Y, f(x))$ are l-equivalences, for every $q \geq 1$.

3.5. **Corollary.** *A continuous map $f : X \to Y$, l-localizes homotopy iff Y is l-local space and for every integer $q \geq 1$, $\mathrm{Ker}\, f_{\#q}$ is $S(l)$-torsion group and for any class $u \in \pi_q(Y, f(x))$, there exists $s \in S(l)$, such that $u^s \in \mathrm{Im}\, f_{\#q}$.*

4.5. *Remark.* Using [7, Proposition 4], the condition a) in Definition 2.5 can be written as follows :

a') Y is a simple space and the group $\pi_n(Y)$, $n \geq 1$ (which are abelian) are l-local.

Applying Theorem 2.2 and Definition 2.5 we obtain :

5.5. **Proposition.** *If X, Y are l-simple topological spaces and $f : X \to Y$ is a continuous map, then f l-localizes homotopy iff f l-localizes homotopy in the usual sense (for example,* [2, p. 31]).

6.5. As usually, a continuous map $f : X \to Y$ is said to l-localize homology if all groups $H_n(Y)$ are l-local and $f_{*n} : H_n(X) \to H_n(Y)$ are l-equivalences for every n.

Using Theorem 2.2 and paragraph 4, we have :

7.5. **Proposition.** *The homomorphism $f_{*n} : H_n(X) \to H_n(Y)$ is an l-equivalence iff the homomorphism*

$$f_{*n} : H_n(X ; Z_l) \to H_n(Y ; Z_l) \text{ is an isomorphism.}$$

8.5. Now, we recall (see, for example [9, Ch. 9]) that if $F \to E \xrightarrow{p} B$ is a Hurewicz fibration, then there exists a contravariant functor from the fundamental groupoid of the space B in the homotopy category of topological spaces which associates to any point $b \in B$ the fibre $F_b = p^{-1}(b)$ and to a class of paths $[\omega]$ a homotopy class $h_{[\omega]} \in [F_{\omega(0)}, F_{\omega(1)}]$. It follows that for a ring R, one can define a contravariant functor from the fundamental groupoid of B to the category of graded R-modules, by associating to a point $b \in B$ the R-module $H_*(F_b ; R)$ and to a class of paths $[\omega]$ the homomorphism $h_{[\omega]*} : H_*(F_{\omega(0)} ; R) \to H_*(F_{\omega(1)} ; R)$. The above fibration is called R-orientable if $h_{[\omega]*} = \mathrm{Id}$, for every class of paths $[\omega]$ in B. This fibration is orientable if it is R-orientable for every ring R. Using Theorem 2.2 and [2, Proposition 5.3, p. 32] we obtain :

9.5. **Proposition.** *Let $F \to E \to B$ and $\overline{F} \to \overline{E} \to \overline{B}$ be two orientable fibrations and consider the following commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \overline{F} & \longrightarrow & \overline{E} & \longrightarrow & \overline{B} \end{array}$$

If two of the maps f, g, h l-localize homology, then so does then third.

10.5. **Theorem.** *Let $n \geq 1$ and let π be a group such that $\pi \in \mathfrak{Ab}$ if $n \geq 2$ and $\pi \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and π_l is abelian, for $n = 1$.*

Let $\hat{q}_l : K(\pi, n) \to K(\pi_l, n)$ be the continuous map induced by the homomorphism $q_l : \pi \to \pi_l$. Then :

a) *$\hat{q}_l$ l-localizes homotopy,* b) *$\hat{q}_l$ l-localizes homology.*

Proof. a) First, we remark that the space $K(\pi_l, n)$ is l-m-simple, for every m, because $\pi_m(K(\pi_l, n)) = \delta_{mn}\pi_l \in \mathrm{Ob}\,\mathfrak{Gr}_l$ and if $n = 1$, then by Proposition 3.4, the space $K(\pi_l, 1)$ is l-1-simple, iff the group $\pi_1(K(\pi_l, 1)) = \pi_l$ is abelian, what we suppose by hypothesis. Then, it is obvious that the homotopy groups of $K(\pi_l, n)$ are l-local. Therefore $K(\pi_l, n)$ is indeed l-local. Also by hypothesis we have $\pi \in \mathrm{Ob}\,\mathfrak{Gr}_l$. Finally, $(\hat{q}_l)_{\# m} : \pi_m(K(\pi, n)) \to \pi_m(K(\pi_l, n))$ is the trivial homomorphism if $m \neq n$ and this coincides with q_l if $m = n$. By Corollary 4.3, it follows that q_l l-localizes homotopy.

b) For $n \geq 12$, $\dot{q}_1$ localizes homology by, Lemma 5.4, of [2, p. 34]. For $n = 1$, the space $K(\pi_l, 1)$ is a simple space and from [2] it follows that the groups $H_m(K(\pi_l, 1))$ are l-local. We consider the homomorphisms $(\hat{q}_l)_{*m} : H_m(K(\pi, 1)) \to H_m(K(\pi_l, 1))$. By Proposition 7.5, in order that these homomorphisms be l-equivalences it is necessary and sufficient that the homomorphisms $(\dot{q}_1)_* : H_m(K(\pi, 1), Z_l) \to H_m(K(\pi_l, 1), Z_l)$ be isomorphisms. But $H_m(K(\pi_l, 1), Z_l) \cong H_m(\pi_l, Z_l)$ and $H_m(K(\pi, 1), Z_l) \cong H_m(\pi, Z_l)$. Then $H_m(\pi, Z_l)$ are the homology groups of the chain complex

$$C(\pi, Z_l) : \ldots \to C_m \xrightarrow{\partial} C_{m-1} \to \ldots$$

with $C_m = C_m(\pi) \otimes Z_l$, where $C_m(\pi)$ is the free abelian group generated by the set of systems $(x_0, x_1, \ldots, x_m)$, with $x_i \in \pi$. One can prove, considering at first the case when π is finite generated and then the general case writting $\pi = \varinjlim \pi'$, with π' all finite generated subgroups of π, that we have an isomorphism between the above chain complex and the complex $C(\pi_l, Z_l) = C(\pi_l) \otimes Z_l$. Thus it follows that $(\hat{q}_l)_*$ are isomorphisms and therefore $(\hat{q}_l)_{*m}$ are l-equivalences.

11.5. **Corollary.** *If $f : K(\pi, n) \to K(\pi', n)$, $n \geq 1$, l-localizes homotopy, then f also l-localizes homology.*

Proof. We must consider only the case $n = 1$, the other cases being treated in [2, Corollary 5.5, p. 37]. But, if $f : K(\pi, 1) \to K(\pi', 1)$ l-localizes homotopy, then we have $\pi_l \cong \pi_l \cong \pi'$ and f is in fact the map $\hat{q}_l : K(\pi, 1) \to K(\pi_l, 1)$, which by Theorem 10.5, b) l-localizes homology. To establish and prove the general result regarding the relation between the homotopy l-localization and the homology l-localization we use a suitable Postnikov decomposition of l-simple space. Namely, closely following the Postnikov decomposition of simple space [9, Ch. 8, §3], we obtain :

12.5. **Theorem.** *If X is an l-simple space, then there exists a Postnikov decomposition*

$$\begin{tikzcd} \dots \arrow[r] & X^n \arrow[r, "p^n"] & X^{n-1} \arrow[r] & \dots & \dots \arrow[r] & X^1 \arrow[r, "p^1"] & \mathrm{pt} \\ & & X \arrow[ul, "g^n"] \arrow[u, "g^{n-1}"'] & & & & \end{tikzcd}$$

where $K(\pi_n(X), n) \to X^n \xrightarrow{p^n} X^{n-1}$ *is the fibre map induced by the Postnikov invariant* $k^n : X^{n-1} \to K(\pi_n(X), n+1)$ *and* $g^n : X \to X^n$ *is an* l-n-*equivalence (Definition* 4.4) *and* $\pi_i(X^n) = 0$, *when* $i > n$.

Now we can prove the following theorem:

13.5. **Theorem.** *If* $f : X \to Y$ *l-localizes homotopy and X is l-simple space, then f also l-localizes homology.*

Proof. Let $\{X^n, p^n, g^n\}$ and $\{Y^n, q^n, h^n\}$ be Postnikov decomposition for X and Y respectively. Then the map f induces $f^n : X^n \to Y^n$ such that $f^n g^n = h^n f$ and having the commutative diagram

$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "p^n"'] & Y^n \arrow[d, "q^n"] \\ X^{n-1} \arrow[r, "f^{n-1}"] & Y^{n-1} \end{tikzcd}$$

Therefore, for any n we have the map $f' = (\widehat{f_{\#n}}) : K(\pi_n(X)_l, n) \to K(\pi_n(Y)_l, n)$. Now we prove, by induction on n, that f^n l-localizes homology. For $n = 1$, $X^1 = K(\pi_1(X), 1)$ and $Y^1 = K(\pi_1(Y), 1)$ and we can apply Corollary 11.5. Suppose that f^{n-1} l-localizes homology and we consider the following commutative diagram:

$$\begin{tikzcd} K(\pi_n(X), n) \arrow[r] \arrow[d, "f'"'] & X^n \arrow[r, "p^n"] \arrow[d, "f^n"'] & X^{n-1} \arrow[d, "f^{n-1}"'] \\ K(\pi_n(Y), n) \arrow[r] & Y^n \arrow[r, "q^n"] & Y^{n-1} \end{tikzcd}$$

Then, by hypothesis f' l-localizes homotopy and by Corollary 11.5 this also l-localizes homology. Finally, we have the commutative diagram

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"] \arrow[d, "g^n_*"'] & H_n(Y) \arrow[d, "h^n_*"] \\ H_n(X^n) \arrow[r, "f^n_*"] & H_n(Y^n) \end{tikzcd}$$

In this diagram h^n_* is a n-equivalence and g^n_* is an l-n-equivalence. It follows that h^n_* is an isomorphism. Therefore, because f^n l-localizes homology and thus $H_n(Y^n)$ is l-local group, it follows that $H_n(Y)$ is also l-local. Moreover, by applying a suitable version Whitehead theorem for the category $\mathfrak{G}_l$, it follows that g^n_* is also an l-equivalence. Using the above diagram and the fact that f^n_* is an l-equivalence, it follows that $f_* : H_n(X) \to H_n(Y)$ is also l-equivalence for every n, and this ends the proof of the theorem.

14.5. **Corollary.** *If X is an l-local space, then it has all homology groups l-local.*

Proof. From [7, Prop. 4], X is a simple space. Applying Theorem 2.2 it follows that X local in the usual sense. Then we can apply [2, Corollary 5.7, p. 39].

6. **Localization of the l-simple spaces. 1.6. Definition.** *A continuous map* $f : X \to Y$ *is called l-localization. if :*

(i) Y *is an l-local space,*

(ii) *If* $g : X \to W$ *is a continuous map, with W an l-local space, then there exists a continuous map* $\bar{g} : Y \to W$, *unique up to homotopy such that* $\bar{g} \circ f = g$.

Imediately it follows:

2.6. **Proposition.** *If* $f : X \to Y$ *and* $f' : X \to Y'$ *are l-localizations, then there exists a unique homotopy equivalence* $h : Y \to Y'$, *such that* $hf = f'$.

3.6. **Proposition.** *If* $f : X \to Y$, *l-localizes homology and Y is l-local space, then f is an l-localization.*

The proof is similar with those of Proposition 6.3 from [2, p. 40].

From Proposition 3.6 and Corolary 13.5, we obtain:

4.6. **Corollary.** *If* $f : X \to Y$, *l-localizes homotopy and if X is an l-local space, then f is an l-localization.*

5.6. **Theorem.** *Given an l-simple space X, there exists an l-local space* X_l *and a continuous map* $u : X \to X_l$ *which is an l-localization.*

Proof. The space X being supposed l-simple, there exists a Postnikov decomposition of its $\{X^n, p^n, g^n, k^n\}$ satisfying the conditions of Theorem 12.5. We construct the sequence $u^n : X^n \to X^n_l$, which l-localizes homology, by induction on n. For $n = 1$, we have $X^1 = K(\pi_1(X, x_0), 1)$ and we take $u^1 = \hat{q}_1$, where $q_l : \pi_1(X, x_0) \to \pi_1(X)_l$ is the localization homomorphism, for a fixed point $x_0 \in X$. From Theorem 10.5, u^1 l-localizes homotopy and homology and by Colollary 4.6 this is even an l-localization.

Suppose that we constructed $u^{n-1} : X^{n-1} \to X^{n-1}_l$ with X^{n-1}_l an l-local space, which is an l-localization. If we denote $\pi_n(X, x_0) = \pi$, we can consider the map $\hat{q}_l k^n : X^{n-1} \to K(\pi_l, n+1)$. The space $K(\pi_l, n+1)$ being l-local and $u^{n-1} : X^{n-1} \to X^{n-1}$ being an l-localization it follows that there exists a continuous map $k^n_l : X^{n-1}_l \to K(\pi_l, n+1)$ such that $\hat{q}_l k^n = k^n_l u^{n-1}$.

We consider the Serre fibration.

$$\Omega K(\pi_l, n+1) \longrightarrow EK(\pi_l, n+1) \longrightarrow K(\pi_l, n+1).$$

Then, the map k^n_l induces a fibre map $\Omega K(\pi_l, n+1) \to X^n_l \xrightarrow{q^n} X^{n-1}_l$ and $\hat{q}_l$ and u^{n-1} induce a map $u^n : X^n \to X^n_l$ such that $u^{n-1} p^n = q^n u^n$. We prove that u^n is an l-localization.

First, $u^n_{\#i} = u^{n-1}_{\#i}$, if $i < n$, therefore $u^n_{\#i}$ is an l-equivalence and the groups $\pi_i(X^{n-1}_l)$ are l-local. For $i = n$, $u^n_{\#n}$ coincides with $(\Omega\hat{q}_l)$ and, because $\hat{q}_l$ l-localizes homotopy, it follows that $\Omega\hat{q}_l$ l-localizes homotopy.

Then we consider the homotopy exact sequence

$$\ldots \to \pi_{i+1}(K(\pi_i, n+1)) \to \pi_i(X_l^n) \to \pi_i(X_l^{n-1}) \to \pi_i(K(\pi_i, n-1)) \to \ldots$$

and using the induction hypothesis, we deduce that $\pi_i(X_l^n)=0$, for $i>n$. Also we have $\pi_i(X^n)=0$ if $i>n$. Thus it follows that $u^n_{\#i}: \pi_i(X^n) \to \pi_i(X_l^n)$ is an l-equivalence, for any i. Then, by [7, Example 11], we have that X_l^n is an l-simple space and by the above considerations, it follows that it is even l-local space. Thus we obtained that $u^n : X^n \to X_l^n$ l-localizes homotopy and since X^n is an l-simple space, it follows, from Corollary 4.7, that u^n is an l-localization.

Now we consider the following inverse system of topological spaces

$$X^1 \xleftarrow{q^2} X^2 \leftarrow \ldots \leftarrow X^{n-1} \xleftarrow{q^n} X^n \leftarrow \ldots$$

and let $X_l = P(\varprojlim_n X^n)$, where, for a space Y, $P(Y)$ denotes the geometric realization of the singular chain complex $S(Y)$. Also, we can define the map $u : X \to X_l$ by composition of the functors $\varprojlim$ and P, that is $u = P(\varprojlim_n u^n)$. Because u^n are l-localizations it follows that u is also an l-localization, what finishes the proof of the theorem.

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ON THE STANLEY-REISNER RING OF A JOIN

BY

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In [4], it was studied the S t a n l e y - R e i s n e r ring of the suspension $S\Delta$ of a given simplicial complex Δ. In this note we consider a similar problem for the join $\Delta_1 * \Delta_2$ of arbitrary simplicial complexes Δ_1, Δ_2. For such a simplicial complex we study the following properties: C o h e n - M a c a u l a y n e s s (CM), double C o h e n - M a c a u l a y n e s s (2-CM) almost C o h e n - M a c a u l a y n e s s (AMC), B u c h s b a u m n e s s and G o r e n s t e i n e s s.

Proposition 1. (see 3.2, p. 167 of [2]). *If Δ_1, Δ_2 are simplicial complexes, then for an arbitrary field k and each integer n there exists the following isomorphism*:

$$\tilde{H}_n(\Delta_1 * \Delta_2 ; k) \cong \bigoplus_{i+j=n-1} \tilde{H}_i(\Delta_1 ; k) \otimes \tilde{H}_j(\Delta_2 ; k). \tag{1}$$

Proof. It is well-known [3, ex : 3, p. 251] and [5, ex : 6.3, p. 109] that if X, Y are polyhedra then $X * Y$ and $S(X \wedge Y)$ are homotopy equivalent. So, for two simplicial complexes Δ_1, Δ_2 it follows the homotopy equivalence.

$$|\Delta_1 * \Delta_2| \simeq S(|\Delta_1| \wedge |\Delta_2|). \tag{2}$$

where $\wedge$ is the smash product.

On the other hand there exists [3, 8.5.5, p. 343] the following isomorphism of reduced chain complexes:

$$\wedge : \bigoplus_{r+s=n} \tilde{C}_r(\Delta_1) \otimes \tilde{C}_s(\Delta_2) \cong \tilde{C}_n(\Delta_1 \wedge \Delta_2). \tag{3}$$

This isomorphism and the Künneth formula imply the following isomorphism

$$\tilde{H}_n(\Delta_1 \wedge \Delta_2 ; k) \cong \bigoplus_{r+s=n} \tilde{H}_r(\Delta_1 ; k) \otimes \tilde{H}_s(\Delta_2 ; k) \oplus \bigoplus_{r+s=n-1} \mathrm{Tor}\,(\tilde{H}_r(\Delta_1 ; k), \tilde{H}_s(\Delta_2 ; k)). \tag{4}$$

By [3, 4.5.10, p. 145] if C is a chain complex and k is a field, then the group $H_*(C \otimes k) = \oplus H_n(C \otimes k)$ becomes a vector space over k. All k-vector spaces are flat, so Tor_n^k vanish for all $n \geq 1$, [6, vol. 1, p. 114]. We deduce $\mathrm{Tor}\,(\tilde{H}_r(\Delta_1 ; k), \tilde{H}_s(\Delta_2 ; k)) = 0$ for all r, s. Using (4), we obtain the following isomorphism of vector spaces: