Corollary 4.6. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is the locally Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^\perp$ if and only if we have $h(X, bY) + \nabla_X^\perp cY \in \Gamma(c\tilde{D} \oplus \nu)$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(D \oplus \tilde{D})$.*

From proposition 4.1. and the assertions V) and VI) of proposition 4.2., we get:

Corollary 4.7. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $\tilde{D}$ and M_2 is a leaf of $D \oplus D^\perp \oplus \{\xi\}$ if and only if we have $h(X, bY) + \nabla_X^\perp cY \in \Gamma(c\tilde{D} \oplus \nu)$ and $u(X, Y) \in \Gamma(D^\perp \oplus \tilde{D})$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(\tilde{D})$.*

5. **Totally umbilical and totally geodesic almost semi-invariant submanifolds.** In the same way with [2] we obtain:

Proposition 5.1. *Any totally umbilical almost semi-invariant submanifold M of a cosymplectic manifold $\tilde{M}$ is a totally geodesic submanifold.*

Thus it remains to study the totally geodesic submanifolds.

From corollary 4.6. it follows:

Proposition 5.2. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $c\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to the normal connection $\nabla^\perp$, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of $D^\perp$.*

From corollary 4.7. it follows:

Proposition 5.3. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $c\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to the normal connection $\nabla^\perp$ and $u(X,Y) \in \Gamma(D^\perp \oplus \tilde{D})$, $\forall X \in \Gamma(\{\xi\}^\perp)$, $Y \in \Gamma(\tilde{D})$, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $\tilde{D}$ and M_2 ia a leaf of $D \oplus D^\perp \oplus \{\xi\}$.*

From corollary 4.5. it follows

Proposition 5.4. *Let M be a totally geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. If the distribution $\tilde{D}$ is $\{\xi\}^\perp$-parallel with respect to ∇, then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a leaf of $D^\perp \oplus \tilde{D} \oplus \{\xi\}$ and M_2 is a leaf of D.*

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Received 29.VI.1984

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, Matematică, 1985, f. 2

INVARIANT FINSLER CONNECTIONS ON VECTOR BUNDLES

BY

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The automorphisms of the Finsler spaces were considered, in a local or global manner, by different authors, as for instance by E.T. Davies, H.C. Vang, K. Yano, V.L. Laptev, M. Matsumoto, R. Miron, Gy. Soós etc. For references see [1], [2].

The purpose of this paper is to present the problem of invariant linear connections of a Finsler structure, from a group theoretical point of view, by following the ideas of K. Nomizu [4] in the classical case of the homogeneous affine connected manifolds.

1. **N-homogeneous vector bundles and Finsler connections.** All the geometrical objects and maps considered below are C^∞-differentiable.

Let $\xi = (E, \pi, M)$ be a vector bundle over the manifold M. As in [3] a Finsler structure over ξ is defined by a non linear connection (N) on E, that is, by a differentiable distribution $u \to N_u \subset E_u$, such that at every $u \in E$, the tangent space E_u is given by $E_u = N_u \oplus E_u^V$, where $E_u^V = \ker \pi_u^T$ is the vertical space at u. The decomposition above produces a corresponding split for the contangent space E_u^*, as well as for the tensorial and exterior algebras over E.

By definition, a Finsler connection on E is a linear connection ∇ on E, such that the horizontal spaces N_u of the distribution (N) are parallel with respect to ∇, and the vertical spaces E_u^V are parallel too. It follows that a Finsler connection ∇ on E determines two covariant derivative laws, the so called ∇^H, respectively ∇^V-covariant derivations, namely the horizontal and the vertical part of ∇.

Now we describe the situation by using the formalism of the vector bundles. If we put $N = \bigcup_{u \in E} N_u$, $V = \bigcup_{u \in E} E_u^V$, $T(E) = \bigcup_{u \in E} E_u$, then by considering the tangent bundle $\xi^T = (T(E), \pi_*, E)$, the horizontal bundle $\varkappa = (N, \pi_N, E)$ and the vertical bundle $\eta = (V, \pi_v, E)$, we have the Whitney sum

$$\xi^T = \varkappa \oplus \eta. \tag{1}$$

Definition 1. *Let $\xi = (E, \pi, M)$ and ξ^T be as above. If there is a Lie group G which acts differentiable and simply transitive on E such that it preserves the distribution (N), then we call ξ^T a N-homogeneous vector bundle with the group G.*

Proposition 1. *If ξ^T is a N-homogeneous vector bundle, then the action induced by G preserves the vertical distribution (E_u^V) too.*

In fact let τ_a, $(\tau_a)_*$ $(a\in G)$ be the actions of G on E respectively of the tangent group $T(G)$ on $T(E)$. For a fixed $a\in G$, τ_a is a diffeomorphism so that $(\tau_a)_*$ induces an isomorphism of the tangent spaces E_u, $E_{\tau_a(u)}$. Then, according to the formula (1), from $N_{\tau_a(u)}=(\tau_a)_*(N_u)$ it results $E^V_{\tau_a(u)}=(\tau_a)_* E^V_u$.

Proposition 2. *If ξ^T is a N-homogeneous vector bundle, then one has $E=G/K$ where $K=\{g\in G,\ \tau_g(u_0)=u_0\}$ is the isotropy group of a fixed element $u_0\in E$.*

The statement is a classical one, based only on the simple transitivity of G on E.

Remark. If a Lie group G acts transitively on a manifold M, the tangent group $T(G)$ doesn't act transitively on $T(M)$. That is the reason for we ask the (simple) transitivity of G on E, because in this case G induces also, via the projection π, a transitive action on the base space M.

Definition 2. *Let G be a Lie group acting on a manifold E. A linear connection ∇ on E is invariant by G (G-invariant) if*

$$(\tau_a)_*(\nabla_X Y)=\nabla_{(\tau_a)_* X}(\tau_a)_* Y, \qquad \forall a\in G. \tag{2}$$

In the following we consider the connections $\bar\nabla^H$, $\bar\nabla^V$ induced by ∇^H, ∇^V on $\varkappa$ respectively on η.

Proposition 3. *If ξ^T is a N-homogeneous vector bundle with the group G, and ∇ is an invariant connection on E, then the induced connections $\bar\nabla^H$, $\bar\nabla^V$ are G-invariant too.*

This follows by restricting (2) on $\varkappa$ respectively on η.

Remark. Conversely, by starting from two G invariant connections $\bar\nabla^H$, $\bar\nabla^V$ on $\varkappa$ respectively on η, one can construct the direct sum connection $\bar\nabla^H\oplus\bar\nabla^V$ which is a G-invariant connection on ξ^T. Of course this is different from the connection ∇ considered in Proposition 3.

2. **Reductive N-homogeneous vector bundles.** Now we proceed by following the ideas of K. Nomizu [4] adapted to the considered situation. Let $\xi=(E,\pi,M)$ be a vector bundle, such that its tangent bundle ξ^T is a N-homogeneous vector bundle with the group G. We denote by $\mathfrak{G}$ the Lie algebra of G and by $\mathfrak{N}$ the Lie algebra of the identity component K_e of the isotropy group K.

Definition 3. *The N-homogeneous vector beundle ξ^T is called reductive if there exists a subspace $\mathfrak{M}\subset\mathfrak{G}$ such that we have*

$$\mathfrak{G}=\mathfrak{M}\oplus\mathfrak{N}, \qquad AdK\mathfrak{M}\subseteq\mathfrak{M}, \tag{3}$$

Remark, The N-homogeneous vector bundle ξ^T is reductive if its base space E is reductive in the sens of [4]. The subspace $\mathfrak{M}$ is called AdK-invariant.

Remark. The vector bundle ξ^T is reductive if the action G on E is propre. In fact, in this case the isotropy group K is compact, so that E is reductive.

Proposition 4. *If a N-homogeneous vector bundle ξ^T with the group G is reductive, i.e. if (3) holds, then there exist the AdK-invariant subspaces $\mathfrak{M}_H$ and $\mathfrak{M}_V$ of $\mathfrak{M}$ such that*

$$\mathfrak{M}=\mathfrak{M}_H\oplus\mathfrak{M}_V. \tag{4}$$

Indeed, the tangent space E_{u_0} is identified with $\mathfrak{M}$ via a certain isomorphism and so (4) follows from (1). It is known that if $k\in K$, then $\tau_k U_0=u_0$ and τ_k induces on E_{u_0} the linear transformation $(\tau_k)_*=Adk|_{\mathfrak{M}}$. We will take into account that $(\tau_k)_*\mathfrak{M}_H\subseteq\mathfrak{M}_H$, $(\tau_k)_*\mathfrak{M}_V\subseteq M_V$ so that by denoting $Ad_H=Ad|_{\mathfrak{M}_H}$ $Ad_V=Ad|_{\mathfrak{M}_V}$ one has $Adk|_{\mathfrak{M}}=Ad_Hk+Ad_Vk$. Now, the fundamental existence theorem ([4], Theorem 8.1) of invariant linear connections leads to the following propositions.

Proposition 5. *If ξ^T is a reductive N homogeneous vector bundle (1) with the group G, then there exists a bijection between the set of the induced invariant connections on $\varkappa$ and the set of the bilinear functions $\bar\varphi:\mathfrak{M}_H\times\mathfrak{M}_H\to\mathfrak{M}_H$ which are Ad_HK invariant, that is*

$$(Ad_Hk)\bar\varphi(X,Y)=\bar\varphi(Ad_HkX,\ Ad_HkY), \qquad \forall k\in K. \tag{5}$$

Proposition 6. *If ξ^T is a reductive N-homogeneous vector bundle (1) with the group G, then there exists a bijection between the set of the induced invariant connections on η and the set of the bilinear functions $\bar\gamma:\mathfrak{M}_V\times\mathfrak{M}_V\to\mathfrak{M}_V$ which are Ad_VK invariant, that is*

$$(Ad_Vk)\bar\gamma(X,Y)=\bar\gamma(Ad_VkX,\ Ad_VkY) \qquad \forall k\in K. \tag{6}$$

The proofs of the two propositions are based essentially upon the considerations in [4], § 8.

3. **Torsion and curvature of the induced invariant connections.** As in [4], the torsion and curvature tensors of the induced invariant connections may be expressed by the bilinear functions $\bar\varphi$ respectively $\bar\gamma$. We have:

Proposition 7. *The torsion and the curvature tensor of the induced invariant connection $\bar\nabla^H$ in $u_0\in E$ are given by*

$$\bar T^H(X,Y)=\bar\varphi(X,Y)-\bar\varphi(X,X)-[X,Y]_H \qquad X,Y\in\mathfrak{M}_H \tag{7.1}$$

$$\bar R^H(X,Y)Z=\bar\varphi(X,\bar\varphi(Y,Z))-\bar\varphi(Y,\bar\varphi(X,Z))-\bar\varphi([X,Y]_H,Z)-[[X,Y]_H,Z] \tag{7.2}$$

where $[X,Y]_H$ denotes the $\mathfrak{M}_H$ — component of $[X,Y]$.

Proposition 8. *The torsion and the curvature tensor of the induced invariant connection $\bar\nabla^V$ in $u_0\in E$ are given by*

$$\bar T^V(X,Y)=\bar\gamma(X,Y)-\bar\gamma(Y,X)-[X,Y]_V \qquad X,Y\in\mathfrak{M}_V \tag{8.1}$$

$$\bar R^V(X,Y)Z=\bar\gamma(X,\bar\gamma(Y,Z))-\bar\gamma(Y,\bar\gamma(X,Z))-\bar\gamma([X,Y]_V,Z)-[[X,Y]_V,Z] \tag{8.2}$$

The proofs of the Propositions 7 and 8 are quite similar to these of [4], §9.

4. **Special connections and final remarks.** Some special Finsler connections can be introduced on a N-homogeneous vector bundle ξ^T with a Lie group G, in a similar way as the canonical affine connections

on a reductive homogeneous space [4]. §10. For the moment, we will call the invariant connection ∇ on E a canonical connection of first kind, if the induced connections $\bar{\nabla}^H$, $\bar{\nabla}^V$ are canonical of the first kind too. For such a connection we have

$$(9)\qquad \bar{T}^H=0,\quad \bar{T}^V=0,\quad \bar{\varphi}(X, Y)=\frac{1}{2}[X, Y]_H,\ \bar{\gamma}(X,Y)=\frac{1}{2}[X, Y]_V.$$

Of course ∇ is not determined by this conditions only. Analogously can be introduced the connection of second kind ∇, by the condition that both the invariant induced connections $\bar{\nabla}^H$, $\bar{\nabla}^V$ should be of the second kind too.

For this connection one has

$$(10)\qquad \bar{T}^H=-[X, Y]_H,\ \bar{T}^V=-[X, Y]_V,\ \bar{R}^H(X, Y)Z=-[[X, Y]_H, Z],$$
$$\bar{R}^V(X, Y)Z=-[[X, Y]_V, Z]$$

Other topics considered in [4], as group spaces, holonomy groups, invariant metrical connections, symmetrical homogeneous spaces, could be also transposed in the Finsler geometry.

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Received 14.IX.1984

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a. Matematică, 1985, f. 2

COHOMOLOGICAL EINSTEIN-KAEHLER SUBMANIFOLDS OF CP^m AND SPECTRAL GEOMETRY[1]

BY

DOMENICO PERRONE

§1. **Introduction.** In this paper all manifolds are assumed to be compact, connected and of complex dimension $n>1$.

Following [2] we say that a Kaehler manifold M is *cohomological Einstein* if $c_1=b\omega$ for some constant b, where ω is the cohomology class represented by the fundamental form and c_1 the first Chern class of M.

The spectrum of the Laplacian acting on the exterior p-forms for a Kaehler manifold M is denoted by ${}^{p}\mathrm{Spec}(M)$.

In [3] we obtain the following

Theorem A. *Let M, M' be two Kaehler manifolds. If M, M' satisfy one of the following conditions:*

(i) ${}^{p}\mathrm{Spec}(M)={}^{p}\mathrm{Spec}(M')$ for $p=0, 2$ and $\dim_C M=4$;

(ii) ${}^{p}\mathrm{Spec}(M)={}^{p}\mathrm{Spec}(M')$ for $p=0, 1, 2$.

then M is cohomologically Einsteinian if, and only if, M' is so.

Now let CP^m be the complex projective m-space with the Fubini-Study metric of constant holomorphic curvature 1. If M is a Kaehler manifold holomorphically isometrically immersed in CP^m, the second fundamental form of the immersion and its covariant derivative are denotod by σ and $\nabla'\sigma$ respectively.

In this short note we give the following

Theorem B. *Let M, M' be two Kaehler manifolds with ${}^{2}\mathrm{Spec}(M) = {}^{2}\mathrm{Spec}(M)'$. If M, M' are holomorphically isometrically immersed in CP^m for some m with $\int_M |\nabla'\sigma|^2 dv = \int_{M'} |\nabla'\sigma'|^2 dv'$ where dv denotes the volume element of M, then M is cohomologically Einsteinian (with $c_1=b\omega$) if and only if, M' is cohomologically Einsteinian (with $c_1'=b\omega'$).*

Remark. An n-dimensional algebraic submanifold M of CP^{n+q} is called a *complete intersection* if M is given as a intersection of q non singular hypersurfaces $M_1, \dots, M_q$ of CP^{n+q} in general position. A complete intersection is a Kaehler manifold with the induced Kaehler metric from CP^{n+q}. Let α_i denote the degree of M_i, $i=1, \dots, q$, then the first Chern class c_1 of M is given by ([1, P. 107])

$$c_1=(1/4\pi)(p+q+1-\Sigma\alpha_i)\omega.$$

Thus, M is cohomological Einstein.

[1] This work was carried out in the framework of a national plan of research financed by M.P.I. 40%.