$$\ldots \to \pi_{i+1}(K(\pi_1, n+1)) \to \pi_i(X_1^n) \to \pi_i(X_1^{n-1}) \to \pi_i(K(\pi_1, n-1)) \to \ldots$$

and using the induction hypothesis, we deduce that $\pi_i(X_1^n)=0$, for $i>n$. Also we have $\pi_i(X^n)=0$ if $i>n$. Thus it follows that $u^n_{\#i} : \pi_i(X^n) \to \pi_i(X_1^n)$ is an l-equivalence, for any i. Then, by [7, Example 11], we have that X_1^n is an l-simple space and by the above considerations, it follows that it is even l-local space. Thus we obtained that $u^n : X^n \to X_1^n$ l-localizes homotopy and since X^n is an l-simple space, it follows, from Corollary 4.7, that u^n is an l-localization.

Now we consider the following inverse system of topological spaces

$$X^1 \xleftarrow{q^2} X^2 \leftarrow \ldots \leftarrow X^{n-1} \xleftarrow{q^n} X^n \leftarrow \ldots$$

and let $X_l = P(\varprojlim_n X^n)$, where, for a space Y, $P(Y)$ denotes the geometric realization of the singular chain complex $S(Y)$. Also, we can define the map $u : X \to X_l$ by composition of the functors $\varprojlim$ and P, that is $u = P(\varprojlim_n u^n)$.

Because u^n are l-localizations it follows that u is also an l-localization, what finishes the proof of the theorem.

REFERENCES

1. Adams, J. F. – *Idempotent functors in homotopy theory*. Manifolds, Tokyo, 1973, University of Tokyo Press, 247–255.
2. Arkowitz, M. – *Localization and H-spaces*. Aarhus Universitet, Lecture Notes Series No. 44, 1976.
3. Hilton, P., Mislin, G., Roitberg, J. – *H-spaces of rank two and noncancellation phenomena*. Inv. Math. 16 (1972), 325–334.
4. Hilton, P., Mislin, G., Roitberg, J. – *Localization of Nilpotent Groups and Spaces*, Nort-Holand Math. Studies, 1975.
5. Hilton, P. J., Wylie, S. – *Homology Theory* (Russian), Mir, 1966.
6. Hu, S. T. – *Homotopy Theory* (Russian), Mir, 1964.
7. Pop, I. – *A generalization of simple spaces*. Revue Roumaine de Math. Pures et Appl. Tom XXV, N° 6 (1980), 925–932.
8. Solian, A. – *Coomologia grupurilor*, Ed. Acad. R.S.R., Buc., 1977.
9. Spanier, E. H. – *Algebraic Topology* (Russian), Mir, 1971.
10. Whitehead, G. W. – *Elements of Homotopy Theory*, Springer-Verlag, 1978.

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ON THE STANLEY-REISNER RING OF A JOIN

BY

CORINA SAVA

In [4], it was studied the Stanley-Reisner ring of the suspension $S\Delta$ of a given simplicial complex Δ. In this note we consider a similar problem for the join $\Delta_1 * \Delta_2$ of arbitrary simplicial complexes Δ_1, Δ_2. For such a simplicial complex we study the following properties: Cohen-Macaulayness (CM), double Cohen-Macaulayness (2-CM), almost Cohen-Macaulayness (AMC), Buchsbaumness and Gorensteinness.

Proposition 1. (see 3.2, p. 167 of [2]). *If Δ_1, Δ_2 are simplicial complexes, then for an arbitrary field k and each integer n there exists the following isomorphism:*

$$\tilde{H}_n(\Delta_1 * \Delta_2 ; k) \cong \bigoplus_{i+j=n-1} \tilde{H}_i(\Delta_1 ; k) \otimes \tilde{H}_j(\Delta_2 ; k). \tag{1}$$

Proof. It is well-known [3, ex: 3, p. 251] and [5, ex: 6.3, p. 109] that if X, Y are polyhedra then $X * Y$ and $S(X \wedge Y)$ are homotopy equivalent. So, for two simplicial complexes Δ_1, Δ_2 it follows the homotopy equivalence,

$$|\Delta_1 * \Delta_2| \simeq S(|\Delta_1| \wedge |\Delta_2|), \tag{2}$$

where $\wedge$ is the smash product.

On the other hand there exists [3, 8.5.5, p. 343] the following isomorphism of reduced chain complexes:

$$\wedge : \bigoplus_{r+s=n} \tilde{C}_r(\Delta_1) \otimes \tilde{C}_s(\Delta_2) \cong \tilde{C}_n(\Delta_1 \wedge \Delta_2). \tag{3}$$

This isomorphism and the Künneth formula imply the following isomorphism

$$\begin{aligned} \tilde{H}_n(\Delta_1 \wedge \Delta_2 ; k) \cong & \bigoplus_{r+s=n} \tilde{H}_r(\Delta_1 ; k) \otimes \tilde{H}_s(\Delta_2 ; k) \oplus \\ & \bigoplus_{r+s=n-1} \operatorname{Tor}(\tilde{H}_r(\Delta_1 ; k), \tilde{H}_s(\Delta_2 ; k)). \end{aligned} \tag{4}$$

By [3, 4.5.10, p. 145] if C is a chain complex and k is a field, then the group $H_*(C \otimes k) = \oplus H_n(C \otimes k)$ becomes a vector space over k. All k-vector spaces are flat, so Tor_n^k vanish for all $n \geqslant 1$. [6, vol. I, p. 114]. We deduce $\operatorname{Tor}(\tilde{H}_r(\Delta_1 ; k), \tilde{H}_s(\Delta_1 ; k)) = 0$ for all r, s. Using (4), we obtain the following isomorphism of vector spaces:

$$(5)\qquad \widetilde{H}_n(\Delta_1\wedge\Delta_2 : k)\equiv \bigoplus_{r+s=n} \widetilde{H}_r(\Delta_1 ; k)\otimes\widetilde{H}_s(\Delta_2 ; k).$$

Then, by suspension isomorphism $\widetilde{H}_{q+1}(S\Delta ; k)\equiv\widetilde{H}_q(\Delta : k)$ and using the homotopy equivalence (2), we obtain the isomorphisms :

$$\widetilde{H}_n(\Delta_1{*}\Delta_2 ; k)\equiv\widetilde{H}_n(S(|\Delta_1|\wedge|\Delta_2|) ; k)\equiv\widetilde{H}_{n-1}(\Delta_1\wedge\Delta_2 ; k)\equiv$$

$$\equiv\bigoplus_{r+s=n-1}\widetilde{H}_r(\Delta_1 ; k)\otimes\widetilde{H}_s(\Delta_2 ;k),\quad \text{q.e.d.}$$

Proposition 2. a) $\dim k[\Delta_1{*}\Delta_2]=\dim k[\Delta_1]+\dim k[\Delta_2]$, *where* $\dim k[\Delta]$ *denotes the* K r u l l *dimension of the* S t a n l e y - R e i s n e r *ring of a simplicial complex* Δ, *over a field* k

b) *if* $F=F_1{*}F_2$ *for* $F_i\in\Delta_i$, $i=1, 2$, *then* $\mathrm{Lk}_{\Delta_1*\Delta_2}F=\mathrm{Lk}_{\Delta_1}{*}F_1\mathrm{Lk}_{\Delta_2}F_2$, *and*

c) $\dim \mathrm{Lk}_{\Delta_1*\Delta_2}F=\dim \mathrm{Lk}_{\Delta_1}F_1+\dim \mathrm{Lk}_{\Delta_2}F_2+1$.

Proof : a) The simplexes of the join $\Delta_1{*}\Delta_2$ are $(a_{i_0},\ldots a_{i_n}, b_{j_0},\ldots b_{j_m})$ with $(a_{i_0},\ldots a_{i_n})\in\Delta_1$, $(b_{j_0},\ldots b_{j_m})\in\Delta_2$. By [8, Th 1.3, p. 63] we have

$$\dim k[\Delta_1{*}\Delta_2]=\dim(\Delta_1{*}\Delta_2)+1=\dim\Delta_1+\dim\Delta_2+1+1=\dim k[\Delta_1]+$$
$$+\dim k[\Delta_2]$$

b) Consider $G\in\mathrm{Lk}_{\Delta_1*\Delta_2}F$. This means that $G\in\Delta_1{*}\Delta_2$, $G\cap F=\emptyset$ and $G\cup F\in\Delta_1*\Delta_2$

Then we can write $G=G_1*G_2$ for $G_i\in\mathrm{Lk}_{\Delta_i}F_i$, $i=1, 2$, that is $G\in\mathrm{Lk}_{\Delta_1}F_1*$ $*\,\mathrm{Lk}_{\Delta_2}F_2$.

Conversely, if $G\in\mathrm{Lk}_{\Delta_1}F_1{*}\mathrm{Lk}_{\Delta_2}F_2$ then it easily results that $G\in\mathrm{Lk}_{\Delta_1*\Delta_2}F$, where $F=F_1{*}F_2$.

c) $\dim \mathrm{Lk}_{\Delta_1*\Delta_2}F=\dim(\mathrm{Lk}_{\Delta_1}F_1{*}\mathrm{Lk}_{\Delta_2}F_2)=\dim \mathrm{Lk}_{\Delta_1}F_1+\dim \mathrm{Lk}_{\Delta_2}F_2+$ $+1$.q.e.d.

Theorem 3. *If* Δ_1, Δ_2 *are* CM *simplicial complexes, then the join* $\Delta_1{*}\Delta_2$ *is* CM.

Proof. We use [8, 6.3, p. 44]. Denote $e=$depth $k[\Delta_1{*}\Delta_2]$, $d=\dim k$ $[\Delta_1{*}\Delta_2]$, $e_i=$depth $k[\Delta_i]$, $d_i=\dim k[\Delta_i]$, $i=1, 2$. Let F be a face of the join $\Delta_1{*}\Delta_2$. Then by Proposition 1 and 2 we obtain the following isomorphisms :

$$\widetilde{H}_{i-|F|-1}(\mathrm{Lk}_{\Delta_1*\Delta_2}F ; k)\equiv\widetilde{H}_{i-|F|-1}(\mathrm{Lk}_{\Delta_1}F_1{*}\mathrm{Lk}_{\Delta_2}F_2 ; k)\equiv$$

$$\equiv\bigoplus_{r+s=i-|F|-2}\widetilde{H}_r(\mathrm{Lk}_{\Delta_1}F_1 ;k)\otimes\widetilde{H}_s(\mathrm{Lk}_{\Delta_2}F_2 ; k).$$

If $H^i(k[\Delta_1{*}\Delta_2])\neq 0$, then by [4, lemma 1]. there exists $F\in\Delta_1{*}\Delta_2$ which satisfies $\widetilde{H}_{i-|F|-1}(\mathrm{Lk}_{\Delta_1*\Delta_2}F ; k)\neq 0$. So, for r_0, s_0 such that $r_0+s_0=i-|F|-2$, $\widetilde{H}_{r_0}(\mathrm{Lk}_{\Delta_1}F_1 ; k)\neq 0$, $\widetilde{H}_{s_0}(\mathrm{Lk}_{\Delta_2}F_2 ; k)\neq 0$. Let be $r_1=r_0+|F_1|+1$, $s_1=s_0+|F_2|+1$,

such that $r_1+s_1=i$. Then $H^{r_1}(k[\Delta_1])\neq 0$, $H^{s_1}(k[\Delta_2])\neq 0$. If $H^e(k[\Delta_1*\Delta_2])\neq 0$ then $H^{r_1}(k[\Delta_1])\neq 0$, $H^{s_1}(k[\Delta_2])\neq 0$ for $r_1+s_1=e$.

So, depth $k[\Delta_1]+$depth $k[\Delta_2]\leqslant r_1+s_1=$depth $k[\Delta_1*\Delta_2]$.

If Δ_i, $i=1, 2$ are CM then $d_i=e_i$ and it follows $\dim k[\Delta_1]+\dim k[\Delta_2]=$ $\dim k[\Delta_1*\Delta_2]\leqslant$depth $k[\Delta_1*\Delta_2]$. Because $\dim k[\Delta_1*\Delta_2]\geqslant$ depth $k[\Delta_1*$ $*\,\Delta_2]$ we obtain that $\Delta_1*\Delta_2$ is CM.

Theorem 4 : *If* Δ_i, $i=1, 2$ *are* 2-CM *complexes, then* $\Delta_1*\Delta_2$ *is* 2-CM *complex.*

Proof. We use [8, p. 82]. If Δ_i, $i=1, 2$ are CM then $\Delta_1*\Delta_2$ is CM. Let x be a vertex for $\Delta_1*\Delta_2$. If x is vertex in Δ_1 then $(\Delta_1*\Delta_2)/x=\Delta_1/x*\Delta_2$, where $\Delta_1/x=\{F\in\Delta_1, x\in F\}$. Because Δ_1/x is CM then $(\Delta_1*\Delta_2)/x$ is CM and $\dim(\Delta_1*\Delta_2)/x=\dim\Delta_1/x+\dim\Delta_2+1=\dim\Delta_1+\dim\Delta_2+1=\dim\Delta_1*\Delta_2$. If x is vertex for Δ_2 a similar proof is made. So, $\Delta_1*\Delta_2$ is 2-CM.

Theorem 5: *If* $\Delta_1*\Delta_2$ *is* ACM *complex then* Δ_1, Δ_2 *are* CM *complexes.*

Proof. : We use [1, §2]. If $\Delta_1*\Delta_2$ is ACM then for every $F\in\Delta_1*\Delta_2$, $F\neq\emptyset$, $\mathrm{Lk}_{\Delta_1*\Delta_2}F$ is CM. We consider $F_1=\emptyset$, $F_2\in\Delta_2$ such that $\dim F_2=$ $=\dim\Delta_2$. Then $\mathrm{Lk}_{\Delta_1}F_1=\Delta_1$, $\mathrm{Lk}_{\Delta_2}F_2=\emptyset$ and $\mathrm{Lk}_{\Delta_1*\Delta_2}F=\Delta_1$ is CM. For Δ_2 we consider $F_1\in\Delta_1$, $\dim F_1=\dim\Delta_1$ and $F_2=\emptyset$. Then $\mathrm{Lk}_{\Delta_1*\Delta_2}F=\Delta_2$ is CM.

Remark. If Δ_i, $i=1, 2$ are ACM complexes we cannot say that $\Delta_1*\Delta_2$ is ACM complex. For example, let F be a face for $\Delta_1*\Delta_2$, $F=F_1*F_2$. If $F_i\neq\emptyset$, $i=1, 2$ then $\mathrm{Lk}_{\Delta_1*\Delta_2}F$ is CM because $\mathrm{Lk}_{\Delta_i}F_i$ are CM. If $F_1=\emptyset$ then $\mathrm{Lk}_{\Delta_1*\Delta_2}F=\Delta_1*\mathrm{Lk}_{\Delta_2}F_2$ is not necessarilly CM if Δ_1 is not CM.

Theorem 6: *If* $\Delta_1*\Delta_2$ *is a Gorenstein complex then* Δ_i, $i=1, 2$ *are also Gorenstein.*

Proof : By [8, 5.1, p. 75] Δ is a G o r e n s t e i n complex over k iff for every $F\in\Gamma$,

$$\widetilde{H}_i(\mathrm{Lk}_\Gamma F ; k)=\begin{cases} k \text{ for } i=\dim \mathrm{Lk}_\Gamma F\\ 0 \text{ for } i\neq \dim \mathrm{Lk}_\Gamma F\end{cases}$$

where $\Gamma=$core $V=\{x\in V,\ \mathrm{st}_\Delta x\neq\Delta\}$ is a subcomplex in Δ generated by all vertices of Δ for which $\mathrm{st}_\Delta x\neq\Delta$, and $\mathrm{st}_\Delta x=\{F\in\Delta, F\cup\{x\}\in\Delta\}$. We can prove easily that $\Gamma_V=\Gamma_{V_1}*\Gamma_{V_2}$ for Δ_i complex on the vertex set V_i, $i=1, 2$.

Suppose that $\Delta_1*\Delta_2$ is a G o r e n s t e i n complex and let x be a vertex in V_1. Then x is in $V=V_1\cup V_2$. For every $F_i\in\Gamma_{V_i}$, $i=1, 2$, such that $\dim F_2=\dim\Gamma_{V_2}$,

$$\mathrm{Lk}_{\Gamma_V}F_1*F_2=\mathrm{Lk}_{\Gamma_{V_1}}F_1 \text{ and } \widetilde{H}_i(\mathrm{Lk}_{\Gamma_V}F_1 ; k)\equiv\widetilde{H}_i(\mathrm{Lk}_{\Gamma_V}F_1*F_2 ; k)\equiv$$

$$\equiv\begin{cases}k \text{ for } i=\dim \mathrm{Lk}_{\Gamma_{V_1}}F_1\\ 0 \text{ for } i\neq\dim \mathrm{Lk}_{\Gamma_{V_1}}F_1\end{cases}$$

Then Δ_1 is a G o r e n s t e i n complex. For Δ_2 we can perform a similar proof taking $F_i\in\Gamma_{V_i}$, $i=1, 2$, such that $\dim F_1=\dim\Gamma_{V_1}$.

Theorem 7. *If* $\Delta_1*\Delta_2$ *is a Buchsbaum complex then* Δ_i, $i=1, 2$ *are Buchsbaum.*

Proof. We use [8, 3.1, p. 84] Let be $F_i\in\Delta_i$, $i=1, 2$, $F_i\neq\emptyset$, $\dim F_2=$ $=\dim\Delta_2$. Then $\mathrm{Lk}_{\Delta_1*\Delta_2}F_1*F_2=\mathrm{Lk}_{\Delta_1}F_1$ and $\widetilde{H}_i(\mathrm{Lk}_{\Delta_1*\Delta_2}F_1*F_2 ; k)=$ $=\widetilde{H}_i(\mathrm{Lk}_{\Delta_1}F_1 ; k)=0$ if $i<\dim \mathrm{Lk}_{\Delta_1}F_1$. Then Δ_1 is B u c h s b a u m complex.

For Δ_2 an analogous proof is made with $F_i \in \Delta_i$, $i=1, 2$ and $\dim F_i = \dim \Delta_i$.

Remarks. 1. By [8, 4.2, p. 70] if Δ is CM complex then Δ is Buchsbaum complex.

2. If Δ_i, $i=1,2$ are Buchsbaum complexes, $\Delta_1 * \Delta_2$ may be not a Buchsbaum complex. If $F_i \neq \varnothing$, $i=1, 2$ for $F = F_1 * F_2 \neq \varnothing$ then $\tilde{H}_i(\mathrm{Lk}_{\Delta_1 * \Delta_2} F; k) = k$ if $i = \dim \mathrm{Lk}_{\Delta_1 * \Delta_2} F$ and it vanishes for $i \neq \dim \mathrm{Lk}_{\Delta_1 * \Delta_2} F$. If $F_1 = \varnothing$ then $\mathrm{Lk}_{\Delta_1 * \Delta_2} F = \Delta_1 * \mathrm{Lk}_{\Delta_2} F_2$ and $\tilde{H}_i(\mathrm{Lk}_{\Delta_1 * \Delta_2} F; k) = \bigoplus_{r+s=i-1} \tilde{H}_r(\Delta_1; k) \otimes \tilde{H}_s(\mathrm{Lk}_{\Delta_2} F_2 : k)$. Because we do not know the homological groups of Δ_1, it results that $\Delta_1 * \Delta_2$ may be not a Buchsbaum complex.

REFERENCES

1. Baclawski, K. – *Combinatorial Decompositions of Rings and Almost Cohen-Macaulay Complexes*, preprint.
2. Baclawski, K. and Garcia, A. – *Combinatorial Decompositions of a Class*. Advances in Mathematics 39, 155–184, 1981.
3. Maunder, C. R. F. – *Algebraic Topology*, Cambridge Univ. Press, 1980.
4. Pop, I. – *On the Stanley-Reisner ring of a suspension*. An. Şt. Univ. Iaşi (to appear).
5. Pop, I. and Sava, C. and He Zheng Xu – *Exerciții de topologie algebrică*, 1983, Univ. Iaşi.
6. Radu, N. – *Inele locale*. Ed. Academiei R.S.R., 1968.
7. Spanier, H. – *Algebraic Topologie*. Mc Graw Hill Book Company 1965.
8. Stanley, R. P. – *Combinatorics and Commutative Algebra*. Birkhäuser, 1983.

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ALMOST SEMI-INVARIANT SUBMANIFOLDS OF A COSYMPLECTIC MANIFOLD

BY

CORNELIA-LIVIA BEJAN

1. Introduction. Recently, A. Bejancu and N. Papaghiuc obtained in [1] fundamental results on the geometry of almost semi-invariant submanifolds of a Sasakian manifold. Also, A. Bejancu and D. Smaranda studied in [2] semi-invariant submanifolds in cosymplectic manifolds. The purpose of the present paper is to obtain some results for almost semi-invariant submanifolds of a cosymplectic manifold. In § 2 the fundamental formulas for the structures inherited by an almost semi-invariant submanifold are given. In § 3 we obtain necessary and sufficient conditions for the integrability of all distributions involved in the definition of an almost semi-invariant submanifold and in § 4 we study the geometry of leaves. The last section is dedicated to the totally umbilical and totally geodesic almost semi-invariant submanifolds.

Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with (F, ξ, η, g) as the almost contact metric structure, where F is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\tilde{M}$ satisfying:

$$F^2 = -I + \eta \otimes \xi\ ;\quad \eta(\xi) = 1\ ;\quad F\xi = 0\ ;\quad \eta F = 0\ ; \tag{1.1}$$

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\tilde{M}), \tag{1.2}$$

where I is the identity morphism on the tangent bundle. On any almost contact metric manifold $\tilde{M}$, we have $g(X, FY) + g(FX, Y) = 0$, $\forall X, Y \in \Gamma(T\tilde{M})$.

$\tilde{M}$ is a cosymplectic manifold if and only if we have:

$$\tilde{\nabla}_X F = 0, \quad \forall X \in \Gamma(T\tilde{M}), \tag{1.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. On any cosymplectic manifold $\tilde{M}$, we have $\tilde{\nabla}_X \xi = 0$, $\forall X \in \Gamma(T\tilde{M})$, (see [2]).

Now, let M be a Riemannian manifold isometrically immersed in a cosymplectic manifold $\tilde{M}$. Then we denote by the same symbol g the Riemannian metric on M and by ∇, $\nabla^\perp$ and h, respectively the Levi-Civita connection on M, the Riemannian connection induced by $\tilde{\nabla}$ on the normal bundle $TM^\perp$ and the second fundamental form of M. Also, for each vector bundle