and a suitable existence and uniqueness result can be established.

Note that if we use the metrization introduced in the proof of Remak 4 then we need to assume that A, b and φ are such that

$$b'A(u_1,\ u_2,\ u_3,\ u_4,\ u_5) \leqslant \varphi(b'u_1,\ b'u_2,\ b'u_3,\ b'u_4,\ b'u_5)$$

for any $u_i \in R^m_+$.

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Institute of Mathematics
University of Gdansk
80—952 Gdansk
Poland

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IDENTITY OF HARTE, TAYLOR AND COMMUTANT SPECTRA

BY

RAM U. VERMA

1. Introduction. Let $A=(A_1, \dots, A_n)$ be a commuting system of bounded linear operators on a complex Hilbert space H. Let $L(H)$ denote the algebra of all bounded linear operators on H. Then the commutant $(A)'$ of (A) is a weakly closed subalgebra of the algebra $L(H)$ containing subalgebra (A) and the identity I, and the double commutant $(A)''$ of (A) is a weakly closed abelian subalgebra containing (A) and the identity I, where (A) is the closed subalgebra of $E(H)$. Note that (A) and $(A)''$ are commutative. We now describe the various notions of the joint spectra and the joint essential spectra for a commuting system $A=(A_1, \dots, A_n) \subset$ $\subset L(H)$. Let $\sigma^H(A)$, the Harte spectrum of A, be defined by

$$\sigma^H(A)=\sigma^l(A)\cup\sigma^r(A),$$

where $\sigma^l(A)$ and $\sigma^r(A)$ denote, respectively, the left spectrum and the right spectrum of A (see [11], [7], [5]). Furthermore, let $\sigma^i(A)$ (resp. $\sigma^i_e(A)$), for $i=l, D, P, T$, denote the commutant (resp. commutant essential) spectrum, the Dash (resp. Dash essential) spectrum the polynomial (resp. polynomial essential) spectrum, the Taylor (resp. Taylor essential) spectrum of A.

Note that σ^l, σ^r and σ^H can be defined even without being A commuting system.

The commutant spectrum $\sigma^l(A)$ of A is defined as the set of all $z=$ $=(z_1, \dots, z_n) \in \mathbf{C}^n$ (where $\mathbf{C}^n$ is the n-dimensional complex space) such that the set $\{A_i - z_i\}_{1 \leq i \leq n}$ is contained in a proper (two sided) ideal of $(A)'$.

The Dash spectrum $\sigma^D(A)$ of A is defined as the set of all $z=(z_1, \dots, z_n) \in$ $\in \mathbf{C}^n$ iff, for all $B_1, \dots, B_n$ in $(A)''$,

$$\Sigma_1^n B_i(A_i - z_i) \neq I.$$

The approximate point spectrum $\sigma_\pi(A)$ of A is defined as the set of all $z=(z_1, \dots, z_n) \in \mathbf{C}^n$ such that for some sequence $\{x_k\}$ of unit vectors in H,

$$\|(A_i - z_i)\, x_k\| \to 0 \ (k \to \infty),\ i=1, \dots, n.$$

Remark 1.1. Note that $z=(z_1, \dots, z_n) \in \sigma_\pi(A)$ iff $0 \in \sigma^H(\sum_1^n (A_i - z_i)^*$ $(A_i - z_i))$.

The point spectrum $\sigma_p(A)$ of A is defined as the set of all $z=(z_1, \ldots, z_n) \in \mathbf{C}^n$ such that for some non-zero vector x in H, $(A_i - z_i)x = 0$, $i=1, \ldots, n$.

Remark 1.2. It is easy to see that $(z_1, \ldots, z_n) \in \sigma_p(A)$ iff

$$0 \in \sigma_p(\Sigma_1^n (A_i - z_i)^* (A_i - z_i)).$$

A point $z=(z_1, \ldots, z_n) \in \mathbf{C}^n$ is in the approximate compression spectrum $\sigma_\delta(A)$ of A iff there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\| (A_i - z_i)^* x_k \| \to 0 \ (k \to \infty), \ i=1, \ldots, n.$$

Remark 1.3. It is well-known that $\sigma^l(A) = \sigma_\pi(A)$ and $\sigma^r(A) = \sigma_\delta(A)$. Furthermore, we see that $\sigma_\pi(A) \cup \sigma_\delta(A) \subset \sigma^D(A)$ and $\sigma_p(A) \subset \sigma_\pi(A) \subset \sigma^D(A)$.

The polynomial spectrum $\sigma^P(A)$ of A is defined as follows:

$$\sigma^P(A) = \{z=(z_1, \ldots, z_n) \in \mathbf{C}^n : P(z) \in \sigma(P(A)) \text{ for all polynomials } P: \mathbf{C}^n \to \mathbf{C}\}.$$

The Taylor spectrum, $\sigma^T(A)$ of A is defined as follows:

$$\sigma^T(A) = \{z=(z_1, \ldots, z_n) \in \mathbf{C}^n : E(H, A-z) \text{ is not exact}\},$$

where $E(H, A-z)$ denotes the Koszul complex on H associated with $A-z$ (see [13]).

Remark 1.4. The above notions of the joint spectra lead to the inclusion relation

$$\sigma^H(A) \subset \sigma^T(A) \subset \sigma'(A) \cup \sigma^D(A) \subset \sigma^P(A).$$

All these notions of the joint spectra do coincide in the case of a single operator as well as in the case of operators on finite-dimensional spaces.

Next, we define the Harte essential spectrum $\sigma_e^H(A)$ of A by the relation

$$\sigma_e^H(A) = \sigma_e^l(A) \subset \sigma_e^r(A),$$

where $\sigma_e^l(A)$ and $\sigma_e^r(A)$ denote, respectively, the left essential spectrum and the right essential spectrum of A. For detail, see [6], [9]. Moreover, if $A=(A_1, \ldots, A_n)$ is essentially commuting, then $\sigma_e^l(A)$, $\sigma_e^r(A)$ and $\sigma_e^H(A)$ are non-empty compact subsets of $\mathbf{C}^n$.

Remark 1.5. Note that $(z_1, \ldots, z_n) \in \sigma_e^l(A)$ iff $0 \in \sigma_e^H(\Sigma_1^n (A_i - z_i)^* (A_i - z_i))$ and $(z_1, \ldots, z_n) \in \sigma_e^r(A)$ iff $0 \in \sigma_e^H(\Sigma_1^n (A_i - z_i)(A_i - z_i)^*)$.

Remark 1.6. We also note that $\sigma_e^T(A) \subset \sigma_e^l(A) \subset \sigma_e^D(A) \subset \sigma_e^P(A)$.

Recently Chō and Takaguchi [3] proved that the Taylor spectrum did coincide with the Dash spectrum for a commuting system of normal operators on a Hilbert space, whereas Curto [5] has established a connection between the Harte and Taylor spectra for a commuting pair. Equivalently, we write, for a commuting pair $A=(A_1, A_2)$ on a Hilbert space, that

$$\sigma^T(A) = \sigma^H(A) \cup \bigcup \{C : C \text{ is a bounded component of } \mathbf{C}^2 \setminus \sigma^H(A) \text{ and } C \cap \sigma^T(A) \neq \varnothing\}.$$

A similar expression for its essential analogue holds.

The aim of this paper is to prove that the Harte spectrum $\sigma^H(A)$ of A coincides with the Taylor spectrum $\sigma^T(A)$ of A when $A=(A_1, \ldots, A_n)$ is a commuting system of normal operators on a Hilbert space H. And, the Taylor spectrum coincides with the commutant spectrum $\sigma'(A)$, and which leads to the identity $\sigma^H(A) = \sigma^T(A) = \sigma'(A)$. Furthermore, we remark that the essential analogue of the spectral identity seems to be a reasonable conjecture. The obtained results complement the spectral identity of Chō and Takaguchi [3] for the Taylor and Dash spectra of a commuting system $A=(A_1, \ldots, A_n) \subset L(H)$ of normal operators.

We recall and prove some supporting results for our main results.

Lemma 1.1. (Dash [7]). *Let $A=(A_1, \ldots, A_n) \subset L(H)$ be a system of operators. Then*

(a) $\sigma^l(A) = \sigma_e^l(A) \cup \sigma_p(A)$, (b) $\sigma^r(A) = \sigma_e^r(A) \cup \sigma_p(A^*)^*$,

and hence

(c) $\sigma^H(A) = \sigma_e^H(A) \cup \sigma_p(A) \cup \sigma_p(A^*)^*$.

Lemma 1.2. (Chō and Takaguchi [3]). *Let $A=(A_1, \ldots, A_n) \subset L(H)$ be a commuting system. If $A=(A_1, \ldots, A_n)$ is non-singular, then $A_1^*A_1 + \ldots + A_n^*A_n$ and $A_1A_1^* + \ldots + A_nA_n^*$ are invertible.*

Lemma 1.3 (Chō and Takaguchi [3]). *Let $A=(A_1, \ldots, A_n) \subset L(H)$ be a commuting system. Then*

$$\sigma_\pi(A) \cup \sigma_\delta(A) \subset \sigma^T(A).$$

2. Main formulae

Theorem 2.1. *Let $A=(A_1, \ldots, A_n) \subset L(H)$ be a commuting normal operator system. Then*

$$\sigma'(A) = \sigma_\pi(A).$$

Proof. Let us first prove that $\sigma'(A) \subset \sigma_\pi(A)$. If $(0, \ldots, 0) \notin \sigma_\pi(A)$, then $\Sigma_1^n A_i^*A_i$ is invertible. It is evident that for each A_k, $k=1, \ldots, n$,

$$(\sum_{i=1}^n A_i^*A_i)^{-1} A_j^*A_k = A_k(\Sigma_1^n A_i^*A_i)^{-1} A_j^*.$$

It follows from this that $((\Sigma_1^n A_i^*A_i)^{-1} A_j^*) \in (A)'$ for each $j=1, \ldots, n$, and satisfies

$$(\Sigma_1^n A_i^*A_i)^{-1} A_1^*A_1 + \ldots + (\Sigma_1^n A_i^*A_i)^{-1} A_n^*A_n = I.$$

This in turn, implies that $(0, \ldots, 0) \notin \sigma'(A)$.

To prove the converse inclusion, suppose that $z=(z_1, \ldots, z_n) \in \sigma_\pi(A)$. Therefore, $(0, \ldots, 0) \in \sigma^H(\Sigma_1^n (A_i - z_i)^*(A_i - z_i))$ (by Remark 1.1 and Remark 1.3). This, further, implies that $(0, \ldots, 0) \in \sigma'(\Sigma_1^n (A_i - z_i)^* (A_i - z_i))$ (by Remark 1.4). This implies that $(z_1, \ldots, z_n) \in \sigma'(A)$. This completes the proof.

Consequently, in view of Lemma 1.3 and Theorem 2.1, we arrive at the following result.

Theorem 2.2. *Let $A=(A_1, \ldots, A_n) \subset L(H)$ be a commuting system of normal operators. Then*

$$\sigma^T(A)=\sigma^1(A)=\sigma_\pi(A)\,.$$

Corollary 2.3. *$A=(A_1,\dots,A_n)\subset L(H)$ is non-singular iff $\Sigma_1^n A_i^*A_i$ is invertible on H.*

Theorem 2.4. *Let $A=(A_1,\dots,A_n)\subset L(H)$ be a commuting system of normal operators. Then*

$$\sigma^H(A)=\sigma_\pi(A)\,.$$

Proof. To prove the theorem, it would be sufficient to establish that $\sigma^H(A)\subset\sigma_\pi(A)$. If $(0,\dots,0)\in\sigma^H(A)$, then $\Sigma_1^n A_i^*A_i$ is not invertible. Hence if $(0,\dots,0)\in\sigma^H(A)$, then $(0,\dots,0)\in\sigma^H(\Sigma_1^n A_i^*A_i)$. Thus, it follows from this that there exists a sequence $\{x_k\}$ of unit vectors in H such that $\|(\Sigma_1^n A_i^*A_i)\,x_k\|\to 0$ as $k\to\infty$. But,

$$\|A_1x_k\|^2+\dots+\|A_nx_k\|^2=\langle A_1x_k,A_1x_k\rangle+\dots+\langle A_nx_k,A_nx_k\rangle=$$
$$=\langle A_1^*A_1x_k,x_k\rangle+\dots+\langle A_n^*A_nx_k,x_k\rangle=\langle(A_1^*A_1+\dots+A_n^*A_n)\,x_k,x_k\rangle$$
$$\leqslant\|(A_1^*A_1+\dots+A_n^*A_n)\,x_k\|\,,$$

i.e.,
$$\sum_{i=1}^n\|A_ix_k\|^2\leqslant\|(\sum_{i=1}^n A_i^*A_i)\,x_k\|\,.$$

From this, it follows that $\sum_{i=1}^n\|A_ix_k\|^2\to 0$ as $k\to\infty$. Therefore, $\sum_{i=1}^n\|A_ix_k\|\to 0$ as $k\to\infty$ for each i, $1\leqslant i\leqslant n$. Hence, $(0,\dots,0)\in\sigma_\pi(A)$. This completes the proof.

Now a combination of Theorems 2.2 and 2.4 gives the following theorem.

Theorem 2.5. *Let $A=(A_1,\dots,A_n)\subset L(H)$ be a commuting system of normal operators. Then*

$$\sigma^H(A)=\sigma^T(A)=\sigma^1(A).$$

Remark 2.6. If we combine Theorem 2.5 and the identity $\sigma^T(A)=\sigma^D(A)$ [3, Theorem 2], we can write

$$\sigma^H(A)=\sigma^T(A)=\sigma^1(A)=\sigma^D(A).$$

Corollary 2.7. *Under the assumptions of Theorem 2.5, we find that*

$$\mathrm{conv}\,(\sigma^H(A))=\mathrm{conv}\,(\sigma^T(A))=\mathrm{conv}\,\sigma^1(A))=\overline{W(A)}\,,$$

where the joint numerical range $W(A)$ of $A=(A_1,\dots,A_n)\subset L(H)$ is defined by the expression

$$W(A)=\{(A_1x,x),\dots,(A_nx,x)):\|x\|=1\}\,,$$

and for a set W in the n-dimensional unitary space $\mathbf{C}^n$, conv (W) *and* $\overline{W}$ *denote, respectively, the convex hull and closure of W.*

Proof. Since conv $(\sigma^D(A))=\overline{W(A)}$ (by Dash [10]), the proof follows from Remark 2.6.

3. Concluding remarks. The question whether the identity

$$\sigma^H(A)=\sigma^T(A)=\sigma^1(A)=\sigma^D(A)=\sigma^P(A) \tag{3.1}$$

(for the polynomial spectrum $\sigma^P(A)$) holds is still open.

Under the assumptions of Theorem 2.5, the following essential analogue of the spectral identity of Remark 2.6 seems to be a reasonable conjecture.

$$\sigma_e^H(A)=\sigma_e^T(A)=\sigma_e^1(A)=\sigma_e^P(A)\,. \tag{3.2}$$

At least what we can see clearly now is, in view of Curto [5], that $\sigma_e^H(A)\subset\sigma_e^T(A)\subset\sigma_e^1(A)\subset\sigma_e^P(A)$. Moreover, for some special normal operator systems, the equality between the Harte essential spectrum and the Taylor essential spectrum can be attained ([5], Example 4.1).

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Department of Mathematics
University of Central Florida Orlando
FL 32816–6990 U.S.A.

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