For Δ_2 an analogous proof is made with $F_i \in \Delta_i$, $i=1, 2$ and $\dim F_i = \dim \Delta_i$.

Remarks. 1. By [8, 4.2, p. 70] if Δ is CM complex then Δ is Buchsbaum complex.

2. If Δ_i, $i=1,2$ are Buchsbaum complexes, $\Delta_1 * \Delta_2$ may be not a Buchsbaum complex. If $F_i \neq \emptyset$, $i=1, 2$ for $F = F_1 * F_2 = \emptyset$ then $\tilde{H}_i(\mathrm{Lk}_{\Delta_1 * \Delta_2} F; k) = k$ if $i = \dim \mathrm{Lk}_{\Delta_1 * \Delta_2} F$ and it vanishes for $i \neq \dim \mathrm{Lk}_{\Delta_1 * \Delta_2} F$. If $F_1 = \emptyset$ then $\mathrm{Lk}_{\Delta_1 * \Delta_2} F = \Delta_1 * \mathrm{Lk}_{\Delta_2} F_2$ and $\tilde{H}_i(\mathrm{Lk}_{\Delta_1 * \Delta_2} F; k) = \bigoplus_{r+s=i-1} \tilde{H}_r(\Delta_1; k) \otimes \tilde{H}_s(\mathrm{Lk}_{\Delta_2} F_2; k)$. Because we do not know the homological groups of Δ_1, it results that $\Delta_1 * \Delta_2$ may be not a Buchsbaum complex.

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ALMOST SEMI-INVARIANT SUBMANIFOLDS OF A COSYMPLECTIC MANIFOLD

BY

CORNELIA-LIVIA BEJAN

1. Introduction. Recently, A. Bejancu and N. Papaghiuc obtained in [1] fundamental results on the geometry of almost semi-invariant submanifolds of a Sasakian manifold. Also, A. Bejancu and D. Smaranda studied in [2] semi-invariant submanifolds in cosymplectic manifolds. The purpose of the present paper is to obtain some results for almost semi-invariant submanifolds of a cosymplectic manifold. In § 2 the fundamental formulas for the structures inherited by an almost semi-invariant submanifold are given. In § 3 we obtain necessary and sufficient conditions for the integrability of all distributions involved in the definition of an almost semi-invariant submanifold and in § 4 we study the geometry of leaves. The last section is dedicated to the totally umbilical and totally geodesic almost semi-invariant submanifolds.

Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with (F, ξ, η, g) as the almost contact metric structure, where F is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\tilde{M}$ satisfying:

$$F^2 = -I + \eta \otimes \xi\,; \quad \eta(\xi) = 1\,; \quad F\xi = 0\,; \quad \eta F = 0\,; \tag{1.1}$$

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\tilde{M}), \tag{1.2}$$

where I is the identity morphism on the tangent bundle. On any almost contact metric manifold $\tilde{M}$, we have $g(X, FY) + g(FX, Y) = 0$, $\forall X, Y \in \Gamma(T\tilde{M})$.

$\tilde{M}$ is a cosymplectic manifold if and only if we have:

$$\tilde{\nabla}_X F = 0, \quad \forall X \in \Gamma(T\tilde{M}), \tag{1.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. On any cosymplectic manifold $\tilde{M}$, we have $\tilde{\nabla}_X \xi = 0$, $\forall X \in \Gamma(T\tilde{M})$, (see [2]).

Now, let M be a Riemannian manifold isometrically immersed in a cosymplectic manifold $\tilde{M}$. Then we denote by the same symbol g the Riemannian metric on M and by ∇, $\nabla^\perp$ and h, respectively the Levi-Civita connection on M, the Riemannian connection induced by $\tilde{\nabla}$ on the normal bundle $TM^\perp$ and the second fundamental form of M. Also, for each vector bundle

K, we denote by $\Gamma(K)$ the module of its differentiable sections. Then we have the equations of Gauss and Weingarten given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and respectively} \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \forall X, Y \in \Gamma(TM),\ N \in \Gamma(TM^{\perp}). \tag{1.5}$$

where A_N is the fundamental tensor of Weingarten with respect to the normal section N satisfying:

$$g(h(X, Y), N) = g(A_N X, Y) = g(A_N Y, X), \quad \forall X, Y \in \Gamma(TM),\ N \in \Gamma(TM^{\perp}). \tag{1.6}$$

Suppose $\xi \in \Gamma(TM)$ and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^{\perp}$ the complementary orthogonal distribution to ξ in TM. For any $X \in \Gamma(TM)$, we have $g(FX, \xi) = 0$. Then we put:

$$FX = bX + cX, \tag{1.7}$$

where $bX \in \Gamma(\{\xi\}^{\perp})$ and $cX \in \Gamma(TM^{\perp})$. Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M.

Next, we suppose M endowed with the following differentiable distributions:

$$D = \{X \in \{\xi\}^{\perp} / cX = 0\}\ ;\ D^{\perp} = \{X \in \{\xi\}^{\perp} / bX = 0\},$$

and let $\tilde{D}$ be the complementary orthogonal distribution to $D \oplus D^{\perp} \oplus \{\xi\}$ in TM. It is easily seen that the distributions D and $D^{\perp}$ are orthogonal to each other. M endowed with the distributions D, $D^{\perp}$ and $\tilde{D}$ is called an almost semi-invariant submanifold of $\tilde{M}$, [1]. It is easily seen that the restriction F/D is an automorphism of D and $b/\tilde{D}$ is an automorphism of $\tilde{D}$.

The projection morphism of TM to the distributions D, $D^{\perp}$ and $\tilde{D}$ are denoted respectively by P, Q and L. Then for each $X \in \Gamma(TM)$ we have:

$$X = PX + QX + LX + \eta(X)\xi. \tag{1.8}$$

For any $N \in \Gamma(TM^{\perp})$ we put:

$$FN = tN + fN, \tag{1.9}$$

where tN is the tangential part of FN and fN is the normal part.

Since $FD^{\perp}$ and $c\tilde{D}$ are orthogonal distributions, denoting by ν the orthogonal complementary vector bundle to $FD^{\perp} \oplus c\tilde{D}$ in $TM^{\perp}$, we have:

$$TM^{\perp} = FD^{\perp} \oplus c\tilde{D} \oplus \nu. \tag{1.10}$$

We have (see [1]):

$$t(TM^{\perp}) = D^{\perp} \oplus \tilde{D}\ ;\ t(FD^{\perp}) = D^{\perp}\ ;\ t(c\tilde{D}) = \tilde{D}\ ;\ f(c\tilde{D}) = c\tilde{D}. \tag{1.11}$$

2. Basic formulas. All the following results are obtained taking M an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$.

Denote by $u(X, Y) = \nabla_X bY - A_{cY}X$, $\forall X, Y \in \Gamma(TM)$. Using (1.3), (1.4), (1.5) we obtain:

$$g(u(X, Y), \xi) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.1}$$

We shall omit the following proofs obtained by usual computations.

Lemma 2.1. *For any $X, Y \in \Gamma(TM)$ we have:*

$$P(u(X, Y)) = FP\nabla_X Y\ ; \tag{2.2}$$

$$Q(u(X, Y)) = Qth(X, Y)\ ; \tag{2.3}$$

$$L(u(X, Y)) = bL\nabla_X Y + Lth(X, Y)\ ; \tag{2.4}$$

$$h(X, bY) + \nabla^{\perp}_X cY = c\nabla_X Y + fh(X, Y). \tag{2.5}$$

Remark. Using (2.1) and (2.3), we get:

$$u(X, Y) - u(Y, X) \in \Gamma(D \oplus \tilde{D}), \quad \forall X, Y \in \Gamma(TM). \tag{2.6}$$

Lemma 2.2. *For any $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^{\perp})$, we have:*

$$\nabla_X \xi = 0\ ;\ h(X, \xi) = 0\ ; \tag{2.7}$$

$$A_N \xi = 0\ ;\ \nabla^{\perp}_{\xi} N = 0\ ; \tag{2.8}$$

$$\eta(\nabla_X Y) = 0\ ; \tag{2.9}$$

$$\eta(A_N X) = 0. \tag{2.10}$$

Lemma 2.3. *For any $X, Y \in \Gamma(D^{\perp})$, we have:*

$$A_{FX} Y = A_{FY} X. \tag{2.11}$$

Lemma 2.4. *Denote by K one of the distributions D, $D^{\perp}$ or $\tilde{D}$. Then we have $\nabla_{\xi} X \in \Gamma(K)$ for every $X \in \Gamma(K)$.*

Lemma 2.5. *Denote by K one of the distributions D, $D^{\perp}$ or $\tilde{D}$. Then* $[X, \xi] \in \Gamma(K)$ for every $X \in \Gamma(K)$.

3. Integrability of distributions on M. In this section we shall study the integrability of all distributions involved in the definition of an almost semi-invariant submanifold M of a cosymplectic manifold $\tilde{M}$.

Recall that any cosymplectic manifold $\tilde{M}$ is locally a Riemannian product $\tilde{M}_1 \times \tilde{\gamma}$, where $\tilde{M}_1$ is a Kähler manifold and $\tilde{\gamma}$ is a curve, [2].

From (2.9) we have:

$$[X, Y] \in \Gamma(\{\xi\}^{\perp}), \quad \forall X, Y \in \Gamma(TM). \tag{3.1}$$

Thus, we obtain that M is locally a Riemannian product $M_1 \times \gamma$, where M_1 is a submanifold of $\tilde{M}_1$ and γ is a portion of $\tilde{\gamma}$. It also follows that $D \oplus D^{\perp} \oplus \tilde{D}$ is a parallel distribution with respect to the Levi-Civita connection on M. Next, we shall study the integrability of the other distributions on M.

From lemma 2.5. it follows:

Proposition 3.1. *Denote by K one of the distributions D, $D^\perp$, $\tilde{D}$ or every direct sum of them. Then K is involutive if and only if $K\oplus\{\xi\}$ is involutive.*

Proposition 3.2. *The distribution $D^\perp$ is always integrable.*

Proof. Using (1.2), (1.3), (1.5), (2.11) we obtain:

$$g([X,Y],Z)=g(F[X,Y],FZ)=g(F\tilde{\nabla}_X Y,FZ)-g(F\tilde{\nabla}_Y X,FZ)=$$
$$=g(A_{FX}Y,FZ)-g(A_{FY}X,FZ)=0,\quad \forall X,Y\in\Gamma(D^\perp),\ Z\in\Gamma(D).$$

From (2.4) it follows $bL\nabla_X Y+Lth(X,Y)+LA_{FY}X=0$. Using (2.11) we get $bL[X,Y]=0$. Thus we have $L[X,Y]=0$, $\forall X,Y\in\Gamma(D^\perp)$, since b is an isomorphism on $\tilde{D}$. Applying (3.1) we complete the proof.

Proposition 3.3. *The distribution D is integrable if and only if we have:*

$$h(X,FY)=h(Y,FX),\quad \forall X,Y\in\Gamma(D). \tag{3.2}$$

Proof. From (1.3), (1.4) it follows: $h(X,FY)=c\nabla_X Y+fh(X,Y)$, $\forall X,Y\in\Gamma(D)$. Using (3.1) we get $c[X,Y]=0$, $\forall X,Y\in\Gamma(D)$ if and only if we have (3.2).

Proposition 3.4. *The distribution $D\oplus D^\perp$ is integrable if and only if we have (3.2) and:*

$$A_{FY}X+\nabla_Y FX\in\Gamma(D),\quad \forall X\in\Gamma(D),\ Y\in\Gamma(D^\perp). \tag{3.3}$$

Remark 3.1. Using (1.3), (1.4), (2.6), (2.11) and the result of the proposition 3.3, we have (3.2) and (3.3) if and only if we have:

$$u(X,Y)-u(Y,X)\in\Gamma(D),\quad \forall X,Y\in\Gamma(D\oplus D^\perp). \tag{3.4}$$

Proof. Using (2.4) and (3.1) for any $X,Y\in\Gamma(D\oplus D^\perp)$ and the fact that $b/\tilde{D}$ is an automorphism of $\tilde{D}$, we obtain the distribution $D\oplus D^\perp$ involutive if and only if (3.4) is satisfied.

Proposition 3.5. *The distribution $\tilde{D}$ is integrable if and only if we have:*

$$u(X,Y)-u(Y,X)\in\Gamma(\tilde{D}),\quad \forall X,Y\in\Gamma(\tilde{D})\ \text{and} \tag{3.5}$$

$$h(bX,Y)-h(X,bY)+\nabla^\perp_Y cX-\nabla^\perp_X cY\in\Gamma(c\tilde{D}),\quad \forall X,Y\in\Gamma(\tilde{D}). \tag{3.6}$$

Proof. Using (3.5) and applying F to (2.2), we obtain:

$$P[X,Y]=FP(u(Y,X)-u(X,Y)),\quad \forall X,Y\in\Gamma(TM). \tag{3.7}$$

By applying F to (2.5) we get:

$$Q[X,Y]=Qt(h(Y,bX)-h(X,bY)+\nabla^\perp_Y cX-\nabla^\perp_X cY),\quad \forall X,Y\in\Gamma(TM). \tag{3.8}$$

Using (3.1), (3.7) and (3.8), if (3.6) is satisfied, then $\tilde{D}$ is involutive.

Conversely, if $\tilde{D}$ is involutive, using (3.7) and the fact that F/D is an automorphism of D, we obtain $P(u(Y,X)-u(X,Y))=0$, $\forall X,Y\in\Gamma(\tilde{D})$. Thus, from (2.1), (2.6), (3.7), it follows (3.5). Using (2.5) for any $X,Y\in\Gamma(\tilde{D})$ we obtain:

$$c\nabla_X Y-fh(X,Y)=h(X,bY)-h(Y,bX)+\nabla^\perp_X cY-\nabla^\perp_Y cX,$$
$$\forall X,Y\in\Gamma(\tilde{D}).$$

Since $\tilde{D}$ is involutive, we find (3.6) and complete the proof.

Proposition 3.6. *The distribution $D^\perp\oplus\tilde{D}$ is integrable if and only if we have* (3.5) *and:*

$$\nabla_X bY-A_{cY}X+A_{FX}Y\in\Gamma(\tilde{D}),\quad \forall X\in\Gamma(D^\perp),\ Y\in\Gamma(\tilde{D}). \tag{3.9}$$

Remark 3.3. Using (2.11), we have (3.5) and (3.9) if and only if we have:

$$u(X,Y)-u(Y,X)\in\Gamma(\tilde{D}),\quad \forall X,Y\in\Gamma(D^\perp\oplus\tilde{D}). \tag{3.10}$$

Proof. From (2.2) we have:

$$FP[X,Y]=P(u(X,Y)-u(Y,X)),\quad \forall X,Y\in\Gamma(TM). \tag{3.11}$$

Using (2.6), (3.1), (3.11) and the fact that F/D is an automorphism of D, we have (3.10) if and only if the distribution $D^\perp\oplus\tilde{D}$ is involutive.

Proposition 3.7. *The distribution $D\oplus\tilde{D}$ is integrable if and only if we have:*

$$h(bX,Y)-h(X,bY)+\nabla^\perp_Y cX-\nabla^\perp_X cY\in\Gamma(c\tilde{D}),\quad \forall X,Y\in\Gamma(D\oplus\tilde{D}). \tag{3.12}$$

Proof. Using (1.8), (1.10), (3.1), we have $[X,Y]\in\Gamma(D\oplus\tilde{D})$, $\forall X,Y\in\Gamma(TM)$ if and only if:

$$c[X,Y]\in\Gamma(c\tilde{D}),\quad \forall X,Y\in\Gamma(TM). \tag{3.13}$$

From (1.3), (1.4), (1.5), (1.7) we have (3.13) if and only if (3.12) is satisfied.

Remark 3.4. The study of the Sasakian case shows that the distributions D, $\tilde{D}$, $D\oplus\tilde{D}$, $D\oplus D^\perp$, $\tilde{D}\otimes D^\perp$, $D\otimes D^\perp\oplus\tilde{D}$ are never integrable.

Proposition 3.8. *Similar with the Sasakian case, $\dim\tilde{D}\neq 1$.*

4. The geometry of leaves on an almost semi-invariant submanifold in a cosymplectic manifold.

Definition. *Let K, K_1, K_2 be one of the distributions D, $D^\perp$, $\tilde{D}$ or every direct sum of them. An almost semi-invariant submanifold of a cosymplectic manifold is called (K_1,K_2)-geodesic (resp. K-geodesic) if $h(X,Y)=0$ for every $X\in\Gamma(K_1)$, $Y\in\Gamma(K_2)$ (resp. $X,Y\in\Gamma(K)$).*

Using lemma 2.4. we obtain:

Proposition 4.1. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$ and denote by K one of the distributions D, $D^\perp$, $\tilde{D}$ or every direct sum of them. Then K is involutive and its leaves are totally geodesic immersed in $\tilde{M}$ if and only if $K\oplus\{\xi\}$ is involutive and its leaves are totally geodesic immersed in M.*

Proposition 4.2. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then we have:*

I) *The conditions i), ii) and iii) are equivalent:*

i) D is involutive and its leaves are totally geodesic immersed in M.

ii) $u(X, Y) \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp}\oplus\tilde{D})$.

iii) $h(X, Y) \in \Gamma(\nu)$ and $\nabla_X V \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X, Z \in \Gamma(D)$, $V \in \Gamma(\tilde{D})$.

II) *The conditions a), b), c) are equivalent:*

a) *$D^{\perp}\oplus\tilde{D}$ is involutive and its leaves are totally geodesic immersed in M.*

b) *$u(X, Y) \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X, Y \in \Gamma(D^{\perp}\oplus\tilde{D})$.*

c) *$h(X, Y) \in \Gamma(\nu)$ and $\nabla_X V \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X \in \Gamma(D^{\perp}\oplus\tilde{D})$, $Z \in \Gamma(D)$, $V \in \Gamma(\tilde{D})$.*

III) *Any leaf of the distribution $D^{\perp}$ is totally geodesic immersed in M if and only if we have $h(X, bY)+\nabla^{\perp}_X cY \in \Gamma(c\tilde{D}\oplus\nu)$, $\forall X \in \Gamma(D^{\perp})$, $Y \in \Gamma(D\oplus\tilde{D})$.*

IV) *The distribution $D\oplus\tilde{D}$ is involutive and its leaves are totally geodesic immersed in M if and only if we have $h(X, bY)+\nabla^{\perp}_X cY \in \Gamma(c\tilde{D}\oplus\nu)$, $\forall X, Y \in \Gamma(D\oplus\tilde{D})$.*

V) *$\tilde{D}$ is involutive and its leaves are totally geodesic immersed in M if and only if $u(X, Y) \in \Gamma(D^{\perp}\oplus\tilde{D})$ and $h(X, bY)+\nabla^{\perp}_X cY \in \Gamma(c\tilde{D}+\nu)$, $\forall X, Y \in \Gamma(\tilde{D})$.*

VI) *$D\oplus D^{\perp}$ is involutive and its leaves are totally geodesic immersed in M if and only if $u(X, Y) \in \Gamma(D^{\perp}\oplus\tilde{D})$ and $h(X, bY)+\nabla^{\perp}_X cY \in \Gamma(c\tilde{D}\oplus\nu)$, $\forall X \in \Gamma(D^{\perp}\oplus\tilde{D})$, $Y \in \Gamma(\tilde{D})$.*

Proof. Let K be one of the distributions D, $D^{\perp}$, $\tilde{D}$ or every direct sum of them. The distribution K is involutive and its leaves are totally geodesic immersed in M if and only if:

$$(4.1) \qquad \nabla_X Y \in \Gamma(K), \quad \forall X, Y \in \Gamma(K).$$

Let K' be the orthogonal distribution of K in $\{\xi\}^{\perp}$. Using (2.9) we have (4.1) if and only if $g(\nabla_X Y, Z)=0$, $\forall X, Y \in \Gamma(K)$, $Z \in \Gamma(K')$. From (1.2), (1.3), (1.4), (1.5), (1.7) we get:

$$(4.2) \quad g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = g(\nabla_X bY - A_{cY} X, bZ) + g(h(X, bY)+\nabla^{\perp}_X cY, cZ), \quad \forall X \in \Gamma(TM),\ Y \in \Gamma(K),\ Z \in \Gamma(K').$$

Next, we shall prove the assertion I). Using (1.6) we obtain that iii) implies ii). Take $K=D^{\perp}\oplus\tilde{D}$, $K'=D$ and $X \in \Gamma(D)$. Thus, from (4.2), (2.9) and (2.10), we get the equivalence of i) and ii). Using again (4.2), if we have ii), we obtain $\nabla_X Y \in \Gamma(D)$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp}\oplus\tilde{D})$. From ii) and (1.6) we get iii).

The assertion II) can be proved following step by step the proof of the assertion I) but taking $X \in \Gamma(D^{\perp}\oplus\tilde{D})$.

The assertion III) follows from (4.2) by taking $K=D\oplus\tilde{D}$, $K'=D^{\perp}$ and $X \in \Gamma(D^{\perp})$. Similarly, IV) follows from (4.2) by taking $X \in \Gamma(D\oplus\tilde{D})$.

Using (2.9), (2.10) and taking in (4.2) $K=\tilde{D}$, $K'=D\oplus D^{\perp}$, $X \in \Gamma(\tilde{D})$, the assertion V) holds. In the same way, but taking $X \in \Gamma(D\oplus D^{\perp})$, we obtain VI).

From proposition 3.3. and I) of proposition 4.2., we get:

Corollary 4.1. *Let M be a D-geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then we have:*

i) D is integrable;

ii) Any leaf of the distribution D is totally geodesic immersed in M if and only if $\nabla_X V \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X \in \Gamma(D)$, $V \in \Gamma(\tilde{D})$.

From the assertion II) of proposition 4.2., we obtain:

Corollary 4.2. *Let M be a $(D, D^{\perp}\oplus\tilde{D})$-geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then the distribution $D^{\perp}\oplus\tilde{D}$ is involutive and its leaves are totally geodesic immersed in M if and only if $\nabla_X V \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X \in \Gamma(D^{\perp}\oplus\tilde{D})$, $V \in \Gamma(\tilde{D})$.*

From proposition 3.2. and the assertion III) of proposition 4.2., by using (2.5), we get:

Corollary 4.3. *Let M be a $(D^{\perp}, D\oplus\tilde{D})$-geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then any leaf of the distribution $D^{\perp}$ is totally geodesic immersed in M if and only if we have $\nabla^{\perp}_X N \in \Gamma(c\tilde{D})$, $\forall X \in \Gamma(D^{\perp})$, $N \in \Gamma(c\tilde{D})$.*

From proposition 3.7. and the assertion IV) of proposition 4.2., by using (2.5), we obtain:

Corollary 4.4. *Let M be a $D\oplus\tilde{D}$-geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then we have:*

i) $D\oplus\tilde{D}$ is integrable if and only if $\nabla^{\perp}_Y cX-\nabla^{\perp}_X cY \in \Gamma(c\tilde{D})$, $\forall X, Y \in \Gamma(D\oplus\tilde{D})$;

ii) $D\oplus\tilde{D}$ is involutive and its leaves are totally geodesic immersed in M if and only if $\nabla^{\perp}_X cY \in \Gamma(c\tilde{D})$, $\forall X, Y \in \Gamma(D\oplus\tilde{D})$.

From proposition 4.1. and the assertions I) and II) of proposition 4.2., we get:

Corollary 4.5. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally the Riemannian product $M_1\times M_2$, where M_1 is the leaf of $D^{\perp}\oplus\tilde{D}\oplus\{\xi\}$ and M_2 is a leaf of D if and only if we have $h(X, Z) \in \Gamma(\nu)$ and $\nabla_X Y \in \Gamma(D^{\perp}\oplus\tilde{D})$, $\forall X \in \Gamma(\{\xi\}^{\perp})$, $Z \in \Gamma(D)$, $Y \in \Gamma(\tilde{D})$.*

From proposition 4.1. and the assertions III) and IV) of proposition 4.2., we get:

Corollary 4.6. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is the locally Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^\perp$ if and only if we have $h(X, bY) + \nabla_Y^\perp cY \in \Gamma(c\tilde{D} \oplus \nu)$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(D \oplus \tilde{D})$.*

From proposition 4.1. and the assertions V) and VI) of proposition 4.2., we get:

Corollary 4.7. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $\tilde{D}$ and M_2 is a leaf of $D \oplus D^\perp \oplus \{\xi\}$ if and only if we have $h(X, bY) + \nabla_X^\perp cY \in \Gamma(c\tilde{D} \oplus \nu)$ and $u(X, Y) \in \Gamma(D^\perp \oplus \tilde{D})$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(\tilde{D})$.*

5. **Totally umbilical and totally geodesic almost semi-invariant submanifolds.** In the same way with [2] we obtain:

Proposition 5.1. *Any totally umbilical almost semi-invariant submanifold M of a cosymplectic manifold $\tilde{M}$ is a totally geodesic submanifold.*

Thus it remains to study the totally geodesic submanifolds.

From corollary 4.6. it follows:

Proposition 5.2. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $c\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to the normal connection $\nabla^\perp$, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^\perp$.*

From corollary 4.7. it follows:

Proposition 5.3. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $c\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to the normal connection $\nabla^\perp$ and $u(X,Y) \in \Gamma(D^\perp \oplus \tilde{D})$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(\tilde{D})$, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $\tilde{D}$ and M_2 ia a leaf of $D \oplus D^\perp \oplus \{\xi\}$.*

From corollary 4.5. it follows

Proposition 5.4. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to ∇, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D^\perp \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of D.*

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INVARIANT FINSLER CONNECTIONS ON VECTOR BUNDLES

BY

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The automorphisms of the Finsler spaces were considered, in a local or global manner, by different authors, as for instance by E.T. Davies, H.C. Vang, K. Yano, V.L. Laptev, M. Matsumoto, R. Miron, Gy. Soós etc. For references see [1], [2].

The purpose of this paper is to present the problem of invariant linear connections of a Finsler structure, from a group theoretical point of view, by following the ideas of K. Nomizu [4] in the classical case of the homogeneous affine connected manifolds.

1. **N-homogeneous vector bundles and Finsler connections.** All the geometrical objects and maps considered below are C^∞-differentiable.

Let $\xi = (E, \pi, M)$ be a vector bundle over the manifold M. As in [3] a Finsler structure over ξ is defined by a non linear connection (N) on E, that is, by a differentiable distribution $u \to N_u \subset E_u$ such that at every $u \in E$, the tangent space E_u is given by $E_u = N_u \oplus E_u^V$, where $E_u^V = \ker \pi_u^T$ is the vertical space at u. The decomposition above produces a corresponding split for the contangent space E_u^*, as well as for the tensorial and exterior algebras over E.

By definition, a Finsler connection on E is a linear connection ∇ on E, such that the horizontal spaces N_u of the distribution (N) are parallel with respect to ∇, and the vertical spaces E_u^V are parallel too. It follows that a Finsler connection ∇ on E determines two covariant derivative laws, the so called ∇^H, respectively ∇^V-covariant derivations, namely the horizontal and the vertical part of ∇.

Now we describe the situation by using the formalism of the vector bundles. If we put $N = \bigcup_{u \in E} N_u$, $V = \bigcup_{u \in E} E_u^V$, $T(E) = \bigcup_{u \in E} E_u$, then by considering the tangent bundle $\xi^T = (T(E), \pi_E, E)$, the horizontal bundle $\varkappa = (N, \pi_N, E)$ and the vertical bundle $\eta = (V, \pi_V, E)$, we have the Whitney sum

$$\xi^T = \varkappa \oplus \eta. \tag{1}$$

Definition 1. *Let $\xi = (E, \pi, M)$ and ξ^T be as above. If there is a Lie group G which acts differentiable and simply transitive on E such that it preserves the distribution (N), then we call ξ^T a N-homogeneous vector bundle with the group G.*

Proposition 1. *If ξ^T is a N-homogeneous vector bundle, then the action induced by G preserves the vertical distribution (E_u^V) too.*