ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" — IAŞI
Tomul XXXII, s. I a, Matematică, 1986 fasc. 2

SOME REMARKS ON NONLINEAR A-CONTRACTIONS

BY

M. KWAPISZ

1. In the numerically oriented literature, constructive generalizations of the well known Banach contraction mapping principle were formulated (see [3], [5], [9], [11] — [14], [17], [20] — [22]). These generalizations were obtained by the use of vector-valued or abstract valued distances instead of real valued metrics as well as by the change of the linear Lipschitz conditions by nonlinear ones.

On the other hand in the literature devoted to the existence and uniqueness problems for different kinds of functional equations, the results mentioned above were and are still used (see [1], [6], [10], [12]—[16], [18]—[19]). In this case the following question arises: is it necessary to use more sophisticated contraction type theorems than the classical Banach contraction principle? As Bessaga's result [2] shows, the answer to this question is no (at least in the most cases). Because of the nonconstructive character of Bessaga's theorem, this answer cannot be considered as satisfactory.

The purpose of the present paper is to exhibit the cases where, by a suitable (constructive) metrization (of the space in which the fixed point problem is being considered) the Banach contraction principle or its B o y d-W o n g's [4] generalization can be applied (cf. also a further generalization in [8]). For the simplicity of our considerations we will confine to the fixed point problems in the so-called R^m-metric spaces [1]. An abstract version of these is to be derived according to the general ideas included on [3] or [18].

2. Let R^m_+ be the subset of all nonnegative elements of R^m. R^m is the real m-dimensional coordinate space with the usual topology and the component-wise partial order, i.e. $(x_k) \subset R^m$, $x_k \to x \in R^m$ iff $\| x_k - x \| \to 0$, $\| \cdot \|$ is any norm in R^m, for $x, y \in R^m$, $x = (x^1, \dots, x^m)'$, $y = (y^1, \dots, y^m)'$, $x \leq y$ iff $x^i \leq y^i$, $i = 1, 2, \dots, m$, $x < y$ iff $x^i < y^i$. Here c' denotes the transpose of c.

A set X is called an R^m-metric space [20], [21], [22] if there is a mapping $d : X \times X \to R^m_+$ having the properties:

$$d(x, y) = 0 \Leftrightarrow x = y, \quad d(x, y) \leq d(x, z) + d(y, z)$$

for any $x, y, z \in X$. (the symmetry property is implied by these assumptions).

For $(x_n) \subset X$, $x_n \in X$, we put $x_n \to x$ iff $d(x_n, x) \to 0$, $n \to \infty$. The R^m-metric space X is called d-complete iff any d-Cauchy sequence in X converges to some $x\in X$. (A sequence $(x_n)\subset X$ is d-Cauchy iff for any $\varepsilon\in R^m_+$, $\varepsilon>0$ there exists $N\in\mathfrak{N}=\{0, 1, \dots\}$ such that

$$d(x_{n+p}, x_n)\leqslant\varepsilon$$

for any $n\geqslant N$, $p\in\mathfrak{N}$.

We say that a mapping $f: X\to X$ is an A-contraction iff there exists a nondecreasing mapping $A: R^m_+\to R^m_+$ such that:

(i) $(u_n)\subset R^m_+$, $u_n\searrow u\Rightarrow A(u_n)\searrow A(u)$,

(ii) for any $q\in R^m_+$ there exists the maximal solution $m(q)\in R^m_+$ of the equation

$$u=A(u)+q,$$

(iii) $m(0)=0$,

(iv) for any $x, y\in X$ $d(f(x), f(y))\leqslant A(d(x, y))$.

Theorem 1. *If (X, d) is a d-complete R^m-metric space and $f: X\to X$ is an A-contraction, then there exists in X a unique fixed point x^* of any f^k, $k\in N\setminus\{0\}$, $f^{k+1}:=f\circ f^k$, $f^1=f$. This fixed point is the limit of the successive approximations $(x_n)\subset X$, $x_n=f^n(x)$, $n\in\mathfrak{N}$, for any $x\in X$.*

Proof. The proof of the assertion of the theorem for $k=1$ can be found in [13]. Clearly a fixed point x^* of f is a fixed point of f^k, $k=1, 2, \dots$. We need only to prove that f^k has no other fixed points. Suppose that for some $k=1, 2, \dots$ there is another fixed point of f^k, say y^*. Then we get

$$d(x^*, y^*)=d(f^k(x^*), f^k(y^*))\leqslant A^k(d(x^*, y^*)).$$

Let m^* be the maximal solution of the equation

$$u=A(u)+d(x^*, y^*).$$

Clearly $m^*\geqslant d(x^*, y^*)$. Put $u^*_{i+1}=A(u^*_i)$, $i\in\mathfrak{N}$ $u^*_0=m^*$. It is easy to see that $u^*_i\searrow 0$ for $i\to+\infty$. By induction we get

$$d(x^*, y^*)\leqslant u^*_{ki}\leqslant u^*_i,\ i\in\mathfrak{N},$$

and this implies that $d(x^*, y^*)=0$ what we need to prove.

Remark 1. Put $A(u)=Pu$ for some $m\times m$ matrix P with nonnegative entries; then the conditions (ii) and (iii) hold iff the spectral radius $\sigma(P)$ of the matrix P is less than one [21]. In this case the A-contraction is called P-contraction [21].

Remark 2. If $X=X_1\times X_2\times\dots\times X_m$, where (X_i, d_i) are complete metric spaces, then (X, d) is a complete R^m-metric space

$$d(x, y)=(d^1(x, y), \dots, d^m(x, y))',\quad d^i(x, y)=d_i(x^i, y^i)$$

for $x=(x^1, \dots, x^m)'$, $y=(y^1, \dots, y^m)'$.

Fixed point equations in such spaces frequently arise in investigations of systems of functional equations [1], [6], [10], [19].

It is clear that on R^m-metric space (X, d) is metrisable in many ways, e.g. we can take for a metric in X

$$d'(x, y)=\max_{1\leqslant i\leqslant m}[d^i(x, y)],\quad d''(x, y)=\sum_{i=1}^{m}d^i(x, y),$$

and so on. Unfortunately, under such a metrization of X, the A-contraction f need not be a clasical contraction or a nonlinear contraction in the Boyd - Wong's [4] sense. However, there exists a metric on (the R^m-metric space) X, with respect to which f becomes a (classical) contraction. This is implied by the following.

Theorem 2. (B e s s a g a [2]). *If $f: X\to X$, X is a nonempty set, and any iteration f^k, $k=1, 2, \dots$, of f has a unique fixed point, then for each $\alpha\in(0, 1)$ there exists a complete metric d_α in X such that f is a contraction in X with respect to d_α.*

In view of Bessaga's theorem the result contained in Theorem 1 is equivalent to the Banach contraction principle in the following sense: if the assumptions of the Banach contraction principle are satisfied then for $m=1$ and $A(u)=\alpha u$, $\alpha\in(0,1)$, the assumptions of Theorem 1 hold. Conversely, if the assumptions of Theorem 1 hold, then there exists a metric on X with respect to which f is a contraction and the assertion of the Banach principle holds.

Unfortunately in the general case we cannot find d_α in a constructive manner. But there are cases where it can be done. The equivalence mentioned for the case of P-contractions was established by B o h l [3].

3. In this section, under some assumptions, we reduce Theorem 1 to the following result:

Theorem 3. (B o y d-W o n g [4]). *Suppose that (X, ρ) is a complete metric space and the mapping $f: X\to X$ satisfies*

$$\rho(f(x), f(y))\leqslant\varphi(\rho(x, y)),\quad x, y\in X,$$

where $\varphi: R_+\to R_+$ is an upper semicontinuous from the right function with $\varphi(t)<t$, $t>0$. Then f has a unique fixed point x^* and $f^n(x)\to x^*$ for each $x\in X$.

Now we can formulate

Theorem 4. *Suppose that the assumptions (iv) is satisfied, and there exist $b\in R^m_+$, $b>0$, as well as an upper semicontinuous from the right function $\varphi: R_+\to R_+$ with*

$$A(tb)\leqslant\varphi(t)\,b,\quad \varphi(t)<t,\ t>0.$$

Then there exists a real-valued metric ρ in X such that all assumptions of Theorem 3 hold.

Proof. According to the general idea formulated in [18] (see also [3]) we define ρ by taking

$$\rho(x, y)=\max_{1\leqslant i\leqslant m}\frac{d^i(x, y)}{b^i},$$

where $d(x, y)=(d^1(x, y), \dots, d^m(x, y))'$, $b=(b^1, \dots, b^m)'$. It is clear that ρ is a metric in X and that (X, ρ) is a complete metric space. By the definition of ρ, we have

$$d(x, y)\leqslant\rho(x, y)\,b,\quad x, y\in X.$$

Now, by of the monotonicity of A and condition (iv), we get

$$d(f(x), f(y))\leqslant A(d(x, y))\leqslant A(\rho(x, y)\,b)\leqslant\varphi(\rho(x, y))\cdot b,$$

hence

$$\rho(f(x), f(y))\leqslant\varphi(\rho(x, y)),\quad x, y\in X,$$

what we need to prove.

Remark 3. The assumptions of Theorem 4 are satisfied if there exists $b\in R^m_+$, $b>0$, such that

$$(^*)\qquad A(b)<b, \quad A(tb)\leqslant tA(b),\ t>0.$$

Indeed, if we put

$$\alpha := \max_{1\leqslant i\leqslant m} \frac{(A(b))^i}{b^i}<1,$$

where $A(b)=((A(b))^1, \ldots, (A(b))^m)^1$ we have

$$A(b)\leqslant \alpha b, \quad A(tb)\leqslant t\,.\,A(b)\leqslant \alpha tb,$$

which means that in this case $\varphi(t)=\alpha t$

Remark 4. If we assume that the condition (iv) is satisfied and there exist $v\in R^m_+$, $b>0$ $\varphi: R_+\to R_+$ and an upper semicontinuous from the right, function such that

$$b'A(u)\leqslant \varphi(b'u), \quad \varphi(t)<t, \quad \mu\in R^m_+, \quad t>0,$$

then there is a real valued metric ρ_1 on X such that all the assumptions of Theorem 3 hold. In fact, if we put (see [6], [10])

$$\rho_1(x, y)=b'd(x, y)=\sum_{i=1}^{m} b^i d^i(x, y),$$

one has

$$\rho_1(f(x), f(y))=b'd(f(x), f(y))\leqslant b'A\ (d(x, y))\leqslant \varphi(b'd(x, y))=\varphi(\rho_1(x, y)).$$

what we have to prove.

In the case $A(u)=Pu$, if $\sigma(P)<1$ then there exists $b\in R^m_+$, $b>0$, such that $P'n\leqslant \alpha b$, $\alpha\in(0,1)$, and we get

$$b'Pu=(Pu)'b=u'P'b\leqslant \alpha u'b=\alpha b'u, \quad u\in R^m_+,$$

i.e., $\varphi(t)=\alpha t$.

It seems that the result of Remark 4 is more natural than that of the paper [10], where the concavity assumption appeared. On the other hand the result of Theorem 4 seems to be more convenient in applications than that of Remark 4.

4. Now the question is how to find an element $b\in R^m_+$, $b>0$, for which (*) holds. There is no problem in the case $A(u)=Pu$ (see Remark 1) and the matrix P has positive entries. In this case according to Perron's theorem [7] there exists a positive eigenvalue r of P such that $\sigma(P)=r$ and the corresponding eigenvector v has positive coordinates. Now if $r=\sigma(P)<1$ then we can take $b=v$ and we get.

$$A(v)=Pv=rv<v, \quad A(tv)=P(tv)=tPv, \quad t>0,$$

that is, $\varphi(t)=\sigma(P)\,.\,t<t$.

(Note that, as a consequence of the continuous dependence of the spectrum of the matrix on its entries, the case of a nonnegative matrix P can be reduced to the mentioned above case). Moreover, the converse is valid too, that is, the element $b\in R^m_+$, $b>0$, fulfiling (*), exists if and only if $\sigma(P)<1$ (see [3] and [7]).

In the nonlinear case the vector b fulfiling the condition (*) can be taken as the maximal solution $u_e\in R^m_+$ of the equation

$$u=A(u)+e$$

for some fixed $e\in R^m_+$, $e>0$, if this solution exists and has the property: $A(tu_e)\leqslant tA(u_e)$ for $t>0$. Because it is rather hard to find the element u_e, assuming the continuity of A we produce the sequence

$$u_{i+1}=A(u_i)+e, \quad u_0=e, \quad i\in \mathfrak{N}.$$

It is clear that $u_i\nearrow u^*$, $u_i\leqslant u_e$ and $u^*=A(u^*)+e$. Consequently $u^*>A(u^*)$ and there exists an index i_0 such that $u_{i_0}>A(u_{i_0})$, and so we can take $b=u_{i_0}$ if $A(tu_{i_0})\leqslant tA(u_{i_0})$ for $t>0$.

In the special case $A(u)=Pu$ we get

$$u_i=\sum_{s=0}^{i} P^s e, \quad (I-P)\,u_i=(I-P^{i+1})e,$$

(here I is the identity matrix) and it is clear that $Pu_i<u_i$ if $\sigma(P)<1$ and the index i is sufficiently large.

5. We say that the mapping $f: X\to X$ is a generalized A-contraction [17] if there exists a monotonically continuous mapping $A: (R^m_+)^n\to R^m_+$ such that:

(i') for each $q\in R^m_+$ there exists the maximal solution $m(q)$ of the equation

$$u=A(u, u, u, 2u, u)+q,$$

(ii') $m(0)=0$,

(iii') for any $x, y\in X$

$$d(f(x), f(y))\leqslant A(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))).$$

In the paper [17] it is proved that: if (X, d) is a d-complete R^m-metric space and f is a generalized A-contraction then f has a unique fixed point x^* in X and $f^n(x)\to x^*$ for each $x\in X$.

Using the same metrization of the space X as in the proof of Theorem 4 it is easy to see that: if there exists $b\in R^m_+$, $b>0$, and a continuous non-decreasing function $\varphi: R^5_+\to R_+$ such that

$$A(t_1b, t_2b, t_3b, t_4b, t_5b)\leqslant \varphi(t_1, t_2, t_3, t_4, t_5)\, b$$

and

$$\varphi(t, t, t, a_1t, a_2t)<t, \quad t>0$$

for any t_i, $t\in R_+$, $i=1, 2, 3, 4, 5$ a_1, $a_2\in\{1, 2\}$ with $a_1+a_2=2$, then the mapping f is a generalized φ-contraction, i.e.

$$\rho(f(x), f(y))\leqslant \varphi(\rho(x, y), \rho(x, f(x)), \rho(y, f(y)), \rho(x, f(y)), \rho(y, f(x)))$$

and the existence and uniquenes results can be established for instance by the result of [8].

If we suppose that

$$A(u_1, \ldots, u_5)=\sum_{i=1}^{5} P_i\,.\,u_i,$$

where P_i are $m\times m$ nonnegative matrices such that there exist $b\in R^m_+$, $b>0$, and nonnegative constants α_i having the property $P_ib\leqslant \alpha_ib$, then we get

$$\varphi(t_1, \ldots, t_5)=\sum_{i=1}^{5}\alpha_i t_i$$

and a suitable existence and uniqueness result can be established.

Note that if we use the metrization introduced in the proof of Remak 4 then we need to assume that A, b and φ are such that

$$b'A(u_1, u_2, u_3, u_4, u_5) \leqslant \varphi(b'u_1, b'u_2, b'u_3, b'u_4, b'u_5)$$

for any $u_i \in R^m_+$.

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Received 22.VII.1985

Institute of Mathematics
University of Gdansk
80—952 Gdansk
Poland

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" — IAŞI
Tomul XXXII, s. I a, Matematică, 1986 fasc. 2

IDENTITY OF HARTE, TAYLOR AND COMMUTANT SPECTRA

BY

RAM U. VERMA

1. Introduction. Let $A=(A_1, \ldots, A_n)$ be a commuting system of bounded linear operators on a complex Hilbert space H. Let $L(H)$ denote the algebra of all bounded linear operators on H. Then the commutant $(A)'$ of (A) is a weakly closed subalgebra of the algebra $L(H)$ containing subalgebra (A) and the identity I, and the double commutant $(A)''$ of (A) is a weakly closed abelian subalgebra containing (A) and the identity I, where (A) is the closed subalgebra of $E(H)$. Note that (A) and $(A)''$ are commutative. We now describe the various notions of the joint spectra and the joint essential spectra for a commuting system $A=(A_1, \ldots, A_n) \subset$ $\subset L(H)$. Let $\sigma^H(A)$, the Harte spectrum of A, be defined by

$$\sigma^H(A) = \sigma^l(A) \cup \sigma^r(A),$$

where $\sigma^l(A)$ and $\sigma^r(A)$ denote, respectively, the left spectrum and the right spectrum of A (see [11], [7], [5]). Furthermore, let $\sigma^i(A)$ (resp. $\sigma^i_e(A)$), for $i=l, D, P, T$, denote the commutant (resp. commutant essential) spectrum, the Dash (resp. Dash essential) spectrum the polynomial (resp. polynomial essential) spectrum, the Taylor (resp. Taylor essential) spectrum of A.

Note that σ^l, σ^r and σ^H can be defined even without being A commuting system.

The commutant spectrum $\sigma^l(A)$ of A is defined as the set of all $z=$ $=(z_1, \ldots, z_n) \in \mathbf{C}^n$ (where $\mathbf{C}^n$ is the n-dimensional complex space) such that the set $\{A_i - z_i\}_{1 \leqslant i \leqslant n}$ is contained in a proper (two sided) ideal of $(A)'$.

The Dash spectrum $\sigma^D(A)$ of A is defined as the set of all $z=(z_1, \ldots, z_n) \in$ $\in \mathbf{C}^n$ iff, for all $B_1, \ldots, B_n$ in $(A)''$,

$$\Sigma_1^n B_i(A_i - z_i) \neq I.$$

The approximate point spectrum $\sigma_\pi(A)$ of A is defined as the set of all $z=(z_1, \ldots, z_n) \in \mathbf{C}^n$ such that for some sequence $\{x_k\}$ of unit vectors in H,

$$\|(A_i - z_i)\, x_k\| \to 0 \quad (k \to \infty), \quad i=1, \ldots, n.$$

Remark 1.1. Note that $z=(z_1, \ldots, z_n) \in \sigma_\pi(A)$ iff $0 \in \sigma^H(\sum_1^n (A_i - z_i)^*$ $(A_i - z_i))$.