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A NOTE ON WARD QUASIGROUPS

BY

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Inspired by M. Ward's paper [13] on postulating the inverse operations in a group (cf. [5], [14], [15]), J. M. Cardoso and C. P. da Silva defined in [2] the notion of Ward quasigroup (briefly W-quasigroup) as a quasigroup $(Q,\cdot)$ which satisfies the following conditions:

(W, 1) There is $0 \in Q$ such that $aa=0$ for all $a \in Q$.

(W, 2) $ab\cdot c=a\cdot(c\cdot 0b)$

for the element 0 postulated in (W, 1) and all $a, b, c, \in Q$.

In [2] it was proved that if $(Q, \cdot)$ is a W-quasigroup and $+$ the binary operation on Q defined by $x+y=x\cdot 0y$, then $(Q, +)$ is a group. Conversely, if $(Q, +)$ is a group and the binary operation on Q defined by $xy=x-y$, then $(Q, \cdot)$ is a W-quasigroup (cf. [13]).

In this paper we shall show that the axioms (W.1) and (W.2) in the definition of a W-quasigroup could be replaced by one axiom, i.e. by the law of right transitivity, [4]:

(W) $\qquad ac\cdot bc=ab$, for all $a, b, c\in Q$.

Since the law of right transitivity is the (13)-conjugate of the associativity (see [4]), it follows that $(Q, \cdot)$ is a W-quasigroup iff its (13)-conjugate is a group.

Similar result holds for the quasigroup which satisfies the left transitivity law (briefly W'-quasigroup)

(W') $\qquad ca\cdot cb=ab \quad$ for all $\quad a, b, c\in Q$,

(cf. [4], [12]).

Moreover, we shall prove that in the definition of W-quasigroup as a right transitive quasigroup, the condition of being quasigroup could be weakened (Propositions 2 and 4).

Proposition 1. *A quasigroup $(Q, \cdot)$ is a W-quasigroup iff the law o right transitivity (W) holds in $(Q, \cdot)$.*

Proof. If $(Q, \cdot)$ is a W-quasigroup, then $00=0$ and $00\cdot c=0\cdot(c\cdot 00)$ is valid for all $c\in Q$, which implies $c=c0$, i.e. 0 is the right unit. Using that fact and the axioms (W.1), (W.2), we show that $(Q, \cdot)$ satisfies the law of right transitivity. Indeed

$$ac\cdot bc=ac\cdot(bc\cdot 0)=ac\cdot[b\cdot(0\cdot 0c)]=(ac\cdot 0c)\cdot b=[a\cdot(0c\cdot 0c)]\cdot b=a0\cdot b=ab.$$

Now, let $(Q,\cdot)$ be a right transitive quasigroup and $bb=0$ for some $b\in Q$. If $a\in Q$, then there is such an element $c\in Q$ for which $b=ac$ and therefore $aa=ac\cdot ac=bb=0$, i.e. (W.1) holds. Further, (W) implies $ab=0\cdot ba$, $ab=ab\cdot 0$, where the second equality shows that 0 is the right unit.

Let be $b, c\in Q$ and $b=xy$ for some $x, y\in Q$. Then

$$(c\cdot 0b)\cdot b=[c\cdot(0\cdot xy)]\cdot xy=(c\cdot yx)\cdot(0\cdot yx)=c0=c,$$

and hence

$$a\cdot(c\cdot 0b)=ab\cdot[(c\cdot 0b)\cdot b]=ab\cdot c,$$

i.e. (W.2) is fulfilled.

If $(Q,\cdot)$ is a quasigroup, then we say that $(Q,+)$ is its (13)-conjugate or (13)-parastrophe, when for all $x, y, z\in Q$ $xy=z$ iff $z+y=x$. Evidently, $(Q,+)$ is a quasigroup the (13)-conjugate of which is $(Q,\cdot)$ and $xy+y=x$, $(x+y)\cdot y=x$ for all $x, y\in Q$.

Further, it is well known that associativity and right transitivity are (13)-conjugate identities (cf. [4], [12]), i.e. if $(Q,+)$ is an associative quasigroup (that is, a group), then its (13)-conjugate $(Q,\cdot)$ is a right transitive quasigroup and viceversa. Indeed, if $(Q,+)$ is a group then

$$(ac\cdot bc)+bc=ac=(ab+b)\cdot c=[ab+(bc+c)]\cdot c=[(ab+bc)+c]\cdot c=ab+bc$$

i.e. $ac\cdot bc=ab$; conversely, if $(Q,\cdot)$ satisfies (W), then

$$[a+(b+c)]\cdot(b+c)=a=(a+b)\cdot b=[((a+b)+c)\cdot c]\cdot[(b+c)\cdot c]=$$
$$=[(a+b)+c]\cdot(b+c),$$

i.e. $a+(b+c)=(a+b)+c$.

Therefore we have the following proposition.

Proposition 2. *Let $(Q,\cdot)$ be a quasigroup and $(Q,+)$ its (13)-conjugate. $(Q,\cdot)$ is a W-quasigroup iff $(Q,+)$ is a group.*

Consequently, if $(Q,+)$ is a group and the binary operation $\cdot$ on Q defined by the equality

$$xy=x-y,$$

then $(Q,\cdot)$ is a W-quasigroup and any W-quasigroup could be constructed in that way (cf. [2], [5], [13], [14]).

Furthermore, if $(Q,\cdot)$ is a W-quasigroup and $(Q,+)$ its (13)-conjugate, then for all $x, y\in Q$ we have

$$x+y=(x+y)\cdot 0=[(x+y)\cdot y]\cdot 0y=x\cdot 0y,$$

which expresses the group operation $+$ by the quasigroup operation. Conversely, if the binary operation $+$ is defined by

$$x+y=x\cdot 0y$$

for a W-quasigroup $(Q,\cdot)$, then $(Q,+)$ is the (13)-conjugate of $(Q,\cdot)$ because

$$(x\cdot 0y)\cdot y=x\cdot[y\cdot(0\cdot 0y)]=x\cdot(y\cdot y0)=x\cdot yy=x0=x$$

holds for all $x, y\in Q$ (cf. [2]).

Obviously, if $(Q,\cdot)$ is a W-quasigroup and $*$ a binary operation on Q defined by the equality

$$x*y=y\cdot 0x,$$

then $(Q,*)$ is a group too (cf. [13]), since $x*y=y+x$.

Remark. Let $(Q,\cdot)$ be a quasigroup and $(Q,\circ)$ its (12)-conjugate, i.e. $xy=z$ iff $y\circ x=z$ for all $x, y, z\in Q$. It is easy to prove that $(Q,\circ)$ is a W-quasigroup i.e. a right transitive quasigroup iff $(Q,\cdot)$ is a left transitive quasigroup i.e. W'-quasigroup which means that (W') holds in $(Q,\cdot)$. Moreover, if $(Q,\cdot)$ is a W'-quasigroup, then its (231)-conjugate $(Q,+)$ is a group, where $xy=z$ iff $z+x=y$ for all $x, y, z\in Q$. Therefore, if $(Q,+)$ is a group and the binary operation $\cdot$ on Q defined by the equality $xy=y-x$, then $(Q,\cdot)$ is a W'-quasigroup and any W'-quasigroup could be obtained in such a way.

Further, if $(Q,\cdot)$ is a W'-quasigroup and $(Q,+)$ is its (231)-conjugate then $x+y=y0\cdot x$ for all $x, y\in Q$, where 0 is the left unit in Q and $xx=0$ for all $x\in Q$. Conversely, if we define the binary operation $+$ in a W-quasigroup $(Q,\cdot)$ by the equality $x+y=y\cdot 0x$, then $(Q,+)$ is a group and the (231)-conjugate of $(Q,\cdot)$.

Now we shall show that the definition of W-quasigroup as a right transitive quasigroup could be stated in a weaker form. Namely, for a right transitive groupoid $(Q,\cdot)$ it is enough to require the right solvability (cf. [5], [8]) or the left cancellativity (cf. [15]) to be a W-quasigroup. Let us mention that H. Furstenberg in [5] introduced the name half-group for a right transitive groupoid.

Proposition 3. *If a right transitive groupoid $(Q,\cdot)$ satisfies one of the following two conditions*

1° For any two $a, b\in Q$, there is $x\in Q$ such that $ax=b$ (the right solvability).

2° For any three $a, b_1, b_2\in Q$, the equality $ab_1=ab_2$ implies $b_1=b_2$ (the left cancellativity).

then $(Q,\cdot)$ is a W-quasigroup.

Proof. a) Let $(Q,\cdot)$ be a right transitive groupoid in which 1° holds. Since $aa=ax\cdot ax=bb$, for $b=ax$, it follows that $(Q,\cdot)$ is a unipotent groupoid. Set $aa=0$ for all $a\in Q$. Then

$$0\cdot ab=ba,\ ab\cdot 0=ab, \tag{1}$$

which implies that 0 is the right unit, i.e.

$$a0=a, \tag{2}$$

because of 1°. Hence

$$0\cdot 0a=a. \tag{3}$$

Let be $ax=b$. Then

$$x=x0=xa\cdot 0a=(0\cdot ax)\cdot 0a=0b\cdot 0a,$$

i.e. the element $x=0b\cdot 0a$ is the unique solution of the equation $ax=b$.

Furthermore, since the equation $ya=b$ is equivalent to $ay=0b$ it follows that $ya=b$ has exactly one solution $y=(0\cdot 0b)\cdot 0a=b\cdot 0a$, i.e. $(Q, \cdot)$ is a quasigroup.

b) Let $(Q, \cdot)$ be a right transitive groupoid in which 2° holds. Then

$$ab\cdot c=(ab\cdot bb)\,(c\cdot bb)=ab\cdot (c\cdot bb)$$

and 2° implies $c=c\cdot bb$, which gives $c\cdot bb=c\cdot aa$ and therefore $bb=aa$ for all $a, b\in Q$. Hence, $(Q, \cdot)$ is unipotent and let be $aa=0$ for all $a\in Q$.

Since $c=c\cdot bb=c0$, it follows that 0 is the right unit. Obviously, the equations (1), (2), (3) are valid and $x=0b\cdot 0a$ is the solution of $ax=b$, since

$$a\cdot (0b\cdot 0a)=(0\cdot 0a)\cdot (0b\cdot 0a)=0\cdot 0b=b,$$

i.e. 1° holds which proves the proposition.

Evidently, W-quasigroup could be defined as a unipotent quasigroup which satisfies the identity $xy\cdot z=x\cdot [z\cdot (yy\cdot y)]$, (cf. [10]). The purpose of the following proposition is to give a characterization of W-quasigroups in the case of unipotent right transitive groupoids (cf. [5], [14], [13]). By the way, let be said that such groupoids were called pseudogroups by J. V. Whittaker in [15].

Proposition 4. *If a unipotent right transitive groupoid $(Q, \cdot)$ satisfies one of the following four conditions*

1° $b\cdot aa=b$ *for all* $a, b\in Q$ *;*

2° $ab=bb$ *implies* $a=b$ *for all* $a, b\in Q$ *;*

3° $aa\cdot a=bb\cdot b$ *implies* $a=b$ *for all* $a, b\in Q$ *;*

4° *For any* $a\in Q$, *there are* $x, y\in Q$ *such that* $a=xy$

then $(Q, \cdot)$ is a W-quasigroup.

Proof. Let be $aa=0$ for all $a\in Q$ and obviously the equalities (1) from the proof of the previous proposition are valid.

a) If 1° holds, then 0 is the right unit and therefore (2) and (3) hold which implies that $y=0b\cdot 0a$ is the solution of the equation $ax=b$, i.e. $(Q, \cdot)$ is right solvable. Hence, $(Q, \cdot)$ is a W-quasigroup by Proposition 3.

b) If 2° holds, then $ab=0$ implies $a=b$, for all $a, b\in Q$. Since

$$x\cdot x0=x0\cdot (x0\cdot 0)=x0\cdot (x0\cdot 00)=x0\cdot x0=0,$$

it follows $x0=x$ for all $x\in Q$, i.e. 1° is valid.

c) If 3° holds, then $0a=0b$ implies $a=b$ for all $a, b\in Q$. Since

$$0\cdot x0=00\cdot x0=0x,$$

the element 0 is the right unit, i.e. 1° is satisfied.

d) If 4° holds, then for any $a\in Q$ there are $x, y\in Q$ such that $a=xy$ and we have

$$a0=xy\cdot 0=xy\cdot yy=xy=a,$$

which proves the proposition.

Now, we remind of the special type of W-quasigroup, i.e. of the subtractive groupoid (cf. [1–3], [8–11]). It is defined as a groupoid $(Q, \cdot)$ in which the following two conditions are satisfied :

$$a\cdot bc=c\cdot ba \quad \text{for all} \quad a, b, c\in Q, \tag{S.1}$$

$$a\cdot bb=a \quad \text{for all} \quad a, b\in Q. \tag{S.2}$$

The condition (S.1) is called Abel-Grassmann's law (see [4]) or right modular law (cf. [6], [7]).

Any subtractive groupoid is a quasigroup, [2], and J. M. Morgado proved in [8] that for any groupoid $(Q, \cdot)$ the system of axioms (S. 1), (S.2) is equivalent to the system (W), (S) where (S) is the Sade's left „keys" law (cf. [4]), i.e.

$$b\cdot ba=a \quad \text{for all} \quad a, b\in Q. \tag{S}$$

It is easy to see that (S) is the (13)-conjugate of commutativity, [4], and now the following two propositions are obvious, because of the propositions 1 and 2.

Proposition 5. *A W-quasigroup $(Q, \cdot)$ is a subtractive quasigroup iff (S) holds in $(Q, \cdot)$.*

Proposition 6. *Let $(Q, \cdot)$ be a quasigroup and $(Q, +)$ its (13)-conjugate. $(Q, \cdot)$ is a subtractive quasigroup iff $(Q, +)$ is an abelian group.*

Remark. In a subtractive quasigroup $(Q, \cdot)$ it holds $ab=cb\cdot (cb\cdot ab)=$ $=cb\cdot ca$, i.e. it satisfies Schweitzer's law

$$cb\cdot ca=ab \quad \text{for all } a, b\in Q. \tag{S W}$$

(cf. [4]). Conversely, if a quasigroup $(Q, \cdot)$ satisfies the condition (S.W) then $ab\cdot ab=bb$, $ab\cdot aa=ab$ for all $a, b\in Q$, i.e. there is such an element $0\in Q$ for which $xx=0$ and $x0=x$ hold for all $x\in Q$. Hence

$$b\cdot ba=b0\cdot ba=a0=a,\quad ab=ac\cdot (ac\cdot ab)=ac\cdot bc,$$

i.e. $(Q, \cdot)$ is a subtractive quasigroup. Therefore, one could define a subtractive quasigroup as a quasigroup $(Q, \cdot)$ in which Schweitzer's law (S.W) holds (cf. [3], [14]).

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CERTAINS PROBLÈMES OUVERTS DE LA THÉORIE DES TREILLIS NON-COMMUTATIFS

PAR

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Comme on connait bien, la notion de treillis peut-être introduite par deux voies distinctes, mais logiquement équivalentes ; soit à l'aide d'un ensemble doté de deux opérations binaires qui satisfont certaines conditions concrétisées sous forme d'axiomes, soit à l'aide d'un ensemble doté d'une relation d'ordre, qui est soumise à la condition que tous sous-ensembles constitués de deux éléments possède une marge inférieure et une marge supérieure par rapport à cette relation.

Les treillis non-commutatifs ont été introduits pour la première fois par P. J o r d a n [5]. Ensuite, en [1], [3] et [4] ont été introduites d'autres classes de treillis non-commutatifs, celles-ci définies dans le langage des opérations binaires. On a constaté que les propriétés de ces classes de treillis non-commutatifs sont semblabes à celles rencontrées dans la théorie des treillis. En ces conditions-là on peut se poser la question si les treillis non-commutatifs ne puissent être définis aussi dans le langage des relations binaires. La réponse à cette question constitue un problème ouvert pres-que pour toutes les classes de treillis non-commutatifs connues jusqu'à présent. Il existe, pourtant, deux exceptions : la première est due à M. D. Gerhardts [4] et la deuxième est présentée pour la première fois en ce travail.

Considérons donc un ensemble M doté de deux opérations binaires $\wedge$ et $\vee$, qui pour tout $a, b, c \in M$ vérifie les axiomes :

$$(A) \qquad \begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c), \\ (a \vee b) \vee c = a \vee (b \vee c), \end{cases}$$

$$(B) \qquad \begin{cases} a \wedge (a \vee b) = a, \\ a \vee (a \wedge b) = a, \end{cases} \qquad (S) \qquad \begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b), \\ a \vee (b \vee c) = a \vee (c \vee b). \end{cases}$$

Le triplet $(M, \wedge, \vee)$, où M est un ensemble non-vide, et $\wedge$ et $\vee$ sont deux opérations binaires définies en M qui vérifient les axiomes (A), (B) et (S) s'appelera un *treillis non-commutatifs de type superplat*. Nous nous, proposons d'étudier les propriétés des treillis non-commutatifs de type superplat et de démontrer que ceux-ci peuvent être caractérisés aussi dans le langage des relations binaires.