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A NOTE ON POINCARÉ'S INTEGRAL INEQUALITY

BY

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1. Introduction. One of the many fundamental mathematical discoveries of *H.* Poincaré is the following interesting integral inequality.

Lemma 1. (Poincaré [1, p. 284]). *Let* $Q=\{x=(x_1, \ldots, x_n)\in R^n : 0\leq x_i\leq \alpha\}$ *and let u be a real-valued function belonging to* $C^1(Q)$. *Then*

$$\int_Q u^2(x)\,dx\leq 1/\alpha^n\Big(\int_Q u(x)\,dx\Big)^2+(n/2)\,\alpha^2\int_Q |\nabla u(x)|^2 dx, \tag{1}$$

where

$$|\nabla u|^2=\left(\frac{\partial u}{\partial x_1}\right)^2+\cdots+\left(\frac{\partial u}{\partial x_n}\right)^2.$$

This result of Poincaré has received considerable attention in the literature, because of its fundamental role in establishing a priori bounds for solutions of various problems in the theory of partial differential equations (see, [1, 2]). In the present note we establish an interesting generalization of the inequality (1) and also make use of the inequality (1) to establish a new integral inequality involving functions and their derivatives.

2. Statement of Results. In this section we state our main results on integral inequalities involving two functions and their first order derivatives.

Theorem 1. *Let Q be as defined in Lemma 1 and let f, g be real-valued functions belonging to* $C^1(Q)$. *Then*

$$\begin{aligned}\int_Q f(x)\,g(x)\,dx\leq (1/\alpha^n)\Big(\int_Q f(x)\,dx\Big)\Big(\int_Q g(x)\,dx\Big)+\\ +(n/4)\,\alpha^2\int_Q [|\nabla f(x)|^2+|\nabla g(x)|^2]\,dx)\end{aligned} \tag{2}$$

where ∇ *is as defined in Lemma* 1.

Theorem 2. *Let Q, f, g be as defined in Theorem* 1. *Then*

$$\begin{aligned}\int_Q [|f(x)|\,|\nabla g(x)|+|g(x)|\,|\nabla f(x)|]\,dx\leq 1/(2\alpha^n)\Big[\Big(\int_Q f(x)\,dx\Big)^2+\\ +\Big(\int_Q g(x)\,dx\Big)^2\Big]+(1/2)\,(n/2\,\alpha^2+1)\int_Q [|\nabla f(x)|^2+|\nabla g(x)|^2]\,dx,\end{aligned} \tag{3}$$

where ∇ *is as defined in Lemma* 1.

Remark 1. We note that in the special case when $g(x)=f(x)$, the inequality established in Theorem 1 reduces to the well known Poincaré's inequality given in Lemma 1.

3. Proofs of Theorems 1 and 2. For any $x, y \in Q$ and $h \in C^1(Q)$ the following identity holds

$$h(x)-h(y)=\sum_{i=1}^{n}\int_{y_i}^{x_i}\frac{\partial}{\partial t_i}h(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\,dt_i. \tag{4}$$

Writing (4) for the functions f and g and then by multiplying the results and using the elementary inequalities $ab \leqslant (1/2)(a^2+b^2)$, $(a_1+\ldots+a_n)^2 \leqslant n(a_1^2+\ldots+a_n^2)$, ($a, b, a_i$ are reals) and Schwarz inequality (see, [3]) we obtain

$$\begin{aligned} f(x)\,g(x)+f(y)\,g(y)-f(x)\,g(y)-f(y)\,g(x) &\leqslant \\ \leqslant (n/2)\,\alpha \sum_{i=1}^{n}\int_0^{\alpha}\Bigg[\left\{\frac{\partial}{\partial t_i}f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}^2 &+ \\ +\left\{\frac{\partial}{\partial t_i}g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}^2\Bigg]dt_i. \end{aligned} \tag{5}$$

Integrating both sides of (5) with respect to $x_1, \ldots, x_n, y_1, \ldots, y_n$ we get

$$\begin{aligned} 2\alpha^n \int_Q f(x)\,g(x)\,dx-2\left(\int_Q f(x)\,dx\right)\left(\int_Q g(x)\,dx\right) &\leqslant \\ \leqslant (n/2)\,\alpha^{n+2}\int_Q [\,|\nabla f(x)|^2+|\nabla g(x)|^2]\,dx. \end{aligned} \tag{6}$$

The desired inequality (2) follows from the inequality (6). This completes the proof of Theorem 1.

By virtue of Schwarz' inequality, Poincaré's inequality (1) and the elementary inequality $a^{1/2}b^{1/2} \leqslant (1/2)(a+b)$, ($a, b \geqslant 0$ reals) we observe that

$$\begin{aligned} &\int_Q [\,|f(x)|\,|\nabla g(x)|+|g(x)|\,|\nabla f(x)|\,]\,dx \leqslant \\ &\leqslant \left[(1/\alpha^n)\left(\int_Q f(x)\,dx\right)^2+(n/2)\,\alpha^2\int_Q |\nabla f(x)|^2\,dx\right]^{1/2}\left[\int_Q |\nabla g(x)|^2\,dx\right]^{1/2}+ \\ &+\left[(1/\alpha^n)\left(\int_Q g(x)dx\right)^2+(n/2)\alpha^2\int_Q |\nabla g(x)|^2\,dx\right]^{1/2}\left[\int_Q |\nabla f(x)|^2\,dx\right]^{1/2} \leqslant \\ &\leqslant (1/2\alpha^n)\left[\left(\int_Q f(x)\,dx\right)^2+\left(\int_Q g(x)\,dx\right)^2\right]+1/2\,((n/2)\alpha^2+1)\int_Q [\,|\nabla f(x)|^2+ \\ &+|\nabla g(x)|^2]\,dx. \end{aligned}$$

This is the desired inequality in (3) and the proof of Theorem 2 is complete.

Remark 2. We note that in [2] the author has used a slightly different version of Poincaré's inequality in two and three dimensions to obtain Sobolev type integral inequalities. However the integral inequality established in Theorem 2 by employing the Poincaré's inequality is different from the inequalities established in [2].

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A LAGRANGIAN THEORY OF RELATIVITY*

BY

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To the memory of Professor G. Vrănceanu, on the occasion of the anniversary of fifty years since the first Romanian non-holonomic unitary theory has been realised.

The geometry based on the notion of lagrangian has been studied by many mathematicians, in order to obtain an adequate framework for the variational calculus. Particularly, the Finsler geometry is founded on this idea.

The term of Lagrange space was introduced by J. Kern [23]. It means a pair $L^n=(M, \mathcal{L})$, where M is a differentiable manifold and $\mathcal{L}$ a general lagrangian which provides a non-degenerate fundamental tensor field. A Finsler space is a Lagrange space, but the converse is not true because the lagrangian $\mathcal{L}$ does not satisfy any homogenity property.

The dependence of a lagrangian $\mathcal{L}(x, y)$ on both the point x and direction y offers the possibility to use it to describe the anisotropic properties of the physical space.

Illustrating examples are the Finslerian models of the relativity given by R. Ingarten [19], J. I. Horvath [13], Y. Takano [59, 60], M. Matsumoto [31], G. S. Asanov [2], H. Ishikawa [20, 21], S. Ikeda [17, 18], H. Rund [49, 50, 51] and others.

For these models the following problems are important:

— An unitary description of the elementary particles.

— To remove the difficulties created by the divergence to Einstein's equations.

— The modification of the structure of the space-time by dependence of the metric on the direction.

Some of these problems were not satisfactory solved.

In the Finslerian theory of relativity the equations of Einstein which have been proposed untill now are based upon one of the three curvature tensors of the space. An open problem is to know which is the most convenient curvature tensor to be used in Einstein's equations. To solve it, an unitary principle in establishing the equations of Einstein in these spaces must be adopted.

But all these problems are also interesting in the general case of the Lagrange spaces, a field which was not yet studied.

In this work we are dealing with the following problems:

1°. The introduction of the notion of the generalized Lagrange space and the elucidation of some important aspects of the geometry of these spaces.

* The second part will appear in the following number.