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$\mathcal{C}$-BINARY RELATIONS AND PAIRWISE NORMALITY

BY

ANGEL GUTIERREZ and SALVADOR ROMAGUERA

1. **Introduction.** The known theorems of U r y s o h n [11], T i e t z e-U r y s o h n [12] and K a t e t o v-T o n g [1], [2], [10] are basic in the study of normal topological spaces. Bitopological versions of that theorems are obtained in K e l l y [3] and L a n e [4]; later, L a n e [5] has given a simplified proof of K a t e t o v-T o n g's theorem and in [7], theorem 2.5, is proved, by using techniques of $\mathcal{C}$-binary relations that (X, T) is normal if and only if given two real-valued functions $g \leq f$ on X, such that f is lower semi-continuous (l.s.c.) on X, g is upper semi-continuous (u.s.c.) on X and f is continuous on the closed $C \subset X$, there is a continuous function h on X such that $g \leq h \leq f$ and $h(x) = f(x)$ when $x \in C$.

The main objective of this paper is to obtain a bitopological extension of the previous theorem. We obtain as corollaries the results of K e l l y [3] and L a n e [4].

In the following, N will denote the set of positive integers.

2. **Preliminaries.** A bitopological space (X, T_1, T_2) is a set endowed with two topologies. A bitopological space is pairwise normal if [3] given a T_i-closed set A and a T_j-closed set B, being $A \cap B = \emptyset$, there are a T_j-open set U containing A and a T_i-open set V containing B with $U \cap V = \emptyset$ $(i \neq j,\ i, j = 1, 2)$.

Definition 1 [6]. *Let X be a set, let $\mathcal{P}(X)$ be the family of parts of X and let $\mathcal{C} \subset \mathcal{P}(X)$ a family stable for countable unions. A binary relation ρ on $\mathcal{P}(X)$ is a $\mathcal{C}$-binary relation if* [8]:

i) *Given two finite sets $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ such that $A \rho B$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a $C \in \mathcal{C}$ such that $A \rho C$ and $C \rho B$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

ii) *Given $D, E, F, G \in \mathcal{P}(X)$ such that $F \subset D$, $D \rho E$ and $E \subset G$, then $F \rho G$ and $D \subset E$.*

The following lemma will be useful later on.

Lemma 1. [8]. *Let ρ be a $\mathcal{C}$-binary relation on $\mathcal{P}(X)$. Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ families in $\mathcal{P}(X)$ and let $A, B \in \mathcal{P}(X)$ such that $\cup\{A_n, n \in N\} \subset A$, $B \subset \cap\{B_n, n \in N\}$, $A_n \rho B$ and $A \rho B_n$ for each $n \in N$. Then there is $C \in \mathcal{C}$ such that $A_n \rho C$ and $C \rho B_n$ for each $n \in N$.*

3. **Results.** In the following, when we are concerned to T_i and T_j, we suppose $i \neq j$ and $i, j = 1, 2$. Firstly, we prove a necessary and sufficient condition of pairwise normality in terms of F_σ and G_δ sets. Given a topological space (X, T), a set $F(G)$ in X is $F_\sigma(G_\delta)$ if it is countable union (intersection) of closed (open) subsets of X.

Theorem 1. *A bitopological space (X, T_1, T_2) is pairwise normal if and only if given two sets F, G in X such that F is T_i-F_σ, G is T_j-G_δ, T_i-cl$(F) \subset G$ and $F \subset T_j$-int (G), there is $H \subset X$ verifying $F \subset T_j$-int (H) and T_i-cl $(H) \subset G$.*

Proof. First we prove the necessity. Let (X, T_1, T_2) pairwise normal, let $F = \cup\{F_n,\ n \in N\}$ and $G = \cap\{G_n,\ n \in N\}$, where F_n is T_i-closed and G_n is T_j-open, for each $n \in N$. Let $\mathcal{C} = \mathcal{P}(X)$ and let the binary relation on $\mathcal{P}(X)$ $A \rho B$ if and only if T_i-cl $(A) \subset T_j$-int (B), for every, $A, B \in \mathcal{P}(X)$: ρ is a $\mathcal{C}$-binary relation.

Given $\mathcal{A} = \{F_n,\ n \in N\}$, $\mathcal{B} = \{G_n,\ n \in N\}$, $A = F$ and $B = G$, they verify the hypothesis of lemma 1, therefore there is an $H \in \mathcal{C}$ such that $F_n \rho H$ and $H \rho G_n$, for $n \in N$, then $F \subset T_j$-int (H) and T_i-cl $(H) \subset G$.

Now we prove the sufficiency. Given a T_i-closed set $A \subset X$ and a T_j-closed set $B \subset X$ such that $A \cap B = \varnothing$, we have that A is T_i-F_σ, $X - B$ is T_j-G_δ, T_i-cl $(A) \subset X - B$ and $A \subset T_j$-int $(X - B)$. Then there is $H \subset X$ such that $A \subset T_j$-int $(H$ and $B \subset X - (T_i$-cl $(H))$.

Let $D = \{d_n,\ n \in N\}$ a dense countable subset of $]0, 1[$.

Corollary 1.1. *A bitopological space (X, T_1, T_2) is pairwise normal if and only if given families $\{F(\alpha),\ \alpha \in D\}$ of T_i—F_σ subsets of X and $\{G(\alpha),\ \alpha \in D\}$ of T_j—G_δ subsets of X such that:*

i) T_i-cl $(F(\alpha)) \subset G(\alpha)$ and $F(\alpha) \subset T_j$-int $(G(\alpha))$ when $\alpha \in D$, and

ii) T_i-cl $(F(\alpha)) \subset F(\beta)$ and $G(\alpha) \subset T_j$-int $(G(\beta))$ when $\alpha < \beta$,

there is a family $\{H(\alpha),\ \alpha \in D\}$ of substes of X such that $F(\alpha) \subset T_j$-int $(H(\alpha)) \subset T_i$-cl $(H(\alpha)) \subset G(\alpha)$ when $\alpha \in D$, and T_i-cl $(H(\alpha)) \subset T_j$-int $(H(\beta))$ when $\alpha < \beta$.

Proof. The sufficiency is similar to theorem 1.

For the necessity, by theorem 1, there is $H(d_1) \subset X$ such that $F(d_1) \subset T_j$-int $(H(d_1))$ and T_i-cl $(H(d_1)) \subset G(d_1)$.

Let us suppose that there are, by induction, subsets $H(d_1)$, $H(d_2)$, ..., $H(d_n)$ in X verifying conditions i) and ii). Let

$$F = F(d_{n+1}) \cup \bigcup\{T_i\text{-cl}\,(H(d_r)) : d_r < d_{n+1},\ 1 \leqslant r \leqslant n\} \text{ and}$$

$$G = G(d_{n+1}) \cap \bigcup\{T_j\text{-int}\ (H(d_s)) : d_s > d_{n+1},\ 1 \leqslant s \leqslant n\ ;$$

by theorem 1, there is $H(d_{n+1}) \subset X$ such that $F \subset T_j$-int $(H(d_{n+1}))$ and T_i-cl $(H(d_{n+1})) \subset G$, therefore $F(d_{n+1}) \subset T_j$-int $(H(d_{n+1}))$ and T_i-cl $(H(d_{n+1})) \subset G(d_{n+1})$. Furthermore, T_i-cl $(H(\alpha)) \subset T_j$-int $(H(d_{n+1}))$ when $\alpha < d_{n+1}$, and T_i-cl $(H(d_{n+1})) \subset T_j$-int $(H(\beta))$ when $d_{n+1} < \beta$. Then, by induction, we obtain the family $\{H(\alpha),\ \alpha \in D\}$.

Let $f : X \to\ ]0,1[$. Let $\{F(\alpha),\ \alpha \in D\}$ a family such that $f^{-1}(]0, \alpha[) \subset F(\alpha) \subset f^{-1}(]0, \alpha])$ for each $\alpha \in D$. Then we have [9] that

$$f(x) = \sup\{\alpha \in D : x \notin F(\alpha)\} = \inf\{\alpha \in D : x \in F(\alpha)\}.$$

We say that f is determined by $\{F(\alpha),\ \alpha \in D\}$. Conversely, given an expansive family $\{F(\alpha),\ \alpha \in D\}$ of subsets of X, we may define $f : X \to [0, 1]$ by

$$f = \begin{cases} \sup\{\alpha \in D : x \notin F(\alpha)\}, & \text{when } \{\alpha \in D : x \notin F(\alpha)\} \neq \varnothing \\ 0 & \text{, when } \{\alpha \in D : x \notin F(\alpha)\} = \varnothing. \end{cases}$$

In particular, $f(x) > 0$ for every $x \in X$ when $\cap\{F(\alpha) : \alpha \in D\} = \varnothing$ and $f(x) < 1$ for every $x \in X$ when $\cup\{F(\alpha) : \alpha \in D\} = X$.

Let $f : (X, T_1, T_2) \to\ ]0, 1[$, determined by $\{F(\alpha),\ \alpha \in D\}$. It is known ([9], 3.1) that f is T_i-l.s.c. in X if and only if T_i-cl $(F(\alpha)) \subset F(\beta)$ when $\alpha < \beta$ and f is T_i-u.s.c. in X if and only if $F(\alpha) \subset T_i$-int $(F(\beta))$ when $\alpha < \beta$.

On the other hand, if g is determined by $\{G(\alpha),\ \alpha \in D\}$, then $g \leqslant f$ if and only if $F(\alpha) \subset G(\beta)$ when $\alpha \leqslant \beta$, if and only if $f^{-1}(]0, \alpha[) \subset g^{-1}(]0, \beta[)$ when $\alpha \leqslant \beta$, if and only if $f^{-1}(]0, \alpha]) \subset g^{-1}(]0, \beta])$ when $\alpha \leqslant \beta$.

Theorem 2. *A bitopological space (X, T_1, T_2) is pairwise normal if and only if given real-valued functions f, g on X, where $g \leqslant f$, f is T_i-l.s.c. on X, g is T_j-u.s.c. on X and f is T_j-u.s.c. on the T_i-closed subset $C \subset X$, there is a real-valued function h T_i-l.s.c. and T_j-u.s.c. on X such that $g \leqslant h \leqslant f$ and $h(x) = f(x)$ for every $x \in C$.*

Proof. For the necessity, it is enough by taking functions from X on $]0, 1[$. Given $f, g : X \to\ ]0,1[$, let $F(\alpha) = f^{-1}(]0, \alpha[)$ and $G(\alpha) = g^{-1}(]0, \alpha[) \cap (f^{-1}(]0, \alpha]) \cup (X - C))$, for each $\alpha \in D$.

$$F(\alpha) = \cup\{f^{-1}(]0, \beta]) : \beta < \alpha\} \text{ is } T_i\text{—}F_\sigma \text{ and}$$

$G(\alpha) = (\cap\{g^{-1}(]0, \beta[) : \beta > \alpha\}) \cap (\cap\{f^{-1}(]0, \beta[) \cap C : \beta > \alpha\} \cup (X - C))$ is T_j—G_δ for each $\alpha \in D$; furthermore, T_i-cl $(F(\alpha)) \subset f^{-1}(]0, \alpha]) \subset g^{-1}(]0, \alpha])$, then T_i-cl $(F(\alpha)) \subset F(\beta) \subset G(\alpha)$ when $\beta > \alpha$, and $g^{-1}(]0, \beta[) \cap (f^{-1}(]0, \beta[) \cup (X - C))$ is T_j-open in X, then $G(\alpha) \subset T_j$-int $(G(\beta))$ when $\beta > \alpha$ and $F(\alpha) \subset T_j$-int $(G(\alpha))$.

Therefore we can apply corollary 1.1 and we obtain a family $\{H(\alpha),\ \alpha \in D\}$; the function h determined by $\{H(\alpha),\ \alpha \in D\}$ is T_i-l.s.c. and T_j-u.s.c. on X and $g \leqslant h \leqslant f$.

Finally, as $f^{-1}(]0, \alpha[) \cap C \subset H(\alpha) \cap C \subset f^{-1}(]0, \alpha]) \cap C$, we have $h(x) = f(x)$ for every $x \in C$. This complete the proof of the necessity.

Given a T_i-closed set A and a T_j-closed set B with $A \cap B = \varnothing$, let us take $f(g)$ the characteristic function of $X - A$ (B). By taking $C = \varnothing$, we obtain the T_i-l.s.c. and T_j-u.s.c. function h such that $g \leqslant h \leqslant f$. Then $h^{-1}(]-\infty, 1/2[)$ and $h^{-1}(]1/2, +\infty[)$ are the desired disjoint neighbourhoods of A and B, respectively.

Corollary 2.1. (L a n e [4], 2.5) *In order for (X, T_1^0, T_2) to be pairwise normal, it is necessary and sufficient that for every pair of real-valued functions f and g defined on X such that f is T_i-l.s.c. and g is T_j-u.s.c., and $g \leqslant f$, there exist a T_i-l.s.c. and T_j-u.s.c. function h on X such that $g \leqslant h \leqslant f$.*

Proof. Take $C = \varnothing$ in theorem 2.

Corollary 2.2. *Let C be a T_1-closed and T_2-closed subset in the pairwise normal space (X, T_1, T_2). Let Φ be an upper (lower) bounded function on C T_i-l.s.c. and T_j-u.s.c. on C. Then, there is real-valued function h on X T_i-l.s.c. and T_j-u.s.c. on X such that $h(x) = \Phi(x)$ whenever $x \in C$.*

Proof. Let us suppose $\Phi \geqslant M$. We define the real-valued functions f and g on X in the following way:

$$g(x) = M \text{ for every } x \in X, \quad \text{and} \quad f(x) = \begin{cases} \Phi(x), & x \in C \\ M, & x \in X - C \end{cases}$$

By theorem 2, there is a T_i-l.s.c. and T_j-u.s.c. function h on X such that $h(x)=f(x)=\Phi(x)$ for every $x \in C$.

If we suppose $\Phi \leq M$, we apply the previous technique to the T_j-l.s.c. and T_i-u.s.c. function $-\Phi$ defined on the T_i-closed set C.

Corollary 2.3. (L a n e [4], 2.6) *Given a T_1-closed and T_2-closed subset C in a pairwise normal space (X, T_1, T_2), every bounded real-valued T_i-l.s.c. and T_j-u.s.c. function on C can be extended to a real-valued T_i-l.s.c. and T_j-u.s.c. function on X.*

The next example shows that the reciprocal of corollary 2.3 is not true.

Example. ([9], ex. 3.1) Let $X=\{a, b, c, d\}$, $T_1=\{\varnothing\} \cup \{A \subset X : a \in A\}$ and $T_2=\{\varnothing, X, \{a, b, c\}\}$. Obviously, $\{d\}$ is the only T_1-closed and T_2-closed subset of X. Any function $\Phi : \{d\} \to R$ can be extended according to conditions of corollary 2.3 (if $\Phi(d)=r$, we define $f(x)=r$ for each $x \in X$), but (X, T_1, T_2) is not pairwise normal.

Corollary 2.4. (K e l l y [3], 2.7) *A space (X, T_1, T_2) is pairwise normal if and only if given a T_i-closed set A and a T_j-closed set B with $A \cap B = \varnothing$, there exists a real-valued function h on X such that*

i) *$h(x)=0$ when $x \in A$, $h(x)=1$ when $x \in B$ and $0 \leq h \leq 1$,*

ii) *h is T_i-l.s.c. and T_j-u.s.c. on X.*

Proof. The sufficiency is obvious. For the necessity, we can apply the corollary 2.1 to the characteristic function of $X-A$, f, the characteristic function of B, g, and the subset $C=\varnothing$.

Corollary 2.5. *A space (X, T_1, T_2) is pairwise normal if and only if given a lower bounded T_i-l.s.c. real-valued function $f : X \to R$ which is T_j-u.s.c. on a T_j-closed set $C \subset X$, there is a real-valued function $h : X \to R$ T_i-l.s.c. and T_j-u.s.c. on X such that $h \leq f$ and $h(x)=f(x)$ for $x \in C$.*

Proof. Let $M=\inf\{f(x) : x \in X\}$; we define

$$g(x)=\begin{cases} M, & x \in X-C \\ f(x), & x \in C \end{cases}$$

If (X, T_1, T_2) is pairwise normal, the functions f and g verify the conditions of corollary 2.1, therefore there is a function h T_i-l.s.c. and T_j-u.s.c. on X such that $g \leq h \leq f$. Obviously, $h(x)=f(x)$ when $x \in C$.

Let A T_i-closed and let B T_j-closed subsets of X with $A \cap B=\varnothing$. Let us take f as the characteristic function of $X-A$; this function is lower bounded, T_i-l.s.c. on X and T_j-u.s.c. on A. Then there is h T_i-l.s.c. and T_j-u.s.c. on X such that $h \leq f$ and $h(x)=f(x)$ when $x \in A$. The sets $h^{-1}(]-\infty, 1/2[)$ and $h^{-1}(]1/2, +\infty[)$ are the desired disjoint neighbourhoods of A and B, therefore (X, T_1, T_2) is pairwise normal.

Remark. The technique of $\mathcal{C}$-binary relations used in this paper also is utilized in [6] in order to obtain an alternative proof for the Tong's theorem on perfect normality (T o n g [10], theorem 3) and a bitopological generalization of this theorem.

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Angel Gutiérrez
Dpto. de Matemáticas
E. U. del Profesorado de E.G.B.
Alcalde Reig, 8
46006 — VALENCIA
Spain

Salvador Romaguera
Matemáticas II
E. T. S. I. Caminos
Camino de Vera s.n.
46022 — VALENCIA
Spain