3. **Proofs of Theorems 1 and 2.** For any $x, y \in Q$ and $h \in C^1(Q)$ the following identity holds

$$h(x)-h(y)=\sum_{i=1}^{n}\int_{y_i}^{x_i}\frac{\partial}{\partial t_i}h(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n)\,\mathrm{d}t_i. \tag{4}$$

Writing (4) for the functions f and g and then by multiplying the results and using the elementary inequalities $ab \leqslant (1/2)(a^2+b^2)$, $(a_1+\ldots+a_n)^2 \leqslant n(a_1^2+\ldots+a_n^2)$, ($a, b, a_i$ are reals) and Schwarz inequality (see, [3]) we obtain

$$\begin{aligned} &f(x)\,g(x)+f(y)\,g(y)-f(x)\,g(y)-f(y)\,g(x) \leqslant \\ &\leqslant (n/2)\,\alpha\sum_{i=1}^{n}\int_{0}^{\alpha}\Big[\Big\{\frac{\partial}{\partial t_i}f(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n)\Big\}^2+ \\ &+\Big\{\frac{\partial}{\partial t_i}g(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n)\Big\}^2\Big]\mathrm{d}t_i. \end{aligned} \tag{5}$$

Integrating both sides of (5) with respect to $x_1, \ldots, x_n, y_1, \ldots, y_n$ we get

$$\begin{aligned} &2\alpha^n\int_Q f(x)\,g(x)\,\mathrm{d}x-2\Big(\int_Q f(x)\,\mathrm{d}x\Big)\Big(\int_Q g(x)\,\mathrm{d}x\Big) \leqslant \\ &\leqslant (n/2)\,\alpha^{n+2}\int_Q [\,|\nabla f(x)|^2+|\nabla g(x)|^2]\,\mathrm{d}x. \end{aligned} \tag{6}$$

The desired inequality (2) follows from the inequality (6). This completes the proof of Theorem 1.

By virtue of Schwarz' inequality, Poincaré's inequality (1) and the elementary inequality $a^{1/2}\,b^{1/2} \leqslant (1/2)(a+b)$, ($a, b \geqslant 0$ reals) we observe that

$$\begin{aligned} &\int_Q [\,|f(x)|\,|\nabla g(x)|+|g(x)|\,|\nabla f(x)|\,]\,\mathrm{d}x \leqslant \\ &\leqslant [(1/\alpha^n)\Big(\int_Q f(x)\,\mathrm{d}x\Big)^2+(n/2)\,\alpha^2\int_Q |\nabla f(x)|^2\,\mathrm{d}x]^{1/2}[\int_Q |\nabla g(x)|^2\,\mathrm{d}x]^{1/2}+ \\ &+[(1/\alpha^n)\Big(\int_Q g(x)\mathrm{d}x\Big)^2+(n/2)\alpha^2\int_Q |\nabla g(x)|^2\,\mathrm{d}x]^{1/2}[\int_Q |\nabla f(x)|^2\,\mathrm{d}x]^{1/2} \leqslant \\ &\leqslant (1/2\alpha^n)[\Big(\int_Q f(x)\,\mathrm{d}x\Big)^2+\Big(\int_Q g(x)\,\mathrm{d}x\Big)^2]+1/2\,((n/2)\alpha^2+1)\int_Q [\,|\nabla f(x)|^2+ \\ &+|\nabla g(x)|^2]\,\mathrm{d}x. \end{aligned}$$

This is the desired inequality in (3) and the proof of Theorem 2 is complete.

Remark 2. We note that in [2] the author has used a slightly different version of Poincaré's inequality in two and three dimensions to obtain Sobolev type integral inequalities. However the integral inequality established in Theorem 2 by employing the Poincaré's inequality is different from the inequalities established in [2].

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A LAGRANGIAN THEORY OF RELATIVITY*

BY

RADU MIRON

To the memory of Professor G. Vrănceanu, on the occasion of the anniversary of fifty years since the first Riemannian non-holonomic unitary theory has been realised.

The geometry based on the notion of lagrangian has been studied by many mathematicians, in order to obtain an adequate framework for the variational calculus. Particularly, the Finsler geometry is founded on this idea.

The term of Lagrange space was introduced by J. K e r n [23]. It means a pair $L^n=(M, \mathcal{L})$, where M is a differentiable manifold and $\mathcal{L}$ a general lagrangian which provides a non-degenerate fundamental tensor field. A Finsler space is a Lagrange space, but the converse is not true because the lagrangian $\mathcal{L}$ does not satisfy any homogenity property.

The dependence of a lagrangian $\mathcal{L}(x, y)$ on both the point x and direction y offers the possibility to use it to describe the anisotropic properties of the physical space.

Illustrating examples are the Finslerian models of the relativity given by R. I n g a r t e n [19], J. I. H o r v a t h [13], Y. T a k a n o [59, 60], M. M a t s u m o t o [31], G. S. A s a n o v [2], H. I s h i k a w a [20, 21], S. I k e d a [17, 18], H. R u n d [49, 50, 51] and others.

For these models the following problems are important:

— An unitary description of the elementary particles.

— To remove the difficulties created by the divergence to Einstein's equations.

— The modification of the structure of the space-time by dependence of the metric on the direction.

Some of these problems were not satisfactory solved.

In the Finslerian theory of relativity the equations of Einstein which have been proposed untill now are based upon one of the three curvature tensors of the space. An open problem is to know which is the most convenient curvature tensor to be used in Einstein's equations. To solve it, an unitary principle in establishing the equations of Einstein in these spaces must be adopted.

But all these problems are also interesting in the general case of the Lagrange spaces, a field which was not yet studied.

In this work we are dealing with the following problems:

1°. The introduction of the notion of the generalized Lagrange space and the elucidation of some important aspects of the geometry of these spaces.

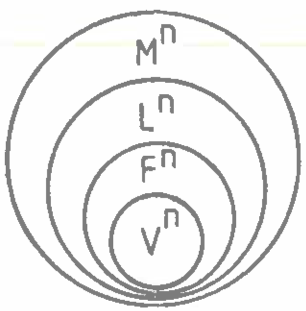
* The second part will appear in the following number.

2°. A correct formulation of Einstein's equations in generalized Lagrange spaces and, in particular, in Lagrange spaces and Finsler spaces.

3°. The establishing of the invariant form of Einstein's equations with respect to the canonical form of the metric of the generalized Lagrange space.

We define a generalized Lagrange space as a pair $M^n=(M, g_{ij}(x, y))$ — (the notation belongs to M. Hashiguchi [9], and S. Watanabe [68]) — where M is an n-dimensional differentiable manifold and $g_{ij}(x, y)$ are the locally components of a distinguished symmetric and non-degenerate, tensor field [34]. Important results in this field were obtained by the author [34, 35, 36], M. Hashiguchi [9], S. Watanabe [68], S. Watanabe, F. Ikeda, S. Ikeda [69, 70], and Miron-Bejancu [38].

The advantage to solve the above problems for the spaces M^n is obvious. These are very general and the signature of the fundamental tensor field $g_{ij}(x, y)$ is at our disposal. Moreover, the Lagrange spaces $L^n=(M, \mathscr{L}(x, y))$, the Finsler spaces $F^n=(M, \mathscr{L}(x, y))$ and the Riemann spaces $V^n=(M, g_{ij}(x))$ are particular M^n spaces. The inclusion relation among these spaces is as follows:



As we shall point out in the following, an M^n-space can be lifted to the tangent bundle TM endowed with an almost Hermitian structure H^{2n} which is called the almost Hermitian model of the space M^n. There are two important results proved by M. Matsumoto [30] and V. Oproiu (private communication): the almost Hermitian model H^{2n} of a Finsler space F^n or a Lagrange space L^n is an almost Kählerian space. This is a reason to use H^{2n} in order to establish correctly Einstein's equations in M^n. But the same idea could be used to study, for example, the generalized Lagrange spaces of constant curvature.

The work is divided in three chapters, corresponding to the above mentioned problems.

In the first chapter we develop the geometry of the generalized Lagrange spaces M^n, pointing out the fundamental geometric objects. There are proved geometric characterizations of the subordinate spaces: Lagrange spaces, Finsler spaces, local Minkowski spaces, Riemann spaces, M^n-spaces, with weakly regular metric, and M^n spaces with regular metric.

In the second chapter the almost Hermitian model H^{2n} is constructed. Einstein's equations of the M^n are introduced as Einstein's equations of the model H^{2n}. These equations, written for Finsler spaces, are totally different from that obtained by the above mentioned authors. Also, the conservation law and the divergence of the energy-momentum tensor field are very different from those known until now.

The last chapter contains the invariant theory of Einstein's equations in generalized Lagrange spaces and their canonical form. For this purpose the method of the non-holonomic coordinates of G. Vrănceanu [66], was adapted and his pseudo-Euclidian tensor ε_{ab} was sistematically used. Here the difficulty was to establish the invariant algorithms of h- and v-covariant derivations.

In the opinion of the author the most convenient way to tackle this subject is by using the tensorial method even if an invariant formulation of the Lagrangian theory of relativity is given.

Some aspects of this theory, in the Finslerian case, will be exposed by the author in a lecture at the Kagoshima University (Japan) at the „XXth Symposium on Finsler Geometry" organized by prof. Masao Hashiguchi.

This work is an hommage paid to the memory of Acad. Prof. Dr. G. Vrănceanu for the creation, fifty years ago, of the first unitary non-holonomic theory and for tireless work in the developping of the Romanian school of geometry, [65].

I. Generalized Lagrange Spaces. The generalized Lagrange spaces have as support manifold the tangent bundle TM of an n-dimensional differentiable manifold M. Consequently, we define the notion of distinguished geometrical objects tensors, connections etc. — on TM and develop some aspects of the geometry of these spaces.

§ 1. d-Connections. Let M be a real C^∞-manifold of dimension n and $\pi: TM \to M$ its tangent bundle. If (U, x^i) is a locally chart on M, then $(\pi^{-1}(U), x^i, y^i)$ is a locally chart on TM. A coordinate transformation on the tangent bundle TM is given by

$$\bar{x}^i=\bar{x}^i(x^1, \ldots, x^n),$$

$$\bar{y}^i=\frac{\partial \bar{x}^i}{\partial x^j} y^j, \quad \det\left\|\frac{\partial \bar{x}^i}{\partial x^j}\right\| \neq 0. \tag{1.1}$$

The indices $i, j, k, \ldots$ run over the set $\{1, \ldots, n\}$ and the Einstein convention of summing is adopted.

A non-linear connection N on TM is a system of functions $N^i_j(x, y)$, given on every $\pi^{-1}(U)$ which, with respect to (1.1), has the transformation law:

$$\bar{N}^i_j=\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^j} N^r_s+\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k}\bar{y}^k. \tag{1.2}$$

When the base manifold M is paracompact, there exist non-linear connections on TM.

One finds that:

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i(x, y)\frac{\partial}{\partial y^j} \tag{1.3}$$

gives a locally base of a distribution N on TM, uniquely determined by the non-linear connection N, which is supplementary to the vertical distribution V on TM. The vertical distribution V has $\{\partial/\partial y^i\}$ as a locally adapted basis. In this way $\{\delta/\delta x^i, \partial/\partial y^i\}$ is a locally basis for the module of the vector fields $\mathscr{X}(TM)$, adapted to the distributions N and V. N is called an horizontal distribution.

The fields of 1-forms $(dx^i, \delta y^i)$:

$$\delta y^i = dy^i + N^i_j(x, y)dx^j \tag{1.4}$$

give the dual basis of the locally basis $\{\delta/\delta x^i, \partial/\partial y^i\}$:

$$\begin{aligned} &< \frac{\delta}{\delta x^j}, dx^i > = \delta^i_j, \quad < \frac{\partial}{\partial y^j}, dx^i > = 0, \\ &< \frac{\delta}{\delta x^j}, \delta y^i > = 0, \quad < \frac{\partial}{\partial y^j}, \delta y^i > = \delta^i_j. \end{aligned} \tag{1.5}$$

Denoting

$$R^i{}_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \quad t^i{}_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j}, \tag{1.6}$$

we remark that :

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i{}_{jk} \frac{\partial}{\partial y^i}, \quad \left[\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right] = -\frac{\partial N^i_k}{\partial y^j} \frac{\partial}{\partial y^i}. \tag{1.7}$$

Then, N is an integrable distribution if and only if $R^i{}_{jk} = 0$. Obviously $R^i{}_{jk}$, $t^i{}_{jk}$ have a geometrical character.

A distinguished geometrical object field — shortly d-object — on the manifold M is a set of the functions $\Omega^A(x, y)$ given on every $\pi^{-1}(U) \subset TM$, which, with respect to (1.1), has the transformation law :

$$\overline{\Omega}^A(\overline{x}, \overline{y}) = \mathcal{F}^A(\Omega^\Sigma(x, y), x^i, \overline{x}^i, \partial_j \overline{x}^i, \ldots, \partial_{j_r} \ldots \partial_{j_1} \overline{x}^i),$$

where $\mathcal{F}^A$ satisfies the known group properties.

We remark that in $\mathcal{F}^A$ we have not explicitely y^i or $\overline{y}^i$. This transformation law is the same as the similar law of transformation of the geometrical object field on the base M. From this point of view the non-linear connection $N(N^i_j)$ is not a d-object, but $\partial N^i_j/\partial y^k$ is . Also, $R^i{}_{jk}$, $t^i{}_{jk}$ from (1.6) are two d-objects.

We call a distinguished connection, d-connection — on M a triplet $L\Gamma = (N^i_j, L^i_{jl}, C^i_{jk})$ formed by a non-linear connection $N^j_i(x, y)$ and by the d-objects $L^i_{jl}(x, y)$, $C^i_{jl}(x, y)$ whose transformations laws with respect to (1.1) are :

$$\begin{aligned} \overline{L}^i_{j\,l} &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} L^r_{st} + \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \overline{x}^j \partial \overline{x}^k}, \\ \overline{C}^i_j &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} C^r_{st}. \end{aligned} \tag{1.8}$$

Remark. In his monograph *Foundations of Finsler Geometry*, M. Matsumoto called the triplet $L\Gamma = (N^i_j, L^i_{jk}, C^i_j)$ a Finsler connection on M.

For example $B\Gamma = (N^i_j, \partial N^i_j/\partial y^k, 0)$ is a d-connection. It is called the Berwald connection.

If $L\Gamma = (N^i_j, L^i_{jl}, C^i_{jk})$ is a d-connection, we can define h- and v-covariante derivatives with respect to $L\Gamma$. Namely, for a d-tensor field $K^i_j(x, y)$ we put :

$$\begin{aligned} K^i_{j|h} &= \frac{\delta K^i_j}{\delta x^h} + K^r_j L^i_{rh} - K^i_r L^r_{jh}, \\ K^i_j|_h &= \frac{\partial K^i_j}{\partial y^h} + K^r_j C^i_{rh} - K^i_r C^r_{jh}, \end{aligned} \tag{1.9}$$

and this rule is a general one.

It is easy to see that $K^i_{j|h}$, $K^i_j|_h$ are two d-tensor fields for which the number of covariance indices are greater with one unity.

The Ricci formulas contain the *curvatures* and *torsions* of $L\Gamma$:

$$\begin{aligned} K^i_{j|l|h} - K^i_{j|h|k} &= K^r_j R_r{}^i{}_{kh} - K^i_r R_j{}^r{}_{kh} - K^i_{j|r} T^r_{kh} - K^i_j|_r R^r{}_{kh}, \\ K^i_{j|k}|_h - K^i_j|_{h|k} &= K^r_j P_r{}^i{}_{kh} - K^i_r P_j{}^r{}_{kh} - K^i_{j|r} C^r_{kh} - K^i_j|_r P^r{}_{kh}, \\ K^i_j|_k|_h - K^i_j|_h|_k &= K^r_j S_r{}^i{}_{kh} - K^i_r S_j{}^r{}_{kh} - K^i_j|_r S^r{}_{kh}, \end{aligned} \tag{1.10}$$

where

$$T^i{}_{jk} = L^i_{jl} - L^i_{kj}, \quad S^i{}_{jk} = C^i_{jk} - C^i_{kj}, \quad P^i{}_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i_{kj}, \quad C^i_{jk}, \quad R^i{}_{jk} \tag{1.11}$$

are *the torsion tensor fields* of $L\Gamma$ and

$$\begin{aligned} R_j{}^i{}_{kh} &= \frac{\delta L^i{}_{jk}}{\delta x^h} - \frac{\delta L^i{}_{jh}}{\delta x^k} + L^r_{jk} L^i_{rh} - L^r_{jh} L^i_{rk} + C^i_{jr} R^r{}_{kh}, \\ P_j{}^i{}_{kh} &= \frac{\partial L^i{}_{jk}}{\partial y^h} - C^i_{jh|k} + C^i_{jr} P^r{}_{kh}, \\ S_j{}^i{}_{kh} &= \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk} \end{aligned} \tag{1.12}$$

are *the curvature tensor fields* of $L\Gamma$.

Using a well known method, we can obtain from (1.10) the Bianchi identities for $L\Gamma$.

If $C(x^i(t), y^i(t))$, $t \in (a, b)$, is a curve on the tangent bundle TM, a d-vector field $X^i(x, y)$ is called parallel along C, with respect to $L\Gamma$ if $X^i(x(t), y(t))$ is a solution of the system of differential equations

$$\frac{dX^i}{dt} = X^i{}_{|k} \frac{dx^k}{dt} + X^i|_k \frac{dy^k}{dt} = 0. \tag{1.13}$$

The d-vector field $X^i(x, y)$ is absolutely parallel if

$$X^i{}_{|k} = 0, \quad X^i|_k = 0.$$

When the last system is completely integrable, it is said that $L\Gamma$ has *absolute parallelism*.

Then the d-connection $L\Gamma$ is with absolute parallelismus if and only if $R_j{}^i{}_{kh} = P_j{}^i{}_{kh} = S_j{}^i{}_{kh} = 0$.

We call

(1.14) $$D^i{}_j = y^k L^i_{kj} - N^i{}_j$$

the deflection tensor field of the **d**-connection $L\Gamma$.

The **d**-connection $L\Gamma$ is called of Cartan type if

(1.15) $$D^i{}_j = 0, \quad C^i_{0j} = 0,$$

where the contraction by y^i is denoted with the index zero.

The **d**-connection $L\Gamma$ is of Cartan type if and only if

(1.16) $$y^i{}_{|k} = 0, \quad y^i|_k = \delta^i_k.$$

Also, when $L\Gamma$ has this property we have

(1.17) $$R^i{}_{jk} = R_0{}^i{}_{jk}, \quad P^i{}_{jk} = P_0{}^i{}_{jk}, \quad S^i{}_{jk} = S_0{}^i{}_{jk}.$$

This result is an immediate consequence of (1.10) applied to (1.16).

The $L\Gamma$ for which the non-linear connection is fixed, will be denoted by $L\Gamma\ (N) = (L^i_{jk},\ C^i_{jk})$.

The transformations $L\Gamma(N) \to L\overline{\Gamma}(N)$ are given by

$$\overline{L}^i_{jk} = L^i_{jk} - B^i_{jk}, \quad \overline{C}^i_{jk} = C^i_{jk} - D^i_{jk},$$

where $B^i_{jk}(x, y)$ and $D^i_{jk}(x, y)$ are **d**-tensor fields. It is very easy to find the relations between the torsions and curvatures of the two **d**-connections $L\Gamma(N)$ and $L\overline{\Gamma}(N)$.

§ 2. **Generalized Lagrange spaces.** The main idea of this paper is based on the following :

Definition 2.1. *A symmetric and non-degenerate* **d**-*tensor field* $g_{ij}(x, y)$ *of type (0,2), is called a generalized Lagrange metric on the manifold* M. *The pair* $(M, g_{ij}(x, y)) = M^n$ *is called a generalized Lagrange space.*

We do not request that $g_{ij}(x, y)$ be homogeneous with respect to y^i and we do not impose any conditions with respect to the signature of *the fundamental* tensor $g_{ij}(x, y)$ of M^n.

Let $g^{ij}(x, y)$ be the **d**-tensor field given by

(2.1) $$g_{ik}(x, y) g^{kj}(x, y) = \delta^j_i.$$

Then, we consider the Obata operators :

(2.2) $$\Omega^{ij}_{hk} = (1/2)\ (\delta^i_h \delta^j_k - g_{hk} g^{ij}), \quad \Omega^{*ij}_{hk} = (1/2)(\delta^i_h \delta^j_k + g_{hk} g^{ij}).$$

They are the supplementary projectors on the module of **d**-tensor fields of the type (p, q), $(p \geq 1,\ q \geq 1)$.

Definition 2.2. *The* **d**-*connection* $L\Gamma(N) = (L^i_{jk},\ C^i_{jk})$ *is called metrical with respect to the generalized Lagrange metric* $g_{ij}(x, y)$ *if*

(2.3) $$g_{ij|k} = 0, \quad g_{ij}|_k = 0.$$

There is a simple geometrical interpretation of the metrical **d**-connections. They preserve the function $g_{ij}(x, y) X^i X^j$ when X^i is transported by parallelism along every curve $C(x^i(t),\ y^i(t))$ and $X^i(x, y)$ is an arbitrary **d**-vector field.

In the paper [34] we have proved :

Theorem 2.1. *There exists an unique metrical* **d**-*connection* $C\Gamma(N) = (L^i_{jk},\ C^i_{jk})$ *having the property* $T^i{}_{jk} = S^i{}_{jk} = 0$. *This connection is given by „the Christoffel symbols"* :

$$L^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right),$$

(2.4) $$C^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h} \right).$$

The uniqueness of $C\Gamma(N)$ allows to call it the N-canonical metrical **d**-connection.

Theorem 2.2. *There exists an unique metrical* **d**-*connection* $L\overset{1}{\Gamma}(N) = (\overset{1}{L}{}^i_{jk},\ \overset{1}{C}{}^i_{jk})$ *for which the torsion tensor fields* $T^i{}_{jk}, S^i{}_{jk}$ *are given a priori. This connection is given by* :

$$\overset{1}{L}{}^i_{jk} = L^i_{jk} + \frac{1}{2} g^{ir} (g_{rh} T^h{}_{jk} - g_{jh} T^h{}_{rk} + g_{kh} T^h{}_{jr}),$$

$$\overset{1}{C}{}^i_{jk} = C^i_{jk} + \frac{1}{2} g^{ir} (g_{rh} S^h{}_{jk} - g_{jh} S^h{}_{rk} + g_{kh} S^h{}_{jr}),$$

where $C\Gamma(N) = (L^i_{jk},\ C^i_{jk})$ *is the* N-*canonical* **d**-*connection.*

Theorem 2.3. *The set of all metrical* **d**-*connections* $L\overline{\Gamma}(N) = (\overline{L}^i_{jk},\ \overline{C}^i_{jk})$ *is given by* :

(2.5) $$\overline{L}^i_{jk} = L^i_{jk} + \Omega^{ir}_{sj} X^s{}_{rk}, \quad \overline{C}^i_{jk} = C^i_{jk} + \Omega^{ir}_{sj} Y^s{}_{rk},$$

where $L\Gamma(N) = (L^i_{jk},\ C^i_{jk})$ *is a fixed metrical* **d**-*connection and* X^i_{jk}, Y^i_{jk} *are* **d**-*arbitrary tensor fields.*

The formulas (2.5) give also the transformations of the metrical d-connections. Let $\mathcal{T}$, be the set of these transformations and denote by „o" the product of transformations.

Theorem 2.4. *The pair* $(\mathcal{T}, \circ)$ *is an abelian group isomorphic to the additive group of the pairs of* **d**-*tensor fields* $(\Omega^{ir}_{sj} X^s{}_{rk},\ \Omega^{ir}_{sj} Y^s{}_{rk})$.

Denoting by

$$d = \det \| g_{ij}(x, y) \|, \quad R_{ijkh} = g_{jr} R_i{}^r{}_{kh}, \ldots,$$

(2.6) $$R_{ij} = R_i{}^r{}_{jr}, \quad S_{ij} = S_i{}^r{}_{jr}, \quad \overset{1}{P}_{ij} = P_i{}^r{}_{jr}, \quad \overset{2}{P}_{ij} = P_i{}^r{}_{rj},$$
$$R = g^{ij} R_{ij}, \quad S = g^{ij} S_{ij},$$

we can prove :

Theorem 2.5. *The* N-*canonical* **d**-*connection* $C\Gamma(N)$ *has the properties* :

$$1°.\ R_{ijkh} + R_{ijhk} = 0, \quad S_{ijkh} + S_{ijhk} = 0,$$

$$2°.\ R_{ijkh} + R_{jikh} = 0, \quad P_{ijkh} + P_{jikh} = 0, \quad S_{ijkh} + S_{jikh} = 0,$$

$$3°.\ R_{i\ kh}^{\ i}=P_{i\ kh}^{\ i}=S_{i\ kh}^{\ i}=0,$$

$$4°\ S_{ij}=S_{ji}\ ;\ \{R^{i}{}_{jk}=0\Rightarrow R_{ij}=R_{ji}\},$$

$$5°.\ \frac{\delta g}{\delta x^{k}}=2gL^{i}_{ik},\ \frac{\partial g}{\partial y^{k}}=2gC^{i}_{ik},$$

$$6°\ t^{i}{}_{jk}=0\Rightarrow\{P^{i}{}_{jk}=P^{i}{}_{kj},\ \overset{1}{P}_{ij}-\overset{1}{P}_{ji}=\overset{2}{P}_{ij}-\overset{2}{P}_{ji}\},$$

$$7°.\ \overset{1}{P}_{jk}=P^{r}{}_{kr|j}-P^{r}{}_{kj|r}+P^{r}{}_{hr}C^{h}_{kj}-P^{r}{}_{hj}C^{h}_{kr},$$

$$\overset{2}{P}_{jk}=C^{r}_{rk|j}-C^{r}_{jk|r}+C^{r}_{jh}P^{h}{}_{rk}-C^{r}_{rh}P^{h}{}_{jk}.$$

Also, we have:

Theorem 2.6. *If $C\Gamma(N)$ is of the Cartan type and $t^{i}{}_{jk}=0$, $P^{i}{}_{0k}=0$, $C^{i}_{jk|0|h}-C^{i}_{jh|0|k}=0$ then $C\Gamma(N)$ has the properties:*

$$1°.\ P_{i\ jk}^{\ h}-P_{i\ kj}^{\ h}=-S_{i\ jk|0}^{\ h},$$

$$2°.\ \overset{1}{P}_{ij}-\overset{2}{P}_{ij}=-S_{ij|0}.$$

Proof. The hypothesis, Theorem 2.5 and the Bianchi identities imply $S_{i\ jk|0}^{\ h}=y^{r}(P_{i\ rk}^{\ h}|_{j}-P_{i\ rj}^{\ h}|_{k})=-(P_{i\ jk}^{\ h}-P_{i\ kj}^{\ h})$. Then 1° is verified. The contraction $h=k$ in 1° gives 2°.

Remark. The last Theorem shows that $\overset{1}{P}_{ij}\neq\overset{2}{P}_{ij}$ for some spaces M^{n}.

§ 3. Particular generalized Lagrange spaces: Lagrange spaces), Finsler spaces, locally Minkowski spaces and Riemann spaces.

Some important classes of generalized Lagrange spaces M^{n} deserve a special attention; namely the Lagrange spaces, the Finsler spaces, etc.

Definition 3.1. *A pair $L^{n}=(M,\mathcal{L})$ where M is an n-dimensional manifold, $\mathcal{L}(x,y)$ is a real differentiable function on the tangent bundle TM, with property that the* **d**-*tensor field*

$$g_{ij}(x,y)=\frac{1}{2}\frac{\partial^{2}\mathcal{L}}{\partial y^{i}\partial y^{j}}\tag{3.1}$$

is non-degenerate is called a Lagrange space.

All our considerations of the previous paragraphs can be applied to the study of the Lagrange spaces L^{n}. But, in this case we can determine a non-linear connection which depends on the fundamental function $\mathcal{L}(x,y)$ only.

Theorem 3.1. (J. Kern). *The functions $N^{i}_{j}(x,y)$, defined on every chart domain $\pi^{-1}(U)$ in TM, by*

$$N^{i}_{j}=\frac{\partial G^{i}}{\partial y^{j}},\quad G^{i}=\frac{1}{4}g^{ij}\left(\frac{\partial^{2}\mathcal{L}}{\partial y^{j}x^{k}}y^{k}-\frac{\partial\mathcal{L}}{\partial x^{j}}\right)\tag{3.2}$$

are the coefficients of a non-linear connection on TM, determined by the fundamental function $\mathcal{L}(x,y)$ of the Lagrange space $L^{n}=(M,\mathcal{L})$ only.

Proof. Taking into account the transformation of the natural basis $\{\partial/\partial x^{i},\partial/\partial y^{i}\}$ with respect to (1.1):

$$\frac{\partial}{\partial x^{i}}=\frac{\partial\bar{x}^{h}}{\partial x^{i}}\frac{\partial}{\partial\bar{x}^{h}}+\frac{\partial^{2}\bar{x}^{h}}{\partial x^{i}\partial x^{j}}y^{j}\frac{\partial}{\partial\bar{y}^{h}};\quad\frac{\partial}{\partial y^{i}}=\frac{\partial\bar{x}^{j}}{\partial x^{i}}\frac{\partial}{\partial\bar{y}^{j}}$$

we obtain for N^{i}_{j} from (3.2) the transformation (1.2).

Remark. The non-linear connection N given by (3.2) can be obtained by applying the variational principle to the Lagrangian $\mathcal{L}(x(t),dx/dt))$ for the parametrizated curves $x^{i}(t)$.

In the theory of the spaces L^{n} we shall use the nonlinear connection (3.2). Thus we have $t^{i}{}_{jk}=0$. Put now:

$$C_{ijk}=g_{jh}C^{h}_{ik}=\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial y^{k}}+\frac{\partial g_{jk}}{\partial y^{i}}-\frac{\partial g_{ik}}{\partial y^{j}}\right).\tag{3.3}$$

Theorem 3.2. *A generalized Lagrange space $M^{n}=(M,g_{ij}(x,y))$ is a Lagrange space if and only if the* **d**-*tensor field $C_{ijk}(x,y)$ from (3.3) is totally symmetric.*

Proof. If (3.1) holds, then $C_{ijk}=1/4\,\partial^{3}\mathcal{L}/\partial y^{i}\partial y^{j}\partial y^{k}$. Conversely, $g_{ij}(x,y)$ being given, by the system of equations with partial derivatives (3.1), we obtain a solution $\mathcal{L}(x,y)$ if and only if

$$\frac{\partial g_{ij}}{\partial y^{k}}=\frac{\partial g_{jk}}{\partial y^{i}}=\frac{\partial g_{ik}}{\partial y^{j}}.$$

But C_{ijk} from (3.3) being totally symmetric, these conditions are satisfied.

The following result communicated to us by prof. Vasile Oproiu, is of interest when the almost Hermitian model of the space M^{n} is considered.

Theorem 3.3. (V. Oproiu) *If $L^{n}=(M,\mathcal{L})$ is a Lagrange space, then d-forms*

$$\omega=\frac{1}{2}\frac{\partial\mathcal{L}}{\partial y^{i}}dx^{i},\quad\theta=g_{ij}\delta y^{i}\wedge dx^{j}\tag{3.4}$$

are globally defined on TM and θ is the exterior differential of ω:

$$\theta=d\omega\tag{3.5}$$

Proof. Evidently ω and θ are globally defined on the tangent bundle TM. Computing the exterior differential $d\omega$ we get

$$d\omega=\frac{1}{4}\left[\left(\frac{\partial\mathcal{L}}{\partial y^{k}}\right)_{|j}-\left(\frac{\partial\mathcal{L}}{\partial y^{j}}\right)_{|k}\right]dx^{j}\wedge dx^{k}+g_{jk}\delta y^{j}\wedge dx^{k}.$$

But from (3.2) it follows $(\partial\mathcal{L}/\partial y^{k})_{|j}-(\partial\mathcal{L}/\partial y^{j})_{|k}=0$. Then $d\omega=\theta$.

Corollary 3.1. *The N-canonical connection $C\Gamma(N)$ of the Lagrange space $L^{n}=(M,\mathcal{L})$ has the properties;*

$$1°.\ R^{h}{}_{ij}g_{kh}+R^{h}{}_{jk}g_{ih}+R^{h}{}_{ki}g_{jh}=0,$$

$$2°.\ P^{h}{}_{ik}g_{jh}-P^{h}{}_{jk}g_{ih}=0.$$

Indeed, from $d\theta=0$ and $d(\delta y^i)=(1/2)R^i{}_{jk}dx^k\wedge dx^j+(\partial N^i_j/\partial y^k)\delta y^k\wedge dx^j$ it follows 1° and 2°.

Definition 3.2. *A Finsler space is a pair $F^n=(M,\ L(x,y))$ formed by a manifold M and a real function $L(x,y)$ on TM, which is positive 1-homogeneous with respect to y^i and the* **d**-*tensor field*

$$g_{ij}(x,y)=\frac{1}{2}\frac{\partial^2L^2}{\partial y^i\partial y^j} \tag{3.6}$$

is non-degenerate.

$L(x,y)$ is called the fundamental function of F^n.

Clearly, every Finsler space F^n is a Lagrange space L^n, but the converse is not true. For example $L^n=(M,\ \mathcal{L})$, with

$$\mathcal{L}=\frac{m}{2}\{(y^1)^2+\ldots+(y^n)^2\}-U(x^1,\ldots,x^n)$$

is a Lagrange space, but it is not a Finsler space. This Lagrangian is known in the analytical mechanics in the form $\mathcal{L}=(m/2)\,v^2-U(x)$.

Theorem 3.4. *The functions $N^i_j(x,y)$ defined on every domain of the chart $\pi^{-1}(U)$ of TM by*

$$N^i_j=\frac{\partial G^i}{\partial y^j};\quad G^i=\frac{1}{2}\gamma^i_{00},\quad \gamma^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{jh}}{\partial x^k}+\frac{\partial g_{kh}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^h}\right) \tag{3.7}$$

determine a non-linear connection on TM, which depends only on the fundamental function $L(x,y)$.

Indeed, taking in (3.2) $\mathcal{L}=[L(x,y)]^2$, we get (3.7).

Then the canonical connection $C\Gamma(N)$, where N is given by (3.7) is just the well-known *Cartan connection* $C\Gamma$ of the Finsler space F^n, [28].

Theorem 3.5. *A generalized Lagrange space $M^n=(M,\ g_{ij}(x,y))$, whose fundamental tensor field $g_{ij}(x,y)$ is positive defined, is a Finsler space $F^n=(M,\ L(x,y))$ if and only if the* **d**-*tensor field $C_{ijk}(x,y)$ in (3.3) is totally symmetric and $C_{0jk}(x,y)=0$.*

Proof. Supposing $M^n=F^n$, from (3.6), we have $C_{ijk}=(1/4)\,\partial^3L^2/\partial y^i\partial y^j\partial y^k$. Then C_{ijk} is totally symmetric and $C_{0jk}=y^iC_{ijk}=0$. Conversely, C_{ijk} being totally symmetric, it follows $C_{ijk}+C_{jki}=\partial g_{ji}/\partial y^k$. From $C_{0jk}=0$ we have $y^j\partial g_{jk}/\partial y^i=0$; $y^i\partial g_{jk}/\partial y^i=0$. Therefore g_{ij} is 0-homogeneous. We define $[L(x,y)]^2=g_{ij}(x,y)\,y^iy^j$ and obtain $g_{ij}(x,y)=(1/2)\,\partial^2L^2/\partial y^i\partial y^j$.

The Cartan connection $C\Gamma$ of a Finsler space F^n has a lot of interesting properties. Some of them are given here:

1°. $D^i{}_j=0,\ C^i_{0j}=0.$

2°. $P^i{}_{j0}=0,\ P_h{}^i{}_{j0}=0,\ t^i{}_{jk}=0,$

3°. $R^i{}_{jk}=R_0{}^i{}_{jk},\ P^i{}_{jk}=P_0{}^i{}_{jk},\ S_0{}^i{}_{jk}=0,$

4°. $S_{hijk}=C^r_{hk}C_{rij}-C^r_{hj}C_{rik},$

5°. $R_{ijk}+R_{jki}+R_{kij}=0,$

6°. $P_{ijk}=C_{ijk|0},\ P_{hijk}=C_{ijk|h}-C_{hjk|i}+C^r_{hj}C_{rik|0}-C^r_{ij}C_{rhk|0}.$

Definition 3.3. *A generalized Lagrange space $M^n=(M,g_{ij}(x,y))$ is called pseudo-Riemannian if the fundamental tensor $g_{ij}(x,y)$ does not depend on y^i.*

Theorem 3.6. *A generalized Lagrange space M^n is pseudo-Riemannian if and only if the* **d**-*tensor field C_{ijk} from (3.3) vanishes.*

Indeed, $\partial g_{ij}/\partial y^k=0$ implies $C_{ijk}=0$. Conversely, $C_{ijk}=0$ shows that $\partial g_{ij}/\partial y^k=0$.

We remark that in this case the non-linear connection (3.7) is reducible to $N^i_j=\gamma^i_{jk}y^k$ and we have for $C\Gamma(N)$, $F^i_{jk}=\gamma^i_{jk}$, $C^i_{jk}=0$.

The spaces M^n, L^n, F^n and V^n are in the inclusion relation described in the introduction of this paper.

Definition 3.4. *A generalized Lagrange space $M^n=(M,g_{ij}(x,y))$ is called a generalized locally Minkowski space — shortly $Mink^n$ — if there exists a local chart around every point $(x,y)\in TM$, such that $g_{ij}(x,y)$ does not depend on the coordinates x^i of the point $\pi(x,y)=x$.*

Certainly, we can speek about Lagrange spaces L^n or Finsler spaces F^n which are locally Minkowski.

Theorem 3.7. (S. W a t a n a b e) *If the N-canonical connection $C\Gamma(N)$ of the generalized Lagrange space M^n has the properties:*

$$D^i{}_j=0,\ R^i{}_{jk}=0,\ R_j{}^i{}_{kh}=0,\ P^i{}_{jk}=0,\ C^i_{jk|h}=0, \tag{3.8}$$

then M^n is a $Mink^n$-space.

Proof. The equalities (3.8) and the expression of P_{jikh} imply $P_{ijkh}+P_{ikjh}=0$. Thus this equation together with $P_{ijkh}+P_{jikh}=0$ give $P_j{}^i{}_{kh}=0$. Then we have $\partial L^i_{jk}/\partial y^h=0$ and $\delta L^i_{jk}/\delta x^h=\partial L^i{}_{jk}/\partial x^h$. But $R_j{}^i{}_{kh}=0$ means

$$\frac{\partial L^i{}_{jk}}{\partial x^h}-\frac{\partial L^i{}_{jh}}{\partial x^k}+L^r_{jk}L^i_{rh}-L^r_{jh}L^i_{rk}=0.$$

As in the Riemannian case, we can find a locally coordinate system such that $L^i_{jk}=0$. Then $D^i{}_j=0$, $L^i_{jk}=0$ imply $N^i_j=0$. Consequently $g_{ij|k}=\partial g_{ij}/\partial x^k=0$, q.e.d.

Corollary 3.2. *If the N-canonical connection $C\Gamma(N)$ of the generalized Lagrange space M^n is of Cartan type and $R_j{}^i{}_{kh}=0$, $P^i{}_{jk}=0$, $C^i_{jk|h}=0$ then M^n is a $Mink^n$-space.*

Theorem 3.8. *A Finsler space F^n is a locally Minkowski space if and only if its Cartan connection $C\Gamma$ has the properties: $R_j{}^i{}_{kh}=0$, $C^i_{jk|h}=0$.*

§ 4. The spaces M^n with regular metric. It is important for us to determine the generalized Lagrange spaces M^n for which the N-canonical connection $C\Gamma(N)$ depends on the fundamental tensor field $g_{ij}(x,y)$ only.

To this aim we consider the function defined on TM:

$$\mathcal{E}(x,y)=g_{ij}(x,y)y^iy^j=g_{00}. \tag{4.1}$$

We say that $\mathcal{E}(x,y)$ is *the absolute energy* of M^n.

We see easely that

$$g^*_{ij}(x,y)=\frac{1}{2}\frac{\partial^2\mathcal{E}}{\partial y^i\partial y^j} \tag{4.2}$$

is a covariant symmetric **d**-tensor field on TM. If rank $\|g^*_{ij}\|=n$, then $(M,\mathcal{E})$

is a Lagrange space associated to the generalized Lagrange space M^n.

Definition 4.1. *It is said that M^n is a space with weakly regular metric if $(M, \mathcal{E})$ is a Lagrange space.*

Theorem 4.1. *If the generalized Lagrange space M^n is with weakly regular metric, then $N^i_j(x, y)$ given by*

$$(4.3)\qquad N^i_j=\frac{\partial G^i}{\partial y^j},\quad G^i=\frac{1}{4}g^{*ij}\left(\frac{\partial^2\mathcal{E}}{\partial y^j\partial x^k}y^k-\frac{\partial\mathcal{E}}{\partial x^j}\right)$$

is a non-linear connection, which depends on the fundamental tensor field $g_{ij}(x, y)$ only.

Indeed, theorem 3.1 is working here.

Therefore $C\Gamma(R)=(L^i_{jk}, C^i_{jk})$ given by (2.4) and (4.3) is *the canonical connection* of the space M^n.

Corollary 4.1. (M. Hashiguchi, S. Watanabe, F. Ikeda). *If the generalized Lagrange space M^n with weakly regular metric is given, then $N^i_j(x, y)$ from the formulas :*

$$(4.4)\qquad N^i_j=\frac{\partial G^i}{\partial y^j},\quad G^i=\frac{1}{2}g^{*ih}\left((g_{hr}\gamma^r_{00}+y^r\frac{\partial C_{h00}}{\partial x^k}\right)$$

is a non-linear connection on TM, which depends only on $g_{ij}(x, y)$.

Theorem 4.2. *A generalized Lagrange space M^n with weakly regular metric is a Mink^n-space with weakly regular metric if and only if its canonical connection $C\Gamma$ has the properties :*

$$(4.5)\qquad D^i_j=0,\ R^i{}_{jk}=0,\ R_j{}^i{}_{kh}=0,\ P^i{}_{jk}=0,\ C^i_{jr|h}=0.$$

Proof. If in the coordinate system $(\pi^{-1}(U), x^i, y^i)$ we have $g_{ij}(x, y)=g_{ij}(y)$, then the absolute energy $\mathcal{E}$ is a function on y^i only and (4.3) shows that $N^i_j=0$. Using the expression (2.4) for $C\Gamma$ it follows $L^i_{jk}=0$, which implies $R^i{}_{jk}=0$, $R_j{}^i{}_{kh}=0$. Also, it results $P^i{}_{jk}=0$. Clearly, we get $C^i_{jk|h}=0$. Consequently (4.5) are verified. The converse follows from (4.5) and W a t a n a b e's theorem 3.7.

Definition 4.2. *The generalized Lagrange space M^n is called with the regular metric if :*

1°. *The pair $(M, \mathcal{E})$ is a Lagrange space.*

2°. $\frac{1}{2}\frac{\partial\mathcal{E}}{\partial y^i}=g_{ij}y^j$.

This notion was introduced by the author in [34].

M. H a s h i g u c h i [9] gave a simple and elegant geometrical interpretation of the conditions of regularity 1° and 2°. He proved that a generalized Lagrange space M^n is with regular metric if and only if the variational problem for the absolute energy $\mathcal{E}(x, y)$ is a regular one and the transversality coincides with the orthogonality.

The condition of 0-homogeneity for the fundamental tensor field $g_{ij}(x, y)$ with respect to y^i is a consequence of 1° and 2°. Indeed, these conditions are equivalent with

$$(4.6)\qquad \operatorname{rank}\|g^*_{ij}\|=n,\quad C_{i00}=0.$$

Then for the spaces M^n with regular metric, we have

$$(4.7)\qquad g^*_{ij}=g_{ij}+\frac{\partial g_{is}}{\partial y^j}y^s.$$

By means of (4.6) we get

$$(4.8)\qquad g^*_{ij}y^iy^j=g_{ij}y^iy^j=\mathcal{E}(x, y).$$

Therefore :

$$(4.9)\qquad y^i\frac{\partial\mathcal{E}}{\partial y^i}=2\mathcal{E}(x, y).$$

We obtain so

Theorem 4.3. (R. M i r o n) *The generalized Lagrange space M^n with regular metric has the fundamental tensor field $g_{ij}(x, y)$ 0-homogeneous with respect to y^i and*

$$(4.10)\qquad N^i_j=\frac{\partial G^i}{\partial y^j},\quad G^i=\frac{1}{2}g^{*ir}g_{rk}\gamma^k_{00}$$

are the coefficients of a non-linear connection depending by $g_{ij}(x, y)$ only.

Theorem 4.4. *A Finsler space F^n has the following properties :*

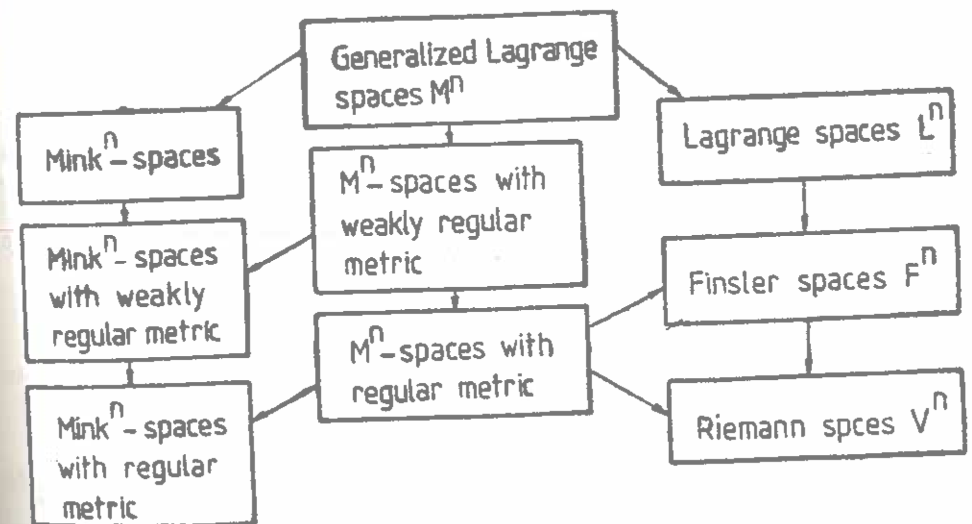
1°. *It is a generalized Lagrange space M^n with regular metric,*

2°. *The absolute energy is $\mathcal{E}(x, y)=[L(x, y)]^2$.*

3°. *The non-linear connection (4.10) is the same as the non-linear connection of Cartan.*

4°. *$C\Gamma(N)$, with N given by (4.10) coincides with the Cartan connection.*

Here is the scheme of the generalized Lagrange spaces M^n projections on the mentioned spaces :



II. Lagrangian theory of relativity. The geometry of a generalized Lagrange space $M^n=(M, g_{ij}(x, y))$ can be identified with the geometry of an almost Hermitian space H^{2n} on tangent bundle TM. The canonical connection $C\Gamma(N)$ of the space M^n is identified with the unique almost Hermitian connection $\overset{N}{\nabla}$ in H^{2n} which is h- and v-symmetric. Consequently, we can define the Einstein equations of the space M^n as Einstein equations of the space H^{2n} for the connection $\overset{N}{\nabla}$. But this connection $\overset{N}{\nabla}$ has non vanishing torsion. This means that the conservation law, generally, is not verified. This is the reason to define the Einstein equations for an almost Hermitian connection of H^{2n} with general torsion. In this way we get the Einstein equations for a metrical distinguished connection with h- and v- non vanishing torsion in M^n, and so we find the most general Lagrangian model of the relativity. But, even in the particular case of Finsler spaces, we obtain the system of Einstein's equations which was not established up to now; it is very different from the known system. This fact is very well seen on the example of the locally Minkowski spaces.

§ 1. **N-connections of Lagrange type on** TM. Let M be an n-dimensional differentiable manifold, TM its tangent bundle and $\pi: TM\to M$ the natural projection. If U is a domain of a coordinate chart, then $\pi^{-1}(U)$ is a domain of a chart on TM and the transformations of the locally coordinates on TM are given by (1.1), I. We consider a non-linear connection N on TM and let V be the vertical distribution on TM. Then

$$\left\{\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i\frac{\partial}{\partial y^j},\ \frac{\partial}{\partial y^i}\right\}\quad (i=1, \ldots, n)$$

is a local basis on $\mathcal{X}(TM)$ adapted to the distributions N and V. As in the first chapter we denote the dual basis by $(dx^i, \delta y^i)$, $\delta y^i=dy^i+N^i_j dx^j$.

Let $J:\mathcal{X}(TM)\to\mathcal{X}(TM)$ be the $\mathcal{F}(TM)$-linear map given by:

$$J\left(\frac{\delta}{\delta x^i}\right)=\frac{\partial}{\partial y^i},\ J\left(\frac{\partial}{\partial y^i}\right)=-\frac{\delta}{\delta x^i},\quad (i=1, \ldots, n). \tag{1.1}$$

Proposition 1.1. 1°. *J is a tensor field of type* (1,1) *on TM,*

2°. $$J\circ J=-I.$$

Proof. Evidently, J is a globally defined tensor field on TM, since we have

$$\frac{\delta}{\delta\bar{x}^i}=\frac{\partial x^j}{\partial\bar{x}^i}\frac{\delta}{\delta x^j},\quad \frac{\partial}{\partial\bar{y}^i}=\frac{\partial x^j}{\partial\bar{x}^i}\frac{\partial}{\partial y^j}.$$

From (1.1), it results $J\circ J=-I$.

J is called the N-almost complex structure on TM.

Theorem 1.1. *The N-almost complex structure J is integrable if and only if $R^i{}_{jk}=t^i{}_{jk}=0$.*

Proof. The Nijenhuis tensor field of the structure J:

$$\mathcal{N}(X, Y)=[X, Y]+J[JX, Y]+J[X, JY]-[JX, JY]\qquad \forall X, Y\in\mathcal{X}(TM)$$

leads to $\mathcal{N}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)=0$, $\mathcal{N}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right)=0$ and $\mathcal{N}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)=0$ iff $R^i{}_{jk}=t^i{}_{jk}=0$ for the non-linear connection N.

Definition 1.1. *A linear connection $\overset{N}{\nabla}$ on the total space TM of the tangent bundle of M is called a N-connection of Lagrange type if:*

1°. *The N-almost complex structure J is absolute parallel by $\overset{N}{\nabla}$,*

2°. *$\overset{N}{\nabla}$ preserves by parallelism the vertical distribution V.*

We have:

Theorem 1.2. *A linear connection $\overset{N}{\nabla}$ on the tangent bundle TM is a N-connection of Lagrange type if and only if in the adapted basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ it has the form:*

$$\left\{\begin{aligned}&\overset{N}{\nabla}_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j}=L^i_{jk}\frac{\delta}{\delta x^i};\quad \overset{N}{\nabla}_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j}=L^i_{jk}\frac{\partial}{\partial y^i},\\ &\overset{N}{\nabla}_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j}=C^i_{jk}\frac{\delta}{\delta x^i},\quad \overset{N}{\nabla}_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}=C^i_{jk}\frac{\partial}{\partial y^i},\end{aligned}\right. \tag{1.2}$$

in which $(L^i_{jk}(x, y), C^i_{jk}(x, y))$ are the coefficients of a d-*connection $L\Gamma(N)$ on the manifold M.*

Proof. Supposing $\overset{N}{\nabla}$ a N-connection of Lagrange type and taking into account (1.1), it follows (1.2). When a coordinate transformation on TM is performed, L^i_{jk}, C^i_{jk} are transformed as in (1.8), I. Conversely, let $\overset{N}{\nabla}$ be given by (1.1), where (L^i_{jk}, C^i_{jk}) are the coefficients of a d-connection on M. It is easy to see that $\overset{N}{\nabla}$ is a N-connection of the Lagrange type on TM.

Corollary 1.1. *There exists a bijective correspondence between the set of* d-*connections $\{L\Gamma(N)\}$ on M and the set of N-connections of Lagrange type $\{\overset{N}{\nabla}\}$ on TM.*

For this reason $\overset{N}{\nabla}$ corresponding to $L\Gamma(N)$ in the previous bijection is called *the lift* of $L\Gamma(N)$.

Corollary 1.2. *A transformation $\overset{N}{\nabla}\to\overset{N}{\overline{\nabla}}$ has as effect the transformation of their coefficients $(L^i_{jk}, C^i_{jk})\to(\overline{L}^i_{jk}, \overline{C}^i_{jk})$ satisfying the law* (1.18), I.

Now, we are looking for the torsion of $\overset{N}{\nabla}$.

Theorem 1.3. *If $\overset{N}{\nabla}$ is the lift of the* d-*connection $L\Gamma(N)$, then its torsion $\overset{N}{T}$ on the adapted basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ takes the values:*

$$\overset{N}{T}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)=T^i{}_{jk}\frac{\delta}{\delta x^i}+R^i{}_{jk}\frac{\partial}{\partial y^i},$$

$$\overset{N}{T}\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right)=C^i_{jk}\frac{\delta}{\delta x^i}+P^i{}_{jk}\frac{\partial}{\partial y^i},\quad \overset{N}{T}\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right)=S^i{}_{jk}\frac{\partial}{\partial y^i}, \tag{1.3}$$

where $T^i{}_{jk}$, $S^i{}_{jk}$, $P^i{}_{jk}$, C^i_{jk}, $R^i{}_{jk}$ are the torsion tensor fields of $L\Gamma(N)$.

Proof. Writing

$$\overset{N}{T}(X, Y)=\overset{N}{\nabla}_X Y-\overset{N}{\nabla}_Y X-[X, Y],\ \ \forall X,\ Y\in\mathcal{X}(TM)$$

in the adapted basis and using (1.2) and [(1,6), (1.10), I] we get the equalities (1.3).

Analogously, the formula

$$\overset{N}{R}(X, Y)Z=[\overset{N}{\nabla}_X,\ \overset{N}{\nabla}_Y]Z-\overset{N}{\nabla}_{[X, Y]}Z,\ \forall X,\ Y,\ Z\in\mathcal{X}(TM)$$

gives the components of the curvature $\overset{N}{R}$ of $\overset{N}{\nabla}$:

Theorem 1.4. *The curvature $\overset{N}{R}$ of the lift $\overset{N}{\nabla}$ of the* d-*connection $L\Gamma(N)=(L^i_{jk}, C^i_{jk})$, in the adapted basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ is given by:*

$$\overset{N}{R}\left(\frac{\delta}{\delta x^l}, \frac{\delta}{\delta x^k}\right)\frac{\delta}{\delta x^j}=R_j{}^i{}_{kl}\frac{\delta}{\delta x^i}, \qquad \overset{N}{R}\left(\frac{\delta}{\delta x^l}, \frac{\delta}{\delta x^k}\right)\frac{\partial}{\partial y^j}=R_j{}^i{}_{kl}\frac{\partial}{\partial y^i},$$

$$\overset{N}{R}\left(\frac{\partial}{\partial y^l}, \frac{\delta}{\delta x^k}\right)\frac{\delta}{\delta x^j}=P_j{}^i{}_{kl}\frac{\delta}{\delta x^i}, \qquad \overset{N}{R}\left(\frac{\partial}{\partial y^l}, \frac{\delta}{\delta x^k}\right)\frac{\partial}{\partial y^j}=P_j{}^i{}_{kl}\frac{\partial}{\partial y^i}, \tag{1.4}$$

$$\overset{N}{R}\left(\frac{\partial}{\partial y^l}, \frac{\partial}{\partial y^k}\right)\frac{\delta}{\delta x^j}=S_j{}^i{}_{kl}\frac{\delta}{x^i}, \qquad \overset{N}{R}\left(\frac{\partial}{\partial y^l}, \frac{\partial}{\partial y^k}\right)\frac{\partial}{\partial y^j}=S^i_{jkl}\frac{\partial}{\partial y^i},$$

where $R_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$, $S_j{}^i{}_{kl}$ are the curvature tensor fields of $L\Gamma(N)$.

The Bianchi identities of $\overset{N}{\nabla}$:

$$\underset{X,Y,Z}{\mathcal{S}}\{\overset{N}{\nabla}_X T(Y, Z)-T([X, Y], Z)-R(X, Y)\,Z\}=0,$$

$$\underset{X,Y,Z}{\mathcal{S}}\{\overset{N}{\nabla}_X R(Y, Z)-R(Y, Z)\,\overset{N}{\nabla}_X-R([X, Y], Z)\}=0.$$

give:

Theorem 1.5. *The Bianchi identities for a* d-*connection $L\Gamma(N)$ are as follows:*

$$\underset{jkl}{\mathcal{S}}\{R_j{}^h{}_{kl}-T^h{}_{jr}T^r{}_{kl}-T^h{}_{jl|l}-C^h_{jr}R^r{}_{kl}\}=0,$$

$$\underset{jkl}{\mathcal{S}}\{R^h{}_{jk|l}+R^h{}_{jr}T^r{}_{kl}+P^h{}_{jr}R^r{}_{kl}\}=0,$$

$$\underset{jkl}{\mathcal{S}}\{R_i{}^h{}_{jk|l}+R_i{}^h{}_{jr}T^r{}_{kl}+P_i{}^h{}_{jr}R^r{}_{kl}\}=0,$$

$$-C^h_{rl}T^r{}_{jk}T^h{}_{jk}|_l+\mathcal{A}_{jk}\{T^h{}_{jr}C^r_{kl}+C^h_{kl|j}+C^h_{jr}P^r{}_{kl}-P_j{}^h{}_{kl}\}=0,$$

$$-P^h{}_{rl}T^r{}_{jk}+S^h{}_{lr}R^r{}_{jk}+R^h{}_{jk}|_l-R_l{}^h{}_{jk}+\mathcal{A}_{jk}\{R^h{}_{jr}C^r_{kl}+P^h{}_{jr}P^r{}_{kl}P^h{}_{kl|j}\}=0, \tag{1.5)'$$

$$-P_i{}^h{}_{rl}T^r{}_{jk}+S^h_{il\,r}R^r{}_{jk}+R_i{}^h{}_{jk}|_l+\mathcal{A}_{jk}\{R_i{}^h{}_{jr}C^r_{kl}+P_i{}^h{}_{jk}P^r{}_{kl}+P_i{}^h{}_{kl|j}\}=0,$$

$$-S_j{}^h{}_{kl}+\mathcal{A}_{kl}\left\{\frac{\partial C^h_{jk}}{\partial y^l}+C^h_{rl}C^r_{jk}\right\}=0,$$

$$-P^h{}_{lr}S^r{}_{jk}+S^h{}_{jk|l}+\mathcal{A}_{jk}\{P^h{}_{rj}C^r_{lk}-S^h{}_{jr}P^r{}_{lk}-P^h{}_{lk}|_j+P_j{}^h{}_{lk}\}=0, \tag{1.5)''$$

$$-P_i{}^h{}_{lr}S^r{}_{jk}+S_i{}^h{}_{jkl}+\mathcal{A}_{jk}\{P_i{}^h{}_{rj}C^r_{lk}-S_i{}^h{}_{jr}P^r{}_{lk}-P_i{}^h{}_{lk}|_j\}=0,$$

$$\underset{jkl}{\mathcal{S}}\{S_j{}^h{}_{kl}-S^h{}_{jr}S^r{}_{kl}-S^h{}_{jk}|_l\}=0,$$

$$\underset{jkl}{\mathcal{S}}\{S_i{}^h{}_{jk}|_l+S_i{}^h{}_{jr}S^r{}_{kl}\}=0. \tag{1.5)'''$$

where $\mathcal{S}_{jkl}$ means the cyclic summation and $\mathcal{A}_{jk}$ the alternate summation. These identities are complicated and very numerous. In the next paragraph we try to simplify them.

§ 2. The components of various geometrical objects in the adaptated frames. The non-linear connection N being fixed, the adaptated basis $\{\delta/\delta x^i, \partial/\partial y^i\}$ to the horizontal N and vertical V distributions are well determined on a domain of chart $\pi^{-1}(U)$ in TM. This basis determines a local frame in $\mathcal{X}(TM)$, denoted by

$$\{X_\alpha\}=\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\},\quad X_i=\frac{\delta}{\delta x^i},\ X_{(i)}=\frac{\partial}{\partial y^i}, \tag{2.1}$$

with $\alpha, \beta, \gamma, \ldots=1, \ldots, 2n$; $i, j, k\ldots=1, \ldots, n$; $(i), (j), (k), \ldots=n+1, \ldots, 2n$.

The dual frame is given by

$$\{\delta\zeta^\alpha\}=\{dx^i,\ \delta y^i\};\ \delta\zeta^i=dx^i,\ \delta\zeta^{(i)}=\delta y^i. \tag{2.1}$$

The lift $\overset{N}{\nabla}$ of the d-connection $L\Gamma(N)=(L^i_{jk}, C^i_{jk})$ has, in the frame $\{X_\alpha\}$ the components:

$$\overset{N}{\nabla}_{X_\gamma}X_\beta=\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}X_\alpha. \tag{2.2}$$

Then, we have

$$\overset{N}{\nabla}_{X_\gamma}\delta\zeta^\alpha=-\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}\delta\zeta^\beta. \tag{2.2)'$$

A vector field $Y=Y^\alpha X_\alpha$ has the covariant derivative:

$$\overset{N}{\nabla}_{X_\gamma}Y=(\overset{N}{\nabla}_\gamma Y^\beta)X_\beta;\ \overset{N}{\nabla}_\gamma Y^\beta=X_\gamma Y^\beta+Y^\alpha\overset{N}{\Gamma}{}^\beta_{\alpha\gamma}. \tag{2.3}$$

A differential 1-form $\omega=\omega_\alpha\delta\zeta^\alpha$ has the covariant derivative:

$$\overset{N}{\nabla}_{X_\gamma}\omega=(\overset{N}{\nabla}_\gamma\omega_\beta)\delta\zeta^\beta;\overset{N}{\nabla}_\gamma\omega_\beta=X_\gamma\omega_\beta-\omega_\alpha\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}. \tag{2.3)'$$

Now, it is clear how $\overset{N}{\nabla}$ operates on an arbitrary tensor field K on TM. For example $K=K^\alpha_\beta X_\alpha\otimes\delta\zeta^\beta$ has the covariante derivative:

$$\overset{N}{\nabla}_{X_\gamma}K=(\overset{N}{\nabla}_\gamma K^\alpha_\beta)X_\alpha\otimes\delta\zeta^\beta,$$

with

$$\overset{N}{\nabla}_\gamma K^\alpha_\beta = X_\gamma K^\alpha_\beta + K^\delta_\beta \overset{N}{\Gamma}{}^\alpha_{\delta\gamma} - K^\alpha_\delta \overset{N}{\Gamma}{}^\delta_{\beta\gamma} \tag{2.4}$$

But $\{X_\alpha\}$ gives rise to a non-holonomic system of coordinates in the sense of G. Vrănceanu [66]. The non-holonomy coefficients of this system $w^\alpha_{\beta\gamma}$ are given by:

$$[X_\beta, X_\gamma] = w^\alpha_{\beta\gamma} X_\alpha \tag{2.5}$$

and the Vrănceanu identities are:

$$\mathcal{S}_{\alpha\beta\gamma}\{X_\alpha w^\delta_{\beta\gamma} - w^\varphi_{\alpha\beta} w^\delta_{\varphi\gamma}\} = 0. \tag{2.6}$$

In the local frames $\{X_\alpha\}$ the torsion $\overset{N}{T}$ and the curvature R of the connection $\overset{N}{\nabla}$ have the components:

$$\overset{N}{T}{}^\alpha_{\beta\gamma} = \overset{N}{\Gamma}{}^\alpha_{\beta\gamma} - \overset{N}{\Gamma}{}^\alpha_{\gamma\beta} + w^\alpha_{\beta\gamma}, \tag{2.7}$$

$$\overset{N}{R}{}^\alpha_{\beta\gamma\delta} = X_\delta \overset{N}{\Gamma}{}^\alpha_{\beta\gamma} - X_\gamma \overset{N}{\Gamma}{}^\alpha_{\beta\delta} + \overset{N}{\Gamma}{}^\varphi_{\beta\gamma} \overset{N}{\Gamma}{}^\alpha_{\varphi\delta} - \overset{N}{\Gamma}{}^\varphi_{\beta\delta} \overset{N}{\Gamma}{}^\alpha_{\varphi\gamma} + \overset{N}{\Gamma}{}^\alpha_{\beta\varphi} w^\varphi_{\gamma\delta}, \tag{2.8}$$

which are obtained from:

$$\overset{N}{T}(X_\gamma, X_\beta) = \overset{N}{T}{}^\alpha_{\beta\gamma} X_\alpha, \quad \overset{N}{R}(X_\delta, X_\gamma) X_\beta = \overset{N}{R}{}^\alpha_{\beta\gamma\delta} X_\alpha. \tag{2.7}$$

Consequently the Bianchi identities of the connection $\overset{N}{\nabla}$ in this frames $\{X_\alpha\}$ are written as:

$$\mathcal{S}_{\alpha\beta\gamma}\{\overset{N}{\nabla}_\alpha \overset{N}{R}_\delta{}^\varphi{}_{\alpha\gamma} + \overset{N}{R}_\delta{}^\varphi{}_{\alpha\psi} \overset{N}{T}{}^\psi_{\beta\gamma}\} = 0, \tag{2.9}$$

$$\mathcal{S}_{\alpha\beta\gamma}\{\overset{N}{\nabla}_\alpha \overset{N}{T}{}^\delta_{\beta\gamma} + \overset{N}{T}{}^\varphi_{\alpha\beta} \overset{N}{T}{}^\delta{}_{\gamma\varphi} - \overset{N}{R}{}^\delta_{\alpha\beta\gamma}\} = 0. \tag{2.9'}$$

In these formulas we separate the components with respect to $X_i = \delta/\delta x^i$ and those to $X_{(i)} = \partial/\partial y^i$

Proposition 2.1. *The N-almost complex structure J has in the local frames $\{X_\alpha\}$ the components J^α_β as follows:*

$$J^i{}_j = 0, \quad J^{(i)}{}_j = \delta^i_j, \quad J^i{}_{(j)} = -\delta^i_j, \quad J^{(i)}{}_{(j)} = 0 \tag{2.10}$$

Proposition 2.2. *The lift $\overset{N}{\nabla}$ of the* **d**-*connection $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ has in the frames $\{X_\alpha\}$ the components $\Gamma^\alpha{}_{\beta\gamma}$:*

$$\begin{aligned} &\overset{N}{\Gamma}{}^i_{jk} = L^i_{jk}, \quad \overset{N}{\Gamma}{}^i{}_{(j)k} = 0, \quad \overset{N}{\Gamma}{}^i_{j(k)} = C^i_{jk}, \quad \overset{N}{\Gamma}{}^i{}_{(j)(k)} = 0, \\ &\overset{N}{\Gamma}{}^{(i)}_{jk} = 0, \quad \overset{N}{\Gamma}{}^{(i)}_{(j)k} = L^i_{jk}, \quad \overset{N}{\Gamma}{}^{(i)}_{j(k)} = 0, \quad \overset{N}{\Gamma}{}^{(i)}_{(j)(k)} = C^i_{jk}. \end{aligned} \tag{2.11}$$

We remark the simplicity of these components of $\overset{N}{\nabla}$ if we compare them with the components of a linear connection ∇ on TM expressed in the natural local basis [28, 63].

Proposition 2.3. *The coefficients $w^\alpha{}_{\beta\gamma}$ of non-holonomy are given by*

$$\begin{aligned} &w^i_{jk} = 0, \quad w^i_{(j)k} = 0, \quad w^i_{j(k)} = 0, \quad w^i_{(j)(k)} = 0, \\ &w^{(i)}_{jk} = R^i{}_{jk}, \quad w^{(i)}_{(j)k} = -\frac{\partial N^i_k}{\partial y^j}, \quad w^{(i)}_{j(k)} = \frac{\partial N^i_j}{\partial y^k}, \quad w^{(i)}_{(j)(k)} = 0. \end{aligned} \tag{2.12}$$

These equalities hold by vertue of (2.5) and (1.7), I.

Proposition 2.4. *The torsion tensor field $\overset{N}{T}$ of the lift $\overset{N}{\nabla}$ of the* **d**-*connection $L\Gamma(N)$ has, in the local frames $\{X_\alpha\}$ the components $\overset{N}{T}{}^\alpha{}_{\beta\gamma}$ as follows:*

$$\begin{aligned} &\overset{N}{T}{}^i{}_{jk} = T^i{}_{jk}, \quad \overset{N}{T}{}^i{}_{(j)k} = -C^i_{kj}, \quad \overset{N}{T}{}^i{}_{j(k)} = C^i_{jk}, \quad \overset{N}{T}{}^i{}_{(j)(k)} = 0, \\ &\overset{N}{T}{}^{(i)}{}_{jk} = R^i{}_{jk}, \quad \overset{N}{T}{}^{(i)}{}_{(j)k} = -P^i{}_{kj}, \quad \overset{N}{T}{}^{(i)}{}_{j(k)} = P^i{}_{jk}, \quad \overset{N}{T}{}^{(i)}{}_{(j)(k)} = S^i{}_{jk}. \end{aligned} \tag{2.13}$$

Proposition 2.5. *The curvature tensor field $\overset{N}{R}$ of the lift $\overset{N}{\nabla}$ of the* **d**-*connection $L\Gamma(N)$ has, in the local frames $\{X_\alpha\}$, the coefficients $\overset{N}{R}{}^\alpha_{\beta\gamma\delta}$:*

$$\begin{aligned} &\overset{N}{R}{}_j{}^i{}_{kl} = R^i_{jkl}, \quad \overset{N}{R}{}^i_{(j)kl} = 0, \quad \overset{N}{R}{}^{(i)}_j{}_{kl} = 0, \quad \overset{N}{R}{}^{(i)}_{(j)kl} = R_j{}^i{}_{kl}, \\ &\overset{N}{R}{}_j{}^i{}_{k(l)} = P_j{}^i{}_{kl}, \quad \overset{N}{R}{}^i_{(j)l(l)} = 0, \quad \overset{N}{R}{}^{(i)}_j{}_{l(l)} = 0, \quad \overset{N}{R}{}^{(i)}_{(j)k(l)} = P_j{}^i{}_{kl}, \\ &\overset{N}{R}{}_j{}^i{}_{(k)(l)} = S_j{}^i{}_{kl}, \quad \overset{N}{R}{}^i_{(j)(k)(l)} = 0, \quad \overset{N}{R}{}^{(i)}_{j(k)(l)} = 0, \quad \overset{N}{R}{}^{(i)}_{(j)(k)(l)} = S_j{}^i{}_{kl}. \end{aligned} \tag{2.14}$$

These formulas hold by vertue of (1.4) and (2.8)'.

We remark in (2.14) the simplicity of the coefficients of the curvature $\overset{N}{R}$ in the adaptated frames $\{X_\alpha\}$. This means that Einstein's equations will appear in a simplified form.

Putting $R_{ij} = R_i{}^h{}_{jh}$ etc. as in (2.6), I, we have

Proposition 2.6. *The Ricci tensor field $\overset{N}{R}_{\alpha\beta}$ of the lift of the* **d**-*connection $L\Gamma(N)$, in the frames $\{X_\alpha\}$ is given by:*

$$\overset{N}{R}_{ij} = R_{ij}, \quad \overset{N}{R}_{(i)j} = \overset{1}{P}_{ij}, \quad \overset{N}{R}_{i(j)} = -\overset{2}{P}_{ij}, \quad \overset{N}{R}_{(i)(j)} = S_{ij}. \tag{2.15}$$

Let $K = K^\alpha_\beta X_\alpha \otimes \delta\zeta^\beta$ be a tensor field on the tangent bundle TM. We remark that every component K^i_j, $K^i_{(j)}$, $K^{(i)}_j$, $K^{(i)}_{(j)}$ is a **d**-tensor field of type (1,1). Then, for the covariant derivative, we have the expressions:

$$\begin{aligned} &\overset{N}{\nabla}_k K^i_j = K^i_{j|k}, \quad \overset{N}{\nabla}_k K^i_{(j)} = K^i_{(j)|k}, \quad \overset{N}{\nabla}_k K^{(i)}_j = K^{(i)}_j{}_{|k}, \quad \overset{N}{\nabla}_k K^{(i)}_{(j)} = K^{(i)}_{(j)|k}, \\ &\overset{N}{\nabla}_{(k)} K^i_j = K^i_j|_k, \quad \overset{N}{\nabla}_{(k)} K^i_{(j)} = K^i_{(j)}|_k, \quad \overset{N}{\nabla}_{(k)} K^{(i)}_j = K^{(i)}_j|_k, \quad \overset{N}{\nabla}_{(k)} K^{(i)}_{(j)} = K^{(i)}_{(j)}|_k. \end{aligned} \tag{2.16}$$

This rule is a general one.

Consequently, the Bianchi identities (2.9) give the Bianchi identities (1.5).

§ 3. **Almost Hermitian N-model** H^{2n} **of the space** M^n. Let us consider a generalized Lagrange space $M^n=(M, g_{ij}(x, y))$ and a fixed non-linear connection N on TM. The fundamental tensor field $g_{ij}(x, y)$ determines a unique Riemannian metric on the tangent bundle TM:

$$G(x, y)=g_{ij}(x, y)dx^i\otimes dx^j+g_{ij}(x, y)\delta y^i\otimes\delta y^j. \tag{3.1}$$

In the Finsler spaces this metric has been considered by *M.* M a t s u m o t o [30].

The metric $G(x, y)$ will be called the N-lift to TM of the metric $g_{ij}(x, y)$.

Proposition 3.1. *The horizontal N and vertical V distributions are orthogonal with respect to the metric* $G(x, y)$.

Proposition 3.2. *In the local frames* $\{X_\alpha\}$ *the metric* $G(x, y)$ *has the components*:

$$G_{ij}=g_{ij},\ G_{i(j)}=0,\ G_{(i)j}=0,\ G_{(i)(j)}=g_{ij}. \tag{3.2}$$

The following important result holds:

Theorem 3.1. *The pair* (G, J), *formed by the lift G of the metric* g_{ij} *and the N-almost complex structure J, is an almost Hermitian structure on the tangent bundle TM.*

Proof. It is easy to see that

$$G_{\alpha\beta}J^\alpha_\gamma J^\beta_\delta=G_{\gamma\delta} \tag{3.3}$$

Conversely, we have:

Theorem 3.2. *Let* $G(x, y)$ *be a Riemannian metric on TM, N the complementary orthogonal distribution to the vertical distribution V with respect to G and J the almost complex structure determined by N. If* (G, J) *is an almost Hermitian structure on TM, then there is an unique generalized Lagrange metric* $g_{ij}(x, y)$ *on M such that its N-lift is* $G(x, y)$.

Proof. Let N^j_i be the functions given by the equations $G(\partial/\partial x^i-N^j_i\partial/\partial y^j,\ \partial/\partial y^k)=0$. Then on TM it is uniquely determined a non-linear connection N for which $\delta/\delta x^i=\partial/\partial x^i-N^j_i\partial/\partial y^j$ is a local adapted frame. Considering the frames $\{X_\alpha\}=\{\delta/\delta x^i, \partial/\partial y^i\}$, G has the coefficients $G_{\alpha\beta}=G(X_\alpha, X_\beta)$ of the form

$$G_{ij}=h_{ij}(x, y),\ G_{i(j)}=0,\ G_{(i)j}=0,\ G_{(i)(j)}=g_{ij},$$

where $h_{ij}(x, y)$ and $g_{ij}(x, y)$ are symmetric non-degenerate covariant **d**-tensor fields on M. In the frames $\{X_\alpha\}$ the N-almost complex structure J has the coefficients (2.10). But (G, J) being an almost Hermitian structure, it results $h_{ij}(x, y)=g_{ij}(x, y)$. It is easy to see that G is the N-lift of g_{ij}. q.e.d.

It follows that we have a bijective mapping between the set of the almost Hermitian structure (G, J) on TM and the set of the pair $(N, g_{ij}(x, y))$ formed by a non-linear connection N on TM and a generalized Lagrange metric on M.

The space $H^{2n}=(M, (G, J))$ will be called the *N-almost Hermitian model* of generalized Lagrange spaces $M^n=(M, g_{ij}(x, y))$ endowed with a fixed non-linear connection N. Shortly we say „H^{2n} is the N-model of M^n".

Proposition 3.3. *The N-model* H^{2n} *of* M^n *is an Hermitian space if and only if the* **d**-*tensors* R^i_{jk} *and* t^i_{jk} *vanish.*

To the almost Hermitian space H^{2n} is associated the 2-form

$$\theta=\frac{1}{2}\theta_{\alpha\beta}\delta\zeta^\alpha\wedge\delta\zeta^\beta,\quad \theta_{\gamma\beta}=G_{\alpha\gamma}J^\alpha_\beta. \tag{3.4}$$

The exterior differential of this 2-form is given by

$$d\theta=\frac{1}{3!}K_{\alpha\beta\gamma}\delta\zeta^\alpha\wedge\delta\zeta^\beta\wedge\delta\zeta^\gamma, \tag{3.5}$$

whose coefficients are:

$$K_{\alpha\beta\gamma}=\underset{\alpha\beta\gamma}{\mathcal{S}}\{\overset{N}{\nabla}_\alpha\theta_{\beta\gamma}+\overset{N}{T}{}^\varphi_{\alpha\beta}\theta_{\gamma\varphi}\}, \tag{3.6}$$

$\overset{N}{\nabla}$ being the lift of the **d**-connection $L\Gamma(N)=(L^i_{jk}, C^i_{jk})$

Obviously, the last equalities are obtained from:

$$d\theta(X, Y, Z)=\underset{X,Y,Z}{\mathcal{S}}\{X\theta(Y, Z)-\theta([X, Y], Z)\},\ \forall X, Y, Z\in\mathcal{X}(TM)$$

expressed in the adaptated frames $\{X_\alpha\}$.

Proposition 3.4. *The coefficients* $\theta_{\alpha\beta}$ *of the 2 form* θ *in the frames* $\{X_\alpha\}$ *are given by*

$$\theta_{ij}=0,\ \theta_{(i)j}=g_{ij},\ \theta_{i(j)}=-g_{ij},\ \theta_{(i)(j)}=0. \tag{3.7}$$

Proposition 3.5. *In the local frames* $\{X_\alpha\}$ *the coefficients* $K_{\alpha\beta\gamma}$ *of the exterior differential* $d\theta$ *are given by the following* **d**-*tensor fields*:

$$\begin{aligned}
K_{ijk}&=-\{R^h{}_{ij}g_{kh}+R^h{}_{jk}g_{ih}+R^h{}_{ki}g_{jh}\},\\
K_{ij(k)}&=-\{g_{ki|j}-g_{kj|i}+g_{ih}P^h{}_{jk}-g_{jh}P^h{}_{ik}-T^h{}_{ij}g_{hk}\},\\
K_{i(j)(k)}&=-\{g_{ij}|_k-g_{ik}|_j+C^h{}_{ik}g_{jh}-C^h{}_{ij}g_{kh}+S^h{}_{jk}g_{ih}\},\\
K_{(i)(j)(k)}&=0,
\end{aligned} \tag{3.8}$$

where $L\Gamma(N)$ is a **d**-connection on M.

Theorem 3.3. *The N-almost Hermitian model* H^{2n} *is an almost Kählerian space on the tangent bundle TM if and only if the canonical connection* $C\Gamma(N)$ *has the properties*:

$$\underset{ijk}{\mathcal{S}}\{R^h{}_{ij}g_{kh}\}=0,\quad g_{ih}P^h{}_{jk}-g_{jh}P^h{}_{ik}=0,$$

C_{ijk} *is totally symmetric.*

Proof. The last proposition shows that $K_{ijk}=K_{ij(k)}=K_{i(j)(k)}=K_{(i)(j)(k)}=0$ are equivalent to the conditions of the theorem.

Corollary 3.1. *If* H^{2n} *is an almost Kählerian space on TM then it is the N-model of a Lagrange space* L^n.

§ 4. **Almost Hermitian connections.** It is interesting to get the linear connections on the tangent bundle TM which are the lifts of the metrical-**d**-connections $L\Gamma(R)$.

Definition 4.1. *A linear connection* $\overset{N}{\nabla}$ *on TM is called an N-almost Hermitian connection if*:

1. $\overset{N}{\nabla}$ *is a N-connection of Lagrange type on TM,*

2°. $\overset{N}{\nabla}$ *is a metrical connection with respect to the N-lift $G(x, y)$ of the fundamental tensor of the generalized Lagrange space $M^n = (M, g_{ij}(x, y))$.*

Theorem 4.1. *There exists a bijective correspondence between the set $\{\overset{N}{\nabla}\}$ of N-almost Hermitian connections and the set $\{L\Gamma(N)\}$ of the* **d**-*metrical connections with respect to the generalized metric $g_{ij}(x, y)$.*

Proof. If $\overset{N}{\nabla}$ is a linear connection of Lagrange type and has in adapted frames $\{X_\alpha\}$ the coefficients (L^i_{jk}, C^i_{jk}) given by (1.2), $\overset{N}{\nabla}$ is a metrical one with respect to G if and only if the **d**-connection $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ is metrical with respect to $g_{ij}(x, y)$; i.e the mapping $\overset{N}{\nabla} \to L\Gamma(N)$ is a bijection.

Corollary 4.1. *The lift $\overset{N}{\nabla}$ of the N-canonical connection $C\Gamma(N)$ characterizes this connection.*

Corollary 4.2. *There exists a unique N-almost Hermitian connection $\overset{N}{\nabla}$ for which the h- and v- components of its torsion $\overset{N}{T}$ are the given* **d**-*antisymmetric tensor fields T^i_{jk}, S^i_{jk}. $\overset{N}{\nabla}$ is the lift of the* **d**-*metrical connection $L\Gamma(N)$ from the Theorem 2.2, I.*

An interesting result is given by the following:

Theorem 4.2. (V. Oproiu). *The almost Hermitian model H^{2n} of a Lagrange space L^n is an almost Kählerian space.*

Proof. In a Lagrange space $L^n = (M, \mathcal{L})$ the non-linear connection N is given by (3.2), I. From the Corollary 3.1, I the canonical connection $C\Gamma$ of the space L^n has the properties imposed by the Theorem 3.3. Therefore H^{2n} is an almost Kahlerian space on TM.

Remark. This result was personally communicate to me by prof. Vasile Oproiu.

Also, we have:

Theorem 4.3. (Makoto Matsumoto) *The almost Hermitian model H^{2n} of a Finsler space F^n is an almost Kählerian space.*

§ 5. **Einstein's equations of generalized Lagrange spaces.** Now we determine the Einstein equations of the generalized Lagrange spaces $M^n = (M, g_{ij}(x, y))$ using the almost Hermitian models H^{2n}. To maintain the generality of the exposition, we consider in H^{2n} those N-almost Hermitian connections which are lifts of the d-metrical connections $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with prescribed h- and v-torsion T^i_{jk}, S^i_{jk}. In this case $L\Gamma(N)$ is the **d**-connection in Theorem 4.2, I.

Consequently, we suppose given:

1°. the non-linear connection N on TM,

2°. the generalized Lagrange metric $g_{ij}(x, y)$,

3°. the **d**-tensor fields $T^i_{jk}(x, y)$, $S^i_{jk}(x, y)$ such that $T^i_{jk} = -T^i_{kj}$, $S^i_{jk} = -S^i_{kj}$,

4°. the **d**-metrical connection $L\Gamma(N)$ with the coefficients:

$$L^i_{jk} = \frac{1}{2} g^{ir}\left\{\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} + g_{rh}T^h_{jk} - g_{jh}T^h_{rk} + g_{kh}T^h_{jr}\right\},$$
$$(5.1)\qquad C^i_{jk} = \frac{1}{2} g^{ir}\left\{\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r} + g_{rh}S^h_{jk} - g_{jh}S^h_{rk} + g_{kh}S^h_{jr}\right\}.$$

It follows:

$$(5.1)' \qquad L^i_{jk} - L^i_{kj} = T^i_{jk}, \quad C^i_{jk} - C^i_{kj} = S^i_{jk}.$$

There are, at all:

$$n^2 + \frac{n(n+1)}{2} + 2\,\frac{n^2(n-1)}{2} = n^3 + \frac{n(n+1)}{2}$$

functions: N^i_j, g_{ij}, T^i_{jk}, S^i_{jk} given in 1°–4°, and this number is greater then $4n^2$ for $n \geq 4$.

If $T^i_{jk} = S^i_{jk} = 0$, from (5.1) it results that $L\Gamma(N)$ coincides with the N-canonical connection $C\Gamma(N)$.

Again, we denote by $\overset{N}{T}{}^\alpha{}_{\beta\gamma}$, $\overset{N}{R}{}_\beta{}^\alpha{}_{\gamma\delta}$ the components in the local frames $\{X_\alpha\} = \left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ of the torsion and curvature of the lift $\overset{N}{\nabla}$. of the d-connection $L\Gamma(N)$. From $\overset{N}{\nabla}_\alpha G_{\beta\gamma} = 0$ we get

$$(5.2) \qquad \overset{N}{R}_{\alpha\beta\gamma\delta} + \overset{N}{R}_{\beta\alpha\gamma\delta} = 0, \quad \overset{N}{R}{}^\alpha_{\alpha\beta\gamma} = 0.$$

In the frames $\{X_\alpha\}$ the Ricci tensor $\overset{N}{R}_{\alpha\beta}$ has the components expressed in (2.15), (2.16):

$$(5.3) \qquad \overset{N}{R}_{ij} = R_{ij}, \quad \overset{N}{R}_{(i)j} = \overset{1}{P}_{ij}, \quad \overset{N}{R}_{i(j)} = -\overset{2}{P}_{ij}, \quad \overset{N}{R}_{(i)(j)} = S_{ij}.$$

Evidently:

$$(5.4) \qquad G^{ij} = g^{ij}, \; G^{(i)j} = G^{i(j)} = 0, \; G^{(i)(j)} = g^{ij}.$$

Then, the tensor field $\overset{N}{R}{}^\alpha{}_\beta = G^{\alpha\gamma}\overset{N}{R}_{\gamma\beta}$ and the scalar curvature $\overset{N}{R} = G^{\alpha\beta}\overset{N}{R}_{\alpha\beta}$ have the components:

$$(5.5) \qquad \overset{N}{R}{}^i{}_j = R^i{}_j, \; \overset{N}{R}{}^{(i)}{}_j = \overset{1}{P}{}^i{}_j, \; \overset{N}{R}{}^i{}_{(j)} = -\overset{2}{P}{}^i{}_j, \; \overset{N}{R}{}^{(i)}{}_{(j)} = S^i{}_j, \; \overset{N}{R} = R + S.$$

Here $R = g^{ij}R_{ij}$, $S = g^{ij}S_{ij}$.

Definition 5.1. *The Einstein equations of the N-almost Hermitian connection $\overset{N}{\nabla}$, which is the lift of the d-metrical connection $L\Gamma(N)$ in (5.1) are called the Einstein equations of the connection $L\Gamma(N)$ of the generalized Lagrange space M^n.*

Then, Einstein's equations of the model H^{2n} are given by

$$(5.6) \qquad \overset{N}{R}_{\alpha\beta} - \frac{1}{2}\overset{N}{R}G_{\alpha\beta} = \varkappa T_{\alpha\beta},$$

where $\varkappa$ is a constant and $T_{\alpha\beta}$ is the energy-momentum tensor field.

Theorem 5.1. *The Einstein equations of the generalized Lagrange space $M^n=(M, g_{ij}(x, y))$, corresponding to the* d*-coonection (5.1) are :*

$$R_{ij}-\frac{1}{2}(R+S)g_{ij}=\varkappa T_{ij},\quad \overset{1}{P}_{ij}=\varkappa T_{(i)j},$$
(5.7)
$$S_{ij}-\frac{1}{2}(S+R)g_{ij}=\varkappa T_{(i)(j)},\quad \overset{2}{P}_{ij}=-\varkappa T_{i(j)},$$

where $T_{ij}, \ldots$ are d*-tensor fields-„fragments" from the energy-momentum tensor field given in the adapted local frames $\{X_\alpha\}$.*

Indeed, the equality (5.6), in the local adapted frames $\{X_\alpha\}$ are equivalent to (5.7).

Corollary 5.1. *In the case of the vacuum, the Einstein equations of the generalized Lagrange space M^n are*

(5.7)' $$R_{ij}=0,\ S_{ij}=0,\ \overset{1}{P}_{ij}=0,\ \overset{2}{P}_{ij}=0.$$

We remark that the d-tensor fields R_{ij}, S_{ij}, $\overset{1}{P}_{ij}$, $\overset{2}{P}_{ij}$ are not generally symmetric. Then (5.7) or (5.7)' contains $4n^2$ equations. But the number of the given functions N^i_j, g_{ij}, $T^i{}_{jk}$, $S^i{}_{jk}$ is $n^3+\dfrac{n(n+1)}{2}\geqslant 4n^2$ for $n\geqslant 4$.

Theorem 5.2. *The conservation law, with respect to the* d*-connection $L\Gamma(N)$ in (5.1), for the generalized Lagrange space M^n, is given by :*

$$\left[R^i{}_j-\frac{1}{2}(R+S)\delta^i_j\right]_{|i}+\overset{1}{P}{}^i{}_j|_i=0,\quad \left[S^i{}_j-\frac{1}{2}(S+R)\delta^i_j\right]|_i-\overset{2}{P}{}^i{}_{j|i}=0.$$

Proof. From $\overset{N}{\nabla}_\alpha\left(\overset{N}{R}{}^\alpha_\beta-\frac{1}{2}\overset{N}{R}\delta^\alpha_\beta\right)=0$ and (5.3), (5.4), and (5.5) the statement results.

Now, we define $T^\alpha_\beta=G^{\alpha\gamma}T_{\gamma\beta}$ and in the adapted frame $\{X_\alpha\}$ we have

(5.9) $$T^i{}_j=g^{ik}T_{kj},\quad T^{(i)}{}_j=g^{ik}T_{(k)j},\ \ldots$$

Theorem 5.3. *The divergence of the energy-momentum tensor field is given by :*

(5.10) $$T^i{}_{j|i}+T^{(i)}{}_j|_i=\frac{1}{\varkappa}U_j=0,\quad T^i{}_{(j)|i}+T^{(i)}{}_{(j)}|_i=\frac{1}{\varkappa}U_{(j)}=0.$$

where U_j and $U_{(j)}$ are the d*-covariant vector fields :*

$$U_j=\frac{1}{2}g^{ik}\{T^l{}_{jr}R_k{}^r{}_{li}-T^l{}_{ir}R_k{}^r{}_{lj}-R^r{}_{jl}P_k{}^l{}_{ir}+R^r{}_{il}P_k{}^l{}_{jr}+C^r_{jl}P_k{}^l{}_{ri}-$$
$$-S^r{}_{il}P_k{}^l{}_{jr}\}-\frac{1}{2}\{R^i{}_rT^r{}_{ji}+\overset{2}{P}{}^i{}_rR^r{}_{ij}+\overset{1}{P}{}^i{}_rC^r{}_{ji}+S^i{}_rP^r{}_{ji}\},$$
(5.11)
$$U_{(j)}=\frac{1}{2}g^{ik}\{-C^l_{rj}R_k{}^r{}_{li}+P^l{}_{rj}P_k{}^r{}_{ik}-T^l{}_{ir}P_k{}^r{}_{lj}+S^l{}_{jr}S_k{}^r{}_{li}-R^l{}_{ir}S_k{}^r{}_{lj}-$$
$$-S^l{}_{ir}S_k{}^r{}_{lj}\}+\frac{1}{2}\{S^i{}_lS^l{}_{ij}+R^i{}_lC^l_{ij}-\overset{2}{P}{}^i{}_lP^l{}_{ij}\}.$$

Indeed, from the Bianchi identities (2.9), (2.9)' we get

$$\overset{N}{\nabla}_\alpha T^\alpha_\beta=\frac{1}{\varkappa}U_\beta,\quad U_\beta=\frac{1}{2}\{G^{\alpha\delta}\overset{N}{R}_\delta{}^\gamma{}_{\varphi\alpha}\overset{N}{T}{}^\varphi{}_{\beta\gamma}-G^{\alpha\delta}\overset{N}{R}_\delta{}^\gamma{}_{\varphi\beta}\overset{N}{T}{}^\varphi{}_{\alpha\gamma}+\overset{N}{R}{}^\alpha{}_\varphi\overset{N}{T}{}^\varphi{}_{\alpha\beta}\},$$

The last identities, in the adapted frames $\{X_\alpha\}$ lead to (5.10) and (5.11).

Finally we remark an interesting aspect :

Theorem 5.4. *The anti-symmetric part of the energy-momentum tensor field $T_{\alpha\beta}$, in the adapted frames $\{X_\alpha\}$, is given by :*

$$T_{ij}-T_{ji}=\frac{1}{\varkappa}\{(T^r{}_{jr}+P^r{}_{jr})_{|i}-(T^r{}_{ir}+P^r{}_{ir})_{|j}+T^r{}_{ij}(T^h{}_{hr}-P^h{}_{rh})+$$
$$+R^r{}_{ij}(T^h{}_{hr}-P^h{}_{rh})+T^r{}_{ij|r}+R^r{}_{ij}|_r\},$$
(5.12)
$$T_{(i)(j)}-T_{(j)(i)}=\frac{1}{\varkappa}\{(S^r{}_{jr}-C^r_{rj})|_i-(S^r{}_{ir}-C^r_{ri})|_j+S^r{}_{ij}|_r+$$
$$+S^r{}_{ij}(C^h_{hr}+S^h{}_{hr})\},$$
$$T_{i(j)}-T_{(j)i}=\frac{1}{\varkappa}\{(S^r{}_{jr}-C^r_{rj})_{|i}-(T^r{}_{ir}+P^r{}_{ir})|_j+C^r_{ij|r}+P^r{}_{ij}|_r+$$
$$+C^r_{ij}(T^l{}_{lr}-P^l{}_{rl})+P^r{}_{ij}(C^l_{lr}+S^l{}_{lr})\}.$$

Proof. These equations are obtained from

$$T_{\alpha\beta}-T_{\beta\alpha}=\frac{1}{\varkappa}\{\overset{N}{\nabla}_\alpha\overset{N}{T}{}^\gamma{}_{\beta\gamma}-\overset{N}{\nabla}_\beta\overset{N}{T}{}^\gamma{}_{\alpha\gamma}+\overset{N}{\nabla}_\gamma\overset{N}{T}{}^\gamma{}_{\alpha\beta}+\overset{N}{T}{}^\varphi{}_{\alpha\beta}\overset{N}{T}{}^\gamma{}_{\gamma\varphi}\},$$

taking into account (2.13) and (2.17).

Remarks. Prof. Y. Takano [59, 60], made a good remark concerning the previous theory : „The covector U_α acts as a source of the energy-momentum tensor $T_{\alpha\beta}$, or acts as a force density. Namely, then energy-momentum inflows or outflows the internal space and the torsion (the dislocation, to speak) of the external space".

— Our theory can be particularized for the Lagrange spaces L^n or for the Finsler spaces F^n. In the last case Y. Takano [59, 60], H. Ishikawa [20, 21] and S. Ikeda [17, 18] have obtained remarkable results.

§ 6. Einstein equations of the generalized locally Minkowski spaces.

Now, our purpose is to apply the previous theory to the case of the generalized locally Minkowski spaces, $\text{Mink}^n=(M,g_{ij})$ with the weakly regular metric. The case when g_{ij} is the fundamental tensor of a Finsler space has been studied very well by G. S. Asanov [2]. Generally, $\text{Mink}^n=(M, g_{ij}(x, y))$ with the weakly regular metric it is not a Lagrange or a Finsler space.

Let $C\Gamma(N)$ be the canonical connection of the fundamental tensor g_{ij}, where N is the non-linear connection given by the equations (4.3). Using the Theorem 42, I the space Mink^n with the weakly regular metric is characterized by

$$D^i{}_j=0,\ R^i{}_{jk}=0,\ R_j{}^i{}_{kh}=0,\ P^i{}_{jk}=0,\ C^i_{jk|h}=0.$$

It follows $N^i_j=0$, $L^i_{jk}=0$, $P_j{}^i{}_{kh}=0$, $T^i{}_{jk}=0$, $S^i{}_{jk}=0$ in a special coordinate system. The connection $C\Gamma(N)$ is of the form $(0,\ 0,\ C^i_{jk}(y))$. The components of the Ricci tensor are :

$$R_{ij}=0,\ \overset{1}{P}_{ij}=0,\ \overset{2}{P}_{ij}=0,\ S_{ij}=S_{ji}.$$

The scalar curvature of the space Mink^n is

$$R=0,\quad S=S(y)=g^{ij}\,S_{ij}.$$

Then, applying the results established in the previous paragraphs we get :

Theorem 6.1. *Einstein's equations of the spaces Mink^n, with weakly regular metric, corresponding to the canonical connection $C\Gamma(0)$ are given by:*

$$S_{ij}-\frac{1}{2}Sg_{ij}=\varkappa T_{(0)(j)};\ -\frac{1}{2}Sg_{ij}=\varkappa T_{ij}. \tag{6.1}$$

Theorem 6.2. *The energy-momentum tensors of the space Mink^n with weakly regular metric are :*

$$T_{ij}=-\frac{1}{\varkappa}Sg_{ij},\ T_{(i)(j)}=\frac{1}{\varkappa}\left\{S_{ij}-\frac{1}{2}Sg_{ij}\right\},\ T_{(i)j}=T_{i(j)}=0. \tag{6.2}$$

Theorem 6.3. *For the spaces Mink^n with weakly regular metric and connection $C\Gamma(0)$, the conservation law is expressed by the equations*

$$\left(S^i{}_j-\frac{1}{2}S\delta^i_j\right)|_i=0\ ;$$

which is identicaly verified, and the divergence is given by $T^i{}_{j|i}=0$. $T^{(i)}{}_{(j)}|_i=0$.

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INDEFINITE QUATERNION SPACE FORMS. II

BY

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1. Introduction. In the first part of this series, [5], F. G. Santos and the author began the study of indefinite quaternion Kaehlerian manifolds obtaining that if the real dimension of such a manifold is $\geqslant 8$, then it is an Einstein manifold.

Let $(M,\ g,\ V)$ be an indefinite quaternion Kaehlerian manifold. For any vector $X\in T_pM$, $p\in M$, the 4-plane $Q(X)=\{X,\ IX,\ JX,\ KX\}$, $\{I,\ J,\ K\}$ being a local basis of V at p is called a quaternionic 4-plane at p. If $\{X,\ Y\}$ is a non-degenerate section of T_pM, $p\in M$, such that $Q(X)=Q(Y)$, the sectional curvature for such a section is called quaternionic sectional curvature. If $Q(X)\perp Q(Y)$, the corresponding section $\{X,\ Y\}$ is called totally real and its sectional curvature is called totally real sectional curvature. If the quaternionic sectional curvature of M does depend neither on the section nor on the point, M is called an indefinite quaternion-space-form. In [5] we proved that an indefinite quaternion Kaehlerian manifold of real dimension$\geqslant 8$ is an indefinite quaternion-space-form of quaternionic sectional curvature c if and only if

$$\begin{aligned}R(X,\ Y)\,Z=\frac{c}{4}\{&g(Y,Z)\,X-g(X,Z)\,Y+g(IY,Z)\,IX-g(IX,Z)\,IY+\\&+2g(X,IY)\,IZ+g(JY,\ Z)\,JX-g(JX,\ Z)\,JY+2g(X,\ JY)\,JZ+\\&+g(KY,\ Z)\ KX-g(KX,\ Z)\,KY+2g(X,KY)\,KZ\}\end{aligned}$$

for any vector fields $X,\ Y,\ Z$ on M.

In [5] we also obtain some characterizations concerning the behaviour of the sectional curvature for certain types of sections and the behaviour of the curvature tensor for null vectors in order to have that an indefinite quaternion Kaehlerian manifold is an indefinite quaternion-space-form.

The purpose of the present paper is to obtain further characterizations in that way. In the following, if no other thing is mentioned we shall suppose that M is an indefinite quaternion Kaehlerian manifold of real dimension $4m\geqslant 12$ and index (signature) $4s$ such that $1\leqslant s\leqslant m-2$. We refer to [5] for notations and further references. Among all results of [5] we mention the following one that will be used later.

Lemma 1.1. *Let M be an indefinite quaternion Kaehlerian manifold of real dimension $4m\geqslant 8$. Then we have*