coordinate system. The connection $C\Gamma(N)$ is of the form $(0,\ 0,\ C^i_{jk}(y))$. The components of the Ricci tensor are:

$$R_{ij}=0,\ \overset{1}{P}_{ij}=0,\ \overset{2}{P}_{ij}=0,\ S_{ij}=S_{ji}.$$

The scalar curvature of the space Mink^n is

$$R=0,\quad S=S(y)=g^{ij}S_{ij}.$$

Then, applying the results established in the previous paragraphs we get:

Theorem 6.1. *Einstein's equations of the spaces Mink^n, with weakly regular metric, corresponding to the canonical connection $C\Gamma(0)$ are given by:*

$$S_{ij}-\frac{1}{2}Sg_{ij}=\varkappa T_{(0)(j)};\quad -\frac{1}{2}Sg_{ij}=\varkappa T_{ij}. \tag{6.1}$$

Theorem 6.2. *The energy-momentum tensors of the space Mink^n with weakly regular metric are:*

$$T_{ij}=-\frac{1}{\varkappa}Sg_{ij},\ T_{(0)(j)}=\frac{1}{\varkappa}\left\{S_{ij}-\frac{1}{2}Sg_{ij}\right\},\ T_{(0)j}=T_{i(0)}=0. \tag{6.2}$$

Theorem 6.3. *For the spaces Mink^n with weakly regular metric and connection $C\Gamma(0)$, the conservation law is expressed by the equations*

$$\left(S^i{}_j-\frac{1}{2}S\delta^i_j\right)\Big|_i=0;$$

which is identicaly verified, and the divergence is given by $T^i{}_{j|i}=0$. $T^{(i)}{}_{(j)}|_i=0$.

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Department of Mathematics
University of Iaşi
6600 — Iaşi, R. S. România

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INDEFINITE QUATERNION SPACE FORMS. II

BY

JUAN DE DIOS PEREZ

1. Introduction. In the first part of this series, [5], F. G. Santos and the author began the study of indefinite quaternion Kaehlerian manifolds obtaining that if the real dimension of such a manifold is ≥ 8, then it is an Einstein manifold.

Let $(M,\ g,\ V)$ be an indefinite quaternion Kaehlerian manifold. For any vector $X\in T_pM$, $p\in M$, the 4-plane $Q(X)=\{X,\ IX,\ JX,\ KX\}$, $\{I,\ J,\ K\}$ being a local basis of V at p is called a quaternionic 4-plane at p. If $\{X,\ Y\}$ is a non-degenerate section of T_pM, $p\in M$, such that $Q(X)=Q(Y)$, the sectional curvature for such a section is called quaternionic sectional curvature. If $Q(X)\perp Q(Y)$, the corresponding section $\{X,\ Y\}$ is called totally real and its sectional curvature is called totally real sectional curvature. If the quaternionic sectional curvature of M does depend neither on the section nor on the point, M is called an indefinite quaternion-space-form. In [5] we proved that an indefinite quaternion Kaehlerian manifold of real dimension≥ 8 is an indefinite quaternion-space-form of quaternionic sectional curvature c if and only if

$$\begin{aligned}R(X,\ Y)Z=\frac{c}{4}\{&g(Y,Z)X-g(X,Z)Y+g(IY,Z)IX-g(IX,Z)IY+\\&+2g(X,IY)IZ+g(JY,Z)JX-g(JX,Z)JY+2g(X,JY)JZ+\\&+g(KY,Z)KX-g(KX,Z)KY+2g(X,KY)KZ\}\end{aligned}$$

for any vector fields $X,\ Y,\ Z$ on M.

In [5] we also obtain some characterizations concerning the behaviour of the sectional curvature for certain types of sections and the behaviour of the curvature tensor for null vectors in order to have that an indefinite quaternion Kaehlerian manifold is an indefinite quaternion-space-form.

The purpose of the present paper is to obtain further characterizations in that way. In the following, if no other thing is mentioned we shall suppose that M is an indefinite quaternion Kaehlerian manifold of real dimension $4m\geq 12$ and index (signature) $4s$ such that $1\leq s\leq m-2$. We refer to [5] for notations and further references. Among all results of [5] we mention the following one that will be used later.

Lemma 1.1. *Let M be an indefinite quaternion Kaehlerian manifold of real dimension $4m\geq 8$. Then we have*

$$[R(X, Y), I] = -[\rho/(m+2)]\,\{g(KX, Y)\,J - g(JX, Y)\,K\},$$

$$[R(X, Y), J] = -[\rho/(m+2)]\,\{-g(KX, Y)I + g(IX, Y)\,K\},$$

$$[R(X, Y), K] = -[\rho/(m+2)]\,\{g(JX, Y)I - g(IX, Y)\,J\},$$

for any vector fields X and Y on M, $4m\rho$ being the scalar curvature of M.

2. First results. To use in the rest of the paper, we shall prove the following

Lemma 2.1. *Let $\{X, Y\}$ be a totally real plane of T_pM, $p \in M$, and $\{I, J, K\}$ a local basis of V at p. Then*

$$R(X, Y, Y, X) + R(X, IY, IY, X) + R(X, JY, JY, X) + R(X, KY, KY, X) = \\ = [\rho/(m+2)]\, g(X, X)\, g(Y, Y).$$

Proof. From Lemma 1.1 and the first identity of Bianchi we have

$$R(X, IX, JY, KY) = R(X, KY, X, KY) + R(X, JY, X, JY) \tag{2.1}$$

and

$$R(X, IX, Y, IY) = R(X, IY, X, IY) + R(X, Y, Y, X). \tag{2.2}$$

But again from Lemma 1.1,

$$R(X, IX, JY, KY) = -R(X, IX, Y, IY) - [\rho/(m+2)]\,g(X, X)\, g(Y, Y). \tag{2.3}$$

The result follows from (2.1), (2.2) and (2.3).

In the following $\{I, J, K\}$ denotes a local basis of V at the considered point.

Theorem 2.2. *M is an indefinite quaternion-space-form if and only if $R(X, Y, \Phi Y, X) = 0$ for any totally real section $\{X, Y\}$ of T_pM, $p \in M$, such that $g(X, X) = 1$ and $g(Y, Y) = 0$, $\Phi = I, J, K$.*

Proof. Consider $X \in T_pM$ such that $g(X, X) = 1$. We shall denote by W the orthogonal complement of $Q(X)$ in T_pM. We define the following bilinear symmetric function f over W

$$f(Y, Z) = R(X, Y, IZ, X) + R(X, IY, Z, X), \quad Y, Z \in W.$$

Then if Y is a null vector of W we have $f(Y, Y) = 2R(X, Y, IY, X) = 0$. Thus $f(Y, Z) = tg\,(Y, Z)$ for some constant t and all Y, $Z \in W$ (see [1] or [3]). Thus $f(Z, IZ) = tg\,(Z, IZ) = 0 = -R(X, Z, Z, X) + R(X, IZ, IZ, X)$ for any $Z \in W$, that is, $R(X, Z, Z, X) = R(X, IZ, IZ, X)$.

By an analogous reasoning using the bilinear symmetric functions on $W f'$ and f'' corresponding to J and K respectively, we get

$$R(X, Z, Z, X) = R(X, IZ, IZ, X) = R(X, JZ, JZ, X) = R(X, KZ, KZ, X) \tag{2.4}$$

for any $Z \in W$.

Choosing now a vector $Z \in W$ such that $g(Z, Z) = 1$ or $g(Z, Z) = -1$, $\{X, Z\}$ is a totally real section of type (‡) or (±) respectively. Then from (2.4) and Lemma 2.1 we have

$$\sigma(X, IZ) = \rho/[4(m+2)], \tag{2.5}$$

that is, there exists a constant d such that $\sigma(X, IZ) \geqslant d$ for any nondegenerate totally real section of type (‡) or (±). Then it is easy to prove similarly as in [3] that M has constant totally real sectional curvature at p. Now it follows ([5], Theorem 5.5) that M is an indefinite quaternion-space-form. The converse is trivial.

By a similar proof we have

Theorem 2.3. *M is an indefinite quaternion-space-form if and only if $R(X, \Phi Y, \Psi Y, X) = 0$ for any totally real section $\{X, Y\}$ of T_pM, $p \in M$, such that $g(X, X) = 1$ and $g(Y, Y) = 0$ and any Φ, $\Psi = I, J, K$, $\Phi \neq \Psi$.*

Theorem 2.4. *If there exists a constant $d > 0$ such that $|R(X, Y, \Phi Y, X)| \leqslant \leqslant d$ for any totally real section of type (±) $\{X, Y\}$ of T_pM, $p \in M$, and $\Phi = I, J, K$, then M is an indefinite quaternion-space-form.*

Proof. Let X be a vector of T_pM such that $g(X, X) = 1$. Let W be as in the proof of Theorem 2.2, and consider again the bilinear symmetric function f over W given by $f(Y, Z) = R(X, Y, IZ, X) + R(X, IY, Z, X)$ for any $Y, Z \in W$. If $Y \in W$ is such that $g(Y, Y) = -1$, $|f(Y, Y)| = 2\,|R(X, Y, IY, X)| \leqslant 2d = a$. Now we have again $f(Y, Z) = tg\,(Y, Z)$ for any Y, $Z \in W$ and some constant t. Following as in the proof of Theorem 2.2 we obtain the result.

As before we also have the following

Theorem 2.5. *If there exists a constant $d > 0$ such that $|R(X, \Phi Y, \Psi Y, X)| \leqslant d$ for any totally real section $\{X, Y\}$ of type (±) of T_pM, $p \in M$, and any Φ, $\Psi = I, J, K$, $\Phi \neq \Psi$, then M is an indefinite quaternion-space-form.*

The following Corollaries follow easily from Theorems 2.4 and 2.5 respectively.

Corollary 2.6. *M is an indefinite quaternion-space-form if and only if $R(X, Y, \Phi Y, X) = 0$ for any totally real section of type (±) $\{X, Y\}$ of T_pM, $p \in M$, $\Phi = I, J, K$.*

Corollary 2.7. *M is an indefinite quaternion-space-form if and only if $R(X, \Phi Y, \Psi Y, X) = 0$ for any totally real section of type (±) $\{X, Y\}$ of T_pM, $p \in M$, Φ, $\Psi = I, J, K$, $\Phi \neq \Psi$.*

Theorem 2.8. *Let M be an indefinite quaternion Kaehlerian manifold of real dimension $4m \geqslant 8$. Then if for any totally real plane $\{X, Y\}$ of T_pM, $p \in M$, the curvartures $R(X, \Phi X, \Psi Y, Y)$ are constant for any Φ, $\Psi = I, J, K$, $\Phi \neq \Psi$, M is an indefinite quaternion-space-form.*

Proof. If $\{X, Y\}$ is a totally real section of T_pM, $\{X, (IY + JY)/\sqrt{2}\}$ is also a totally real section. Choosing $\Phi = I$, $\Psi = J$, we have that $\frac{1}{2}\,R(X, IX, -KY - Y, IY + JY)$ is a constant, say b. Developping this curvature we have that $R(X, IX, IY, Y) = b$. If we take now the section $\{X, (JY + KY)/\sqrt{2}\}$, $\Phi = J$, $\Psi = K$ (respectively, the section $\{X, (KY + IY)/\sqrt{2}\}$, $\Phi = K$, $\Psi = I$) the result follows from Theorem 6.1 of [5].

3. Results concerning null vectors. Theorem 3.1. *M is an indefinite quaternion-space-form if and only if $R(Z, \Phi Z, \Psi Z, Z) = 0$ for any null vector $Z \in T_pM$, $p \in M$, and any Φ, $\Psi = I, J, K$, $\Phi \neq \Psi$.*

Proof. Let us suppose that X and Y are unit vectors of T_pM spanning a totally real section of type (±). Then $X + Y$ and $X - Y$ are null vectors of T_pM. Then, from the assumption and taking $\Phi=I$, $\Psi=J$, developping $R(X+Y, IX+IY, JX+JY, X+Y)$ and $R(X-Y, IX-IY, JX-JY, X-Y)$ and adding the results we have

$$0=R(X, IX, JX, X)+R(X, IX, JY, Y)+4R(X, IY, JX, Y)+ \\ +R(X, IY, JX, X)+R(Y, IY, JY, Y). \tag{3.1}$$

By the same reasoning applied to $X+IY$ and $X-IY$ we obtain

$$0=R(X, IX, JX, X)-R(X, IX, JY, Y)-4R(X, Y, JX, IY)+ \\ +R(Y, IY, JX, X)-R(Y, IY, JY, Y). \tag{3.2}$$

From (3.1) and (3.2), applying the identity of Bianchi it follows

$$0=R(X, IX, JX, X)+3R(X, JX, IY, Y) \tag{3.3}$$

and

$$0=R(X, IX, JY, Y)+R(Y, IY, JY, Y)+4R(X, Y, X, KY)+ \\ +2R(X, JX, IY, Y), \tag{3.4}$$

Now, using the vectors $X+JY$ and $X-JY$ we get

$$0=R(X, IX, JX, X)+R(X, IX, JY, Y)-4R(X, KY, Y, X)- \\ R(Y, IY, JX, X)-R(Y, IY, JY, Y). \tag{3.5}$$

From (3.2), (3.5) and Lemma 1.1 it follows

$$R(X, IX, JX, X)=R(Y, IY, JY, Y). \tag{3.6}$$

From (3.6) and (3.3) we have

$$0=R(Y, IY, JY, Y)+3R(X, JX, IY, Y), \tag{3.7}$$

and changing X by Y in (3.3), we get

$$0=R(Y, IY, JY, Y)+3R(Y, JY, IX, X). \tag{3.8}$$

From (3.7) and (3.8),

$$R(X, JX, IY, Y)=R(Y, JY, IX, X). \tag{3.9}$$

Finally, from (3.3), (3.4) and (3.9) we have

$$R(X, Y, KY, X)=0 \tag{3.10}$$

for any totally real plane of type $(\pm)$ $\{X, Y\}$ of T_pM, $p\in M$. Taking $\Phi=I$ and $\Psi=K$ (respectively $\Phi=J$ and $\Psi=K$) applying the same reasoning we obtain

$$R(X, Y, JY, X)=R(X, Y, IY, X)=0, \tag{3.11}$$

for any totally real plane of type $(\pm)$ $\{X, Y\}$ of T_pM, $p\in M$. The result follows from (3.10) and (3.11) having in mind Corollary 2.6.

Corollary 3.2. *M is an indefinite quaternion-space-form if and only if $R(X, \Phi X, \Psi Y, Y)=0$ for any null vectors $X, Y\in T_pM$, $p\in M$, $\Phi\neq\Psi$.*

4. Mean curvature for totally real q-planes. A q-plane π $(2\leqslant q\leqslant m)$ of T_pM, $p\in M$, is called totally real if there exists a basis of π $\{X_1, X_2, \ldots, X_q\}$ such that $Q(X_i)\perp Q(X_j)$ for any $i\neq j$, $i, j\in\{1, \ldots, q\}$. Given a subspace S^q of T_pM, $p\in M$, let us suppose that $\{E_1, \ldots, E_{4m}\}$ is an orthonormal basis of T_pM such that $\{E_1, \ldots, E_q\}$ is a basis of S^q (for an orthonormal basis we understand a basis $\{X_1, \ldots, X_{4m}\}$ such that $|g(X_i, X_i)|=1$ and $g(X_i, X_j)=0$ for any $i\neq j$, $i, j\in\{1, \ldots, 4m\}$). The mean curvature of S is then defined by

$$m(S)=1/[q(4m-q)]\sum_{i=1}^{q}\sum_{\alpha=q+1}^{4m}\sigma(E_i, E_\alpha).$$

For a quaternion Kaehlerian manifold it is known that if the mean curvature for a totally real q-plane is independent of the choice of totally real q-planes at each point for $2\leqslant q\leqslant m$, then M is a quaternion-space-form, [2]. Here we shall prove the following

Theorem 4.1. *If the mean curvature for totally real non-degenerate q-planes of null index is independent of the choice of such q-planes at each point for $2\leqslant q\leqslant m-s$, then M is an indefinite quaternion-space-form.*

Proof. Let π be a totally real non-degenerate q-plane of null index of T_pM, $p\in M$. Let us consider the orthonormal basis $\{E_1, \ldots, E_q, E_{q+1}, \ldots, E_m, IE_1, \ldots, IE_m, JE_1, \ldots, JE_m, KE_1, \ldots, KE_m\}$ of T_pM such that $\{E_1, \ldots, E_q\}$ is a basis of π.

Then we have

$$m(\pi)=1/[q(4m-q)][\sum_{i=1}^{q}\sum_{\alpha=q+1}^{m}\{\sigma(E_i, E_\alpha)+\sigma(E_i, IE_\alpha)+\sigma(E_i, JE_\alpha)+$$

$$+\sigma(E_i, KE_\alpha)\}+\sum_{i=1}^{q}\{\sigma(E_i, IE_i)+\sigma(E_i, JE_i)+\sigma(E_i, KE_i)\}+$$

$$+\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq 1}}^{q}\{\sigma(E_i, IE_j)+\sigma(E_i, JE_j)+\sigma(E_i, KE_j)\}]=$$

$$q(m-q)\rho/[q(4m-q)(m+2)]+1[(q(4m-q)][\sum_{i=1}^{q}\{\sigma(E_i, IE_i)+\sigma(E_i, JE_i)+$$

$$+\sigma(E_i, KE_i)\}+\sum_{i=1}\sum_{\substack{j=1\\ j\neq 1}}\{(E_i, IE_j)+\sigma(E_i, JE_j)+\sigma(E_i, KE_j)\}].$$

But in our conditions it is easy to see that $\sigma(E_i, IE_i)+\sigma(E_i, JE_i)+$ $+\sigma(E_i, KE_i)=\dfrac{3\rho}{m+2}$. Then from the above expression and Lemma 2.1 we have

$$m(\pi)=[\rho/(4m-q)]-\sum_{i,j=1}^{q}\sigma(E_i, E_j).$$

Then, if $m(\pi)$ is a constant, the above expression proves that $\sum_{i,j=1}^{q}\sigma(E_i, E_j)$ is a constant. Then it follows, [3], [4], that M is an indefinite quaternion-space-form. The converse is easy to see.

the canonical projection of TM to M. Also, denote by $d\pi : TTM \to TM$ the differential of π and define the vertical subbundle VTM of TTM as the kernel of $d\pi$. A complementary distribution N to VTM in TTM is called a *non-linear connection* on TM (see M a t s u m o t o [15]). The fibres of both vector bundles VTM and N are of the same dimension n. In this section we have the following range of indices: $\alpha, \beta, \gamma, \ldots = 1, \ldots, n$ and $a, b, c, \ldots = 1, \ldots, p$.

Now, we consider a canonical chart (U, φ, R^{2n}) on TM, where U is an open set of TM and $\varphi : U \to R^{2n}$ is a diffeomorphism of U on $\varphi(U)$. For each $z \in U$ denote by $(u^1, \ldots, u^n ; v^1, \ldots, v^n)$ its coordinates in this chart. Then the non-linear connection N is determined by n^2 differentiable functions N^β_α on each U satisfying

$$N^{\beta'}_{\alpha'} = \frac{\partial u^{\beta'}}{\partial u^\beta}\left(\frac{\partial u^\alpha}{\partial u^{\alpha'}} N^\beta_\alpha + \frac{\partial^2 u^\beta}{\partial u^{\alpha'} \partial u^{\gamma'}} v^{\gamma'}\right), \tag{1.1}$$

where $N^{\beta'}_{\alpha'}$ are the corresponding functions on the domain U' of another local chart (U', φ', R^{2n}) with $U \cap U' \neq \varnothing$. Moreover, if $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ is the natural local field of frames on TM then $\{\delta/\delta u^\alpha\}$ given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta}, \quad \alpha = 1, \ldots, n, \tag{1.2}$$

is a local field of frames on the distribution N.

Next, let $\pi_E : E \to TM$ be a vector bundle over TM with standard fibre R^p. Denote by $\Gamma(E)$ and $\Gamma(TTM)$ the module of differentiable sections of E and respectively TTM. Suppose $\nabla : \Gamma(TTM) \times \Gamma(E) \to \Gamma(E)$ is a linear connection on E, that is, we have

$$\begin{cases} \nabla_X(fV + W) = (Xf)\, V + f\nabla_X V + \nabla_X W, \\ \nabla_{fX+Y}(V) = f\nabla_X V + \nabla_Y V, \end{cases} \tag{1.3}$$

for any differentiable function f on TM, $X, Y \in \Gamma(TTM)$ and $V, W \in \Gamma(E)$.

The pair $VFC = (N, \nabla)$ of a non-linear connection on TM and a linear connection on a vector bundle E over TM is called a *vectorial Finsler connection* on E.

Taking into account that for many cases in Finslerian geometry the computations are locally made we need to express locally a vectorial Finsler connection. In order to do this we consider a vectorial chart (U, Φ, R^p) on the vector bundle E, where U is an open set of TM and $\Phi : \pi_E^{-1}(U) \to U \times R^p$ is a diffeomorphism. Let (U', Φ', R^p) be another vectorial chart such that $U \cap U' \neq \varnothing$. Denote by $\{S_a\}$ (resp. $\{S_{a'}\}$) the local basis of differentiable sections of E on U (resp. U'). Then there exist p^2 differentiable functions $G^a_{a'}$ defined on $U \cap U'$ and satisfying

$$S_{a'} = G^a_{a'}\, S_a. \tag{1.4}$$

Now, suppose (N, ∇) be a vectorial Finsler connection on the vector bundle E. Then since N is a non-linear connection on TM there exist n^2 differentiable functions N^β_α on U satisfying (1.1). On the other hand, taking into account that $\{\delta/\delta u^\alpha, \partial/\partial v^\alpha\}$ is a local field of frames on TM, we define on U the differentiable functions $F_a{}^b{}_\alpha$ and $C_a{}^b{}_\alpha$ by

$$\nabla_{\frac{\delta}{\delta u^\alpha}} S_a = F_a{}^b{}_\alpha S_b\,; \qquad \nabla_{\frac{\partial}{\partial v^\alpha}} S_a = C_a{}^b{}_\alpha S_b. \tag{1.5}$$

Thus (N, ∇) induces a triplet of differentiable functions $(N^\beta_\alpha, F_a{}^b{}_\alpha, C_a{}^b{}_\alpha)$ on each domain of a canonical chart of TM. By a similar computation with that one for linear connections we obtain that the existence of a vectorial Finsler connection (N, ∇) on E is equivalent with the existence of a triplet of differentiable functions $(N^\beta_\alpha, F_a{}^b{}_\alpha, C_a{}^b{}_\alpha)$ on each domain U of TM satisfying (1.1) and

$$\begin{cases} F_{a'}{}^{b'}{}_{\alpha'} = \dfrac{\partial u^\alpha}{\partial u^{\alpha'}} \bar{G}^{b'}_b \left(G^a_{a'} F_a{}^b{}_\alpha + \dfrac{\delta G^b_{a'}}{\delta u^\alpha}\right), \\ C_{a'}{}^{b'}{}_{\alpha'} = \dfrac{\partial u^\alpha}{\partial u^{\alpha'}} \bar{G}^{b'}_b \left(G^a_{a'} C_a{}^b{}_\alpha + \dfrac{\partial G^b_{a'}}{\partial v^\alpha}\right), \end{cases} \tag{1.6}$$

for any change of both, the canonical chart and the vectorial chart on TM and respectively E, where $[\bar{G}^{b'}_b]$ is the inverse matrix of $[G^b_{b'}]$.

Now, let $VFC = (N, \nabla)$ be a vectorial Finsler connection on E and R^* be the curvature tensor field of ∇ given by

$$R^*(X, Y)\, V = \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V, \tag{1.7}$$

for any $X, Y \in \Gamma(TTM)$ and $V \in \Gamma(E)$. Then the local components $R_a{}^b{}_{\alpha\beta}$, $P_a{}^b{}_{\alpha\beta}$ and $S_a{}^b{}_{\alpha\beta}$ given by

$$\begin{gathered} R^*\left(\frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha}\right) S_a = R_a{}^b{}_{\alpha\beta} S_b\,; \qquad R^*\left(\frac{\partial}{\partial v^\beta}, \frac{\delta}{\delta u^\alpha}\right) S_a = P_a{}^b{}_{\alpha\beta} S_b, \\ R^*\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) S_a = S_a{}^b{}_{\alpha\beta} S_b \end{gathered} \tag{1.8}$$

are called the *Finsler curvature tensors* of the vectorial Finsler connection VFC. By a direct computation using (1.5), (1.7) and (1.8) we obtain

$$\begin{aligned} R_a{}^b{}_{\alpha\beta} &= \frac{\delta F_a{}^b{}_\alpha}{\delta u^\beta} - \frac{\delta F_a{}^b{}_\beta}{\delta u^\alpha} + F_a{}^c{}_\alpha F_c{}^b{}_\beta - F_a{}^c{}_\beta F_c{}^b{}_\alpha + C_a{}^b{}_\gamma R^\gamma{}_{\alpha\beta}, \\ P_a{}^b{}_{\alpha\beta} &= \frac{\partial F_a{}^b{}_\alpha}{\partial v^\beta} - \frac{\delta C_a{}^b{}_\beta}{\delta u^\alpha} + F_a{}^c{}_\alpha C_c{}^b{}_\beta - C_a{}^c{}_\beta F_c{}^b{}_\alpha + \frac{\partial N^\gamma_\alpha}{\partial v^\beta} C_a{}^b{}_\gamma, \\ S_a{}^b{}_{\alpha\beta} &= \frac{\partial C_a{}^b{}_\alpha}{\partial v^\beta} - \frac{\partial C_a{}^b{}_\beta}{\partial v^\alpha} + C_a{}^c{}_\alpha C_c{}^b{}_\beta - C_a{}^c{}_\beta C_c{}^b{}_\alpha, \end{aligned} \tag{1.9}$$

where $R^\gamma{}_{\alpha\beta}$ is given by

$$R^\gamma{}_{\alpha\beta} = \frac{\delta N^\gamma_\alpha}{\delta u^\beta} - \frac{\delta N^\gamma_\beta}{\delta u^\alpha}. \tag{1.10}$$

Next, we show that each Finsler connection on M is a vectorial Finsler connection on VTM. In fact, a Finsler connection on M is given by triplets $(N^\beta_\alpha, F_\alpha{}^\gamma{}_\beta, C_\alpha{}^\gamma{}_\beta)$ defined on each $U \subset TM$ and satisfying (1.1) and

$$(1.11)\quad \begin{cases} F_{\alpha'}{}^{\gamma'}{}_{\beta'} = \dfrac{\partial u^{\gamma'}}{\partial u^{\gamma}}\left(\dfrac{\partial u^{\alpha}}{\partial u^{\alpha'}} F_{\alpha}{}^{\gamma}{}_{\beta} \dfrac{\partial u^{\beta}}{\partial u^{\beta'}} + \dfrac{\partial^2 u^{\gamma}}{\partial u^{\alpha'}\partial u^{\beta'}}\right), \\ C_{\alpha'}{}^{\gamma'}{}_{\beta'} = \dfrac{\partial u^{\gamma'}}{\partial u^{\gamma}} \dfrac{\partial u^{\alpha}}{\partial u^{\alpha'}} \dfrac{\partial u^{\beta}}{\partial u^{\beta'}} C_{\alpha}{}^{\gamma}{}_{\beta}. \end{cases}$$

On the other hand, in this case a local basis in VTM is just $\{\partial/\partial v^1, \ldots, \partial/\partial v^n\}$ Hence we have

$$(1.12)\qquad G^{a}_{a'} = G^{\alpha}_{\alpha'} = \frac{\partial u^{\alpha}}{\partial u^{\alpha'}}.$$

Thus taking account of (1.11) and (1.12) we obtain (1.6). Also, each vectorial Finsler connection on VTM is a Finsler connection on M. In fact, in this case, from (1.6), by using (1.12) we get (1.11). We have to remark that such a passing from Finsler connections on M to linear connections on VTM was also studied by Oproiu [21].

We finish this section with some remarks on covariant derivatives induced by vectorial Finsler connections. Suppose $VFC=(N^{\beta}{}_{\alpha}, F_a{}^b{}_{\alpha}, C_a{}^b{}_{\alpha})$ is a vectorial Finsler connection on E and $W=W^aS_a$ is a differentiable section on E. Then the *(h)-covariant derivative* and the *(v)-covariant derivative* of W are given by

$$(1.13)\qquad W^a{}_{|\alpha} = \frac{\delta W^a}{\delta u^{\alpha}} + F_b{}^a{}_{\alpha} W^b \quad \text{and respectively}$$

$$(1.14)\qquad W^a|_{\alpha} = \frac{\partial W^a}{\partial v^{\alpha}} + C_b{}^a{}_{\alpha} W^b.$$

For a differentiable functions f on TM we define

$$(1.15)\qquad f_{|\alpha} = \frac{\delta f}{\delta u^{\alpha}}; \quad f|_{\alpha} = \frac{\partial f}{\partial v^{\alpha}}.$$

Now suppose there are defined $(np)^2$ differentiable functions $T^{\alpha a}_{\beta b}$ on the domain U of each canonical chart (U, φ, R^{2n}) of TM. We say that $T^{\alpha a}_{\beta b}$ define a *mixed Finsler tensor field* on M with respect to the vector bundle E if for a change of both the canonical chart on TM and the vectorial chart on E we have

$$(1.16)\qquad T^{\alpha' a'}_{\beta' b'} = \frac{\partial u^{\alpha'}}{\partial u^{\alpha}} \frac{\partial u^{\beta}}{\partial u^{\beta'}} \bar G^{a'}_{a} G^{b}_{b'} T^{\alpha a}_{\beta b}.$$

Now, we consider a Finsler connection $FC=(N^{\beta}_{\alpha}, F_{\alpha}{}^{\beta}{}_{\gamma}, C_{\alpha}{}^{\beta}{}_{\gamma})$ on M and a vectorial Finsler connection $VFC=(N^{\beta}{}_{\alpha}, F_a{}^b{}_{\alpha}, C_a{}^b{}_{\alpha})$ on the vector bundle E. We have to remark that these Finsler connections have the same non linear connection $N=(N^{\beta}_{\alpha})$. Then we define the *(h)-covariant derivative* of $T^{\alpha a}_{\beta b}$ by

$$(1.17)\qquad T^{\alpha a}_{\beta b|\gamma} = \frac{\delta T^{\alpha a}_{\beta b}}{\delta u^{\gamma}} + T^{\varphi a}_{\beta b} F_{\varphi}{}^{\alpha}{}_{\gamma} - T^{\alpha a}_{\varphi b} F_{\beta}{}^{\varphi}{}_{\gamma} + T^{\alpha c}_{\beta b} F_c{}^a{}_{\gamma} - T^{\alpha a}_{\beta c} F_b{}^c{}_{\gamma}$$

and the *(v)-covariant derivative* of $T^{\alpha a}_{\beta b}$ by

$$(1.18)\qquad T^{\alpha a}_{\beta b}|_{\gamma} = \frac{\partial T^{\alpha a}_{\beta b}}{\partial v^{\gamma}} + T^{\varphi a}_{\beta b} C_{\varphi}{}^{\alpha}{}_{\gamma} - T^{\alpha a}_{\varphi b} C_{\beta}{}^{\varphi}{}_{\gamma} + T^{\alpha c}_{\beta b} C_c{}^a{}_{\gamma} - T^{\alpha a}_{\beta c} C_b{}^c{}_{\gamma}.$$

Thus each pair (VFC, FC) of a vectorial Finsler connection on E and a Finsler connection on M defines a *relative differentiation* given by (1.17) and (1.18). Moreover, by a direct computation, using (1.13), (1.14), (1.17) and (1.18) we obtain the following commutation formulas

$$(1.19)\quad \begin{cases} W^a{}_{|\alpha|\beta} - W^a{}_{|\beta|\alpha} = R_b{}^a{}_{\alpha\beta} W^b - R^{\varphi}{}_{\alpha\beta} W^a|_{\varphi} - T_{\alpha}{}^{\varphi}{}_{\beta} W^a{}_{|\varphi}, \\ W^a{}_{|\alpha}|_{\beta} - W^a|_{\beta|\alpha} = P_b{}^a{}_{\alpha\beta} W^b - P^{\varphi}_{\alpha\beta} W^a|_{\varphi} - C_{\alpha}{}^{\varphi}{}_{\beta} W^a{}_{|\varphi}, \\ W^a|_{\alpha}|_{\beta} - W^a|_{\beta}|_{\alpha} = S_b{}^a{}_{\alpha\beta} W^b - S^{\varphi}_{\alpha\beta} W^a|_{\varphi}. \end{cases}$$

where $T_{\alpha}{}^{\varphi}{}_{\beta}$, $P^{\varphi}_{\alpha\beta}$ and $S^{\varphi}_{\alpha\beta}$ are three from the torsion tensors of FC and they are given by

$$(1.20)\quad \begin{cases} T_{\alpha}{}^{\varphi}{}_{\beta} = F_{\alpha}{}^{\varphi}{}_{\beta} - F_{\beta}{}^{\varphi}{}_{\alpha}, \quad P^{\varphi}_{\alpha\beta} = \dfrac{\partial N^{\varphi}_{\alpha}}{\partial v^{\beta}} - F_{\beta}{}^{\varphi}{}_{\alpha}, \\ S^{\varphi}_{\alpha\beta} = C_{\alpha}{}^{\varphi}{}_{\beta} - C_{\beta}{}^{\varphi}{}_{\alpha}. \end{cases}$$

§2. The Finsler normal bundle to a Finsler subspace. Let $F^m = (\tilde M, \tilde L(x, y))$ be a Finsler space, where $\tilde M$ is a real m-dimensional differentiable manifold and $\tilde L(x, y)$ is the fundamental function of F^m (see Matsumoto [15]). Denote by $g_{ij} = (1/2)(\partial^2 L^2/\partial y^i \partial y^j)$ the fundamental tensor field of $\tilde M$. In the present paper we only suppose $g_{ij}(x, y)$ be positive definite. Thus our theory holds good also for subspaces of generalized Finsler spaces (see Miron [19], Hashiguchi [11], Watanabe-Ikeda [26]).

Next, let M be an n-dimensional submanifold of $\tilde M$ locally given by equations

$$(2.1)\qquad x^i = x^i(u^1, \ldots, u^n); \quad \operatorname{rank}\left[\frac{\partial x^i}{\partial u^{\alpha}}\right] = n.$$

In this section and in the sequel we use the following range of indices: $i, j, k, \ldots = 1, \ldots, m$; $\alpha, \beta, \gamma, \ldots = 1, \ldots, n$; $a, b, c, \ldots = n+1, \ldots, m$.

The immersion i of M in $\tilde M$ induces an immersion di of TM in $T\tilde M$. Thus locally, if (u, v) of TM is carried by di in $(x^i(u), y^i(u, v))$ of $T\tilde M$ we have

$$(2.2)\qquad y^i = B^i_{\alpha} v^{\alpha} \quad \text{where} \quad B^i_{\alpha} = \frac{\partial x^i}{\partial u^{\alpha}}.$$

Since (y^i) is the supporting element of $\tilde M$ we call (v^{α}) the supporting element of M. We denote

$$(2.3)\qquad B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^{\alpha}\partial u^{\beta}}; \quad B^i_{0\beta} = B^i_{\alpha\beta} v^{\alpha}.$$

The natural local field of frames $\{\partial/\partial u^{\alpha}, \partial/\partial v^{\alpha}\}$ and $\{\partial/\partial x^i, \partial/\partial y^i\}$ on TM and respectively $T\tilde M$ are related by

$$(2.4)\qquad \begin{cases} \dfrac{\partial}{\partial u^\alpha} = B^i_\alpha \dfrac{\partial}{\partial x^i} + B^i_{0\alpha}\dfrac{\partial}{\partial y^i}\,, \\ \dfrac{\partial}{\partial v^\alpha} = B^i_\alpha \dfrac{\partial}{\partial y^i}\,. \end{cases}$$

Next, the fundamental function $\tilde L$ on F^m induces a fundamental function L on M by

$$(2.5)\qquad L(u, v) = \tilde L(x(u),\ y(u, v)),$$

and further we have a fundamental tensor field $g_{\alpha\beta}(u, v)$ on M given by

$$(2.6)\qquad g_{\alpha\beta}(u, v) = g_{ij}(x(u),\ y(u, v))\, B^i_\alpha B^j_\beta = \frac{1}{2}\ \frac{\partial^2 L^2}{\partial v^\alpha \partial v^\beta}\,.$$

In this way M inherites a Finsler structure and from now on we shall investigate the geometry of the Finsler space $F^n = (M,\ L(u, v))$ immersed in F^m.

First, it is easy to see that in fact the fundamental tensor $g_{ij}(x, y)$ defines a Riemannian metric g on the vertical subbundle $VT\tilde M$ of $TT\tilde M$ by

$$(2.7)\qquad g(X, Y) = g_{ij} X^i Y^j,$$

for any $X = X^i\partial/\partial y^i$ and $Y = Y^i\,\partial/\partial y^i$. We denote by $VT\tilde M^*$ the restriction of the vertical subbundle $VT\tilde M$ to the points of TM. Then by the second line of equations in (2.4) we see that VTM is a vector subbundle of $VT\tilde M^*$. The complementary orthogonal subbundle $VTM^\perp$ to VTM in $VT\tilde M^*$ with respect to the Riemannian metric g is called the *Finsler normal bundle* to F^n.

Let $\{B_a = B^i_a\,\partial/\partial y^i\}$ be a local field of orthonormal frames in the Finsler normal bundle $VTM^\perp$. Thus we have

$$(2.8)\qquad g_{ij}(x(u),\ y(u, v))\ B^i_a(u, v) B^j_b(u, v) = \delta_{ab}.$$

On the other hand, since $VTM^\perp$ is orthogonal to VTM, by the second line of equations (2.4) we get

$$(2.9)\qquad g_{ij}(x(u),\ y(u, v))\ B^i_\alpha(u)\ B^j_a(u, v) = 0.$$

The local basis $\{\partial/\partial v^\alpha = B^i_\alpha\,\partial/\partial y^i,\ B_a = B^i_a\,\partial/\partial y^i\}$ in $VT\tilde M^*$ is called *the Finsler field of frames on F^m along to F^n*. We denote by $[\tilde B^\alpha_i\ \tilde B^a_j]$ the inverse matrix of $[B^i_\alpha\ B^j_a]$. Thus we have

$$(2.10)\qquad \begin{cases} \tilde B^\alpha_i\, B^i_\beta = \delta^\alpha_\beta\,; \quad \tilde B^a_i\, B^i_\alpha = 0\,; \quad \tilde B^\alpha_i\, B^i_a = 0\,; \quad B^i_a\, \tilde B^b_i = \delta^b_a\,; \\ B^i_\alpha\, \tilde B^\alpha_j + B^i_a\, \tilde B^a_j = \delta^i_j\,. \end{cases}$$

Then by (2.6), (2.9) and (2.10) we get

$$(2.11)\qquad g_{\alpha\beta}\ \tilde B^\alpha_i = g_{ij} B^j_\beta\,.$$

§3. The induced Finsler connections by a Finsler connection on the ambient space. Let $F^n = (M, g_{\alpha\beta})$ be a Finsler subspace of $F^m = (\tilde M, g_{ij})$. Suppose $FC = (\tilde N, \tilde\nabla) = (N^i_j,\ F^j_{ik},\ C^j_{ik})$ is a Finsler connection on F^m, where $\tilde N$ is a non-linear connection on $T\tilde M$ given locally by N^i_j and $\tilde\nabla$ is a linear connection on $VT\tilde M$ given locally by F^j_{ik} and C^j_{ik}.

First we prove

Theorem 3.1. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC = (\tilde N,\ \tilde\nabla)$. Then there exists a unique non-linear conection N on TM satisfying*

$$(3.1)\qquad N_z \subset VTM^\perp_z \oplus \tilde N^*_z \quad (\textit{direct sum}), \quad \forall z \in TM,$$

where $\tilde N^*$ is the restriction of $\tilde N$ to TM.

Proof. We define locally the differentiable functions

$$(3.2)\qquad N^\beta_\alpha = \tilde B^\beta_i (B^i_{0\alpha} + N^i_j B^j_\alpha).$$

and it is easy to check (1.1). Thus we have a non-linear connection N on TM defined locally by N^β_α. Next, from the second line of equation in (2.4) we get

$$(3.3)\qquad \frac{\partial}{\partial y^i} = \tilde B^\alpha_i \frac{\partial}{\partial v^\alpha} + \tilde B^a_i\, B_a.$$

The distribution $\tilde N$ on $T\tilde M$ is spanned locally by $\dfrac{\delta}{\delta x^i}$ given by

$$(3.4)\qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}\,.$$

Then by using (1.2), (1.4), (3.2) – (3.4) we obtain

$$(3.5)\qquad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i} + H^a_\alpha B_a\,,$$

where H^a_α are given by

$$(3.6)\qquad H^a_\alpha = \tilde B^a_j\, (B^i_\alpha N^j_i + B^j_{0\alpha}).$$

Thus by (3.5) we have the existence of N satisfying (3.1).

Now suppose $\bar N = (\bar N^\beta_\alpha)$ is another non-linear connection on TM satisfying (3.1), that is, (3.5) with H^a_α as arbitrary functions. Then by using (1.2) for $\bar N^\beta_\alpha$, (2.4) and (3.4) we have

$$(3.7)\qquad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i} + (N^j_i B^i_\alpha + B^j_{0\alpha} - \bar N^\beta_\alpha B^j_\beta)\frac{\partial}{\partial y^j}\,.$$

Thus from (3.5) and (3.7) we obtain

$$(3.8)\qquad H^a_\alpha B^j_a = N^j_i B^i_\alpha + B^j_{0\alpha} - \bar N^\beta_\alpha B^j_\beta.$$

Finally, by contracting (3.8) with $\tilde{B}^\gamma_j$ we obtain $\tilde{N}^\beta_\alpha = N^\beta_\alpha$ given by (3.2) and by contracting (3.8) with $\tilde{B}^a_j$ we get (3.6). Therefore, we have the uniqueness of the non-linear connection given by (3.2).

The non-linear connection N on TM given by (3.2) is called *the induced non-linear connection* by $\tilde{N}$ on $T\tilde{M}$.

Further, the linear connection $\tilde{\nabla} = (F_j{}^i{}_k, C_j{}^i{}_k)$ on $VT\tilde{M}$ induces a linear connection $\overset{*}{\nabla}$ on $VT\tilde{M}^*$ given by

$$\overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial y^i} = \tilde{\nabla}_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial y^i} = F_i{}^j{}_\alpha \frac{\partial}{\partial y^j} \quad \text{and} \tag{3.9}$$

$$\overset{*}{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial y^i} = \tilde{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial y^i} = C_i{}^j{}_\alpha \frac{\partial}{\partial y^j}, \tag{3.10}$$

where $\frac{\delta}{\delta u^\alpha}$ are given by (1.2) with N^β_α from (3.2). Therefore we have a vectorial Finsler connection $IVFC = (N^\beta_\alpha, F_i{}^j{}_\alpha, C_i{}^j{}_\alpha)$ on $VT\tilde{M}^*$ which we call *the induced vectorial Finsler connection*. Of course, it is important for us to find the coefficients $F_i{}^j{}_\alpha$ and $C_i{}^j{}_\alpha$ by means of the elements of the Finsler connection on F^m. Thus we prove

Theorem 3.2. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC = (\tilde{N}, \tilde{\nabla})$. Then there exists a vectorial Finsler connection $IVFC = (N^\beta_\alpha, F_i{}^j{}_\alpha, C_i{}^j{}_\alpha)$ on the vector bundle $VT\tilde{M}^*$, where N^β_α are given by (3.2) and $F_i{}^j{}_\alpha$, $C_i{}^j{}_\alpha$ are given by*

$$F_i{}^j{}_\alpha = F_i{}^j{}_k B^k_\alpha + C_i{}^j{}_k B^k_a H^a_\alpha, \tag{3.11}$$

$$C_i{}^j{}_\alpha = C_i{}^j{}_k B^k_\alpha. \tag{3.12}$$

Proof. First, we see that the linear connection $\tilde{\nabla}$ on $VT\tilde{M}$ is given by

$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = F_j{}^k{}_i \frac{\partial}{\partial y^k} \quad \text{and} \quad (3.14) \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = C_j{}^k{}_i \frac{\partial}{\partial y^k}. \tag{3.13}$$

Then by using (3.5), (3.13) and (3.14) we get

$$\tilde{\nabla}_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial y^i} = (B^k_\alpha F_i{}^j{}_k + H^a_\alpha B^k_a C_i{}^j{}_k) \frac{\partial}{\partial y^j}. \tag{3.15}$$

Hence (3.11) follows from (3.9) and (3.15). In a similar way, by using the second line of equations (2.4) and (3.14) we infer

$$\tilde{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial y^i} = B^k_\alpha C_i{}^j{}_k \frac{\partial}{\partial y^j}. \tag{3.16}$$

Thus, (3.12) follows from (3.16) and (3.10).

By the definition of the Finsler normal bundle we have

$$VT\tilde{M}^* = VTM \oplus VTM^\perp. \tag{3.17}$$

Then we put

$$\overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial v^\beta} = F_\beta{}^\gamma{}_\alpha \frac{\partial}{\partial v^\gamma} + H^a{}_{\beta\alpha} B_a \quad \text{and} \tag{3.18}$$

$$\overset{*}{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial v^\beta} = C_\beta{}^\gamma{}_\alpha \frac{\partial}{\partial v^\gamma} + K^a{}_{\beta\alpha} B_a. \tag{3.19}$$

Thus we have a linear connection ∇ on VTM defined by

$$\nabla_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial v^\beta} = F_\beta{}^\gamma{}_\alpha \frac{\partial}{\partial v^\gamma} \quad \text{and} \quad (3.21) \quad \nabla_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial v^\beta} = C_\beta{}^\gamma{}_\alpha \frac{\partial}{\partial v^\gamma}. \tag{3.20}$$

Next we state

Theorem 3.3. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC = (\tilde{N}, \tilde{\nabla})$. Then there exists a Finsler connection $IFC = (N, \nabla) = (N^\beta_\alpha, F_\alpha{}^\beta{}_\gamma, C_\alpha{}^\beta{}_\gamma)$ on M where N^β_α are given by (3.2) and $F_\alpha{}^\beta{}_\gamma$, $C_\alpha{}^\beta{}_\gamma$ are given by*

$$F_\alpha{}^\beta{}_\gamma = \tilde{B}^\beta_j (B^j_{\alpha\gamma} + F_i{}^j{}_\gamma B^i_\alpha), \text{ respectively} \tag{3.22}$$

$$C_\alpha{}^\beta{}_\gamma = \tilde{B}^\beta_j C_i{}^j{}_\gamma B^i_\alpha. \tag{3.23}$$

Proof. Certainly, $IFC = (N, \nabla)$ is a Finsler connection on M since it is a vectorial Finsler connection on VTM. Moreover, by using (3.9) and (3.3) we get

$$\begin{aligned} \overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}} \frac{\partial}{\partial v^\beta} &= \overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}} \left(B^i_\beta \frac{\partial}{\partial y^i} \right) = (B^j_{\alpha\beta} + F_i{}^j{}_\alpha B^i_\beta) \frac{\partial}{\partial y^j} = \\ &= \tilde{B}^\gamma_j (B^j_{\alpha\beta} + F_i{}^j{}_\alpha B^i_\beta) \frac{\partial}{\partial v^\gamma} + \tilde{B}^a_j (B^j_{\alpha\beta} + F_i{}^j{}_\alpha B^i_\beta) B_a. \end{aligned} \tag{3.24}$$

By comparing (3.24) with (3.18) we get (3.22) and

$$H^a{}_{\beta\alpha} = \tilde{B}^a_j (B^j_{\alpha\beta} + F_i{}^j{}_\alpha B^i_\beta). \tag{3.25}$$

In a similar way, by using (3.10) and (3.3) we infer

$$\overset{*}{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial v^\beta} = \overset{*}{\nabla}_{\frac{\partial}{\partial v^\alpha}} \left(B^j_\beta \frac{\partial}{\partial y^j} \right) = B^j_\beta C_j{}^k{}_\alpha \frac{\partial}{\partial y^k} = B^j_\beta C_j{}^k{}_\alpha \tilde{B}^\gamma_k \frac{\partial}{\partial v^\gamma} + B^j_\beta C_j{}^k{}_\alpha \tilde{B}^a_k B_a. \tag{3.26}$$

Hence, by comparing (3.26) with (3.19) we get (3.23) and

$$K^a{}_{\beta\alpha} = B^j_\beta C_j{}^k{}_\alpha \tilde{B}^a_k. \tag{3.27}$$

The Finsler connection $IFC = (N^\beta_\alpha, F_\alpha{}^\beta{}_\gamma, C_\alpha{}^\beta{}_\gamma)$ on M is called *the induced Finsler connection* on F^n. The tensor fields $H^a{}_{\alpha\beta}$ and $K^a{}_{\alpha\beta}$ $(a = n+1, \ldots, m)$, given by (3.25) and (3.27) are called the *(h)-second fundamental forms* and respectively the *(v)-second fundamental forms* of the Finsler subspace F^n. Also, according to the theory of Riemannian subspaces we call (3.18) and (3.19) *Gauss' formulas* for the immersion of F^n in F^m.

Remark 3.1. If in particular $n=m-1$, then our induced Finsler connection is just that one obtained by Matsumoto in [14] for hypersurfaces in Finsler spaces.

Thus summing up our results, the Finsler connection $FC=(\tilde{N},\tilde{\nabla})=(N^i_j, F_j{}^i{}_k, C_j{}^i{}_k)$ on the ambient Finsler space F^m induces in a natural way a vectorial Finsler connection $IVFC=(N,\overset{*}{\nabla})=(N^\beta_\alpha, F_j{}^i{}_\alpha, C_j{}^i{}_\alpha)$ on $VT\tilde{M}^*$ and a Finsler connection $IFC=(N,\nabla)=(N^\beta_\alpha, F_\alpha{}^\beta{}_\gamma, C_\alpha{}^\beta{}_\gamma)$ on F^n. Thus by the theory from §1 we have a relative differentiation of tensor fields $T^{\alpha i}_{\beta j}$ given by

$$T^{\alpha i}_{\beta j|\gamma}=\frac{\delta T^{\alpha i}_{\beta j}}{\delta u^\gamma}+T^{\varphi i}_{\beta j}F_\varphi{}^\alpha{}_\gamma-T^{\alpha i}_{\varphi j}F_\beta{}^\varphi{}_\gamma+T^{\alpha k}_{\beta j}F_k{}^i{}_\gamma-T^{\alpha i}_{\beta k}F_j{}^k{}_\gamma \tag{3.28}$$

$$T^{\alpha i}_{\beta j}|_\gamma=\frac{\partial T^{\alpha i}_{\beta j}}{\partial v^\gamma}+T^{\varphi i}_{\beta j}C_\varphi{}^\alpha{}_\gamma-T^{\alpha i}_{\varphi j}C_\beta{}^\varphi{}_\gamma+T^{\alpha k}_{\beta j}C_k{}^i{}_\gamma-T^{\alpha i}_{\beta k}C_j{}^k{}_\gamma . \tag{3.29}$$

As an application of (3.28) and (3.29), taking account of (3.25), (3.27) and (3.6) we obtain

$$B^i_{\alpha|\beta}=H^a{}_{\alpha\beta}B^i_a\,;\quad B^i_\alpha|_\beta=K^a{}_{\alpha\beta}B^i_a \quad \text{and} \tag{3.30}$$

$$g_{ij|\alpha}=g_{ij|k}B^k_\alpha+g_{ij}|_kB^k_aH^a_\alpha\,, \tag{3.31}$$
$$g_{ij}|_\alpha=g_{ij}|_kB^k_\alpha .$$

§4. The Finsler normal connection. Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC=(\tilde{N},\tilde{\nabla})=(N^i_j, C_j{}^i{}_k, C_j{}^i{}_k)$. Taking account that $\overset{*}{\nabla}$ is a linear connection on $VT\tilde{M}^*$ by (3.17) we have

$$\overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}}B_a=-\bar{H}_a{}^\varphi{}_\alpha\frac{\partial}{\partial v^\varphi}+F_a{}^b{}_\alpha B_b \quad \text{and} \tag{4.1}$$

$$\overset{*}{\nabla}_{\frac{\partial}{\partial v^\alpha}}B_a=-\bar{K}_a{}^\varphi{}_\alpha\frac{\partial}{\partial v^\varphi}+C_a{}^b{}_\alpha B_b\,. \tag{4.2}$$

Thus we have a linear connection $\nabla^\perp$ on $VTM^\perp$ defined by

$$\nabla^\perp_{\frac{\delta}{\delta u^\alpha}}B_a=F_a{}^b{}_\alpha B_b\,;\qquad \nabla^\perp_{\frac{\partial}{\partial v^\alpha}}B_a=C_a{}^b{}_\alpha B_b. \tag{4.3}$$

Theorem 4.1. *Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC=(\tilde{N},\tilde{\nabla})$. Then there exists a vectorial Finsler connection $FNC=(N,\nabla^\perp)=(N^\beta_\alpha, F_a{}^b{}_\alpha, C_a{}^b{}_\alpha)$ on the Finsler normal bundle $VTM^\perp$, where N^β_α are given by (3.2) and $F_a{}^b{}_\alpha$, $C_a{}^b{}_\alpha$ are given by*

$$F_a{}^b{}_\alpha=\bar{B}^b_i\left(\frac{\delta B^i_a}{\delta u^\alpha}+F_j{}^i{}_\alpha B^j_a\right)=\bar{B}^b_iB^i_{a|\alpha}\,, \tag{4.4}$$

$$C_a{}^b{}_\alpha=\bar{B}^b_i\left(\frac{\partial B^i_a}{\partial v^\alpha}+C_j{}^i{}_\alpha B^j_a\right)=\bar{B}^b_iB^i_a|_\alpha\,. \tag{4.5}$$

Proof. By using (3.3) and (3.9) we get

$$\overset{*}{\nabla}_{\frac{\delta}{\delta u^\alpha}}B_a=\left(\frac{\delta B^i_a}{\delta u^\alpha}+B^j_aF_j{}^i{}_\alpha\right)\bar{B}^\beta_i\frac{\partial}{\partial v^\beta}+\left(\frac{\delta B^i_a}{\delta u^\alpha}+B^j_aF_j{}^i{}_\alpha\right)\bar{B}^b_iB_b. \tag{4.6}$$

Hence by comparing (4.6) and (4.1) we obtain (4.4) and

$$\bar{H}_a{}^\beta{}_\alpha=-\bar{B}^\beta_i\left(\frac{\delta B^i_a}{\delta u^\alpha}+B^j_aF_j{}^i{}_\alpha\right)=-\bar{B}^\beta_iB^i_{a|\alpha}\,. \tag{4.7}$$

In a similar way we infer (4.5) and

$$\bar{K}_a{}^\beta{}_\alpha=-\bar{B}^\beta_i\left(\frac{\partial B^i_a}{\partial v^\alpha}+B^j_aC_j{}^i{}_\alpha\right)=-\bar{B}^\beta_iB^i_a|_\alpha. \tag{4.8}$$

The vectorial Finsler connection $FNC=(N,\nabla^\perp)$ whose coefficients are given by (3.2), (4.4) and (4.5) is called the *Finsler normal connection* on $TM^\perp$. Also, according to the Riemannian case, we call (4.1) and (4.2) the *Weingarten's formulas*.

Next, by differentiating (h)-covariantly (2.9) and taking account of (3.30) we obtain

$$g_{ij|\beta}B^i_\alpha B^j_a+g_{ij}H^b{}_{\alpha\beta}B^i_bB^j_a+g_{ij}B^i_\alpha B^j_{a|\beta}=0. \tag{4.9}$$

Now we put

$$\begin{cases} H_{a\alpha\beta}=\delta_{ab}H^b{}_{\alpha\beta}, \\ K_{a\alpha\beta}=\delta_{ab}K^b{}_{\alpha\beta}, \end{cases}\qquad \begin{cases} H_a{}^\beta{}_\alpha=g^{\beta\varphi}H_{a\varphi\alpha}, \\ K_a{}^\beta{}_\alpha=g^{\beta\varphi}K_{a\varphi\alpha}. \end{cases} \tag{4.10}$$

Then by using (2.11) and (4.10) in (4.9) we get

$$g_{ij|\beta}B^i_\alpha B^j_a+H_{a\alpha\beta}+g_{\varphi\alpha}\bar{B}^\varphi_jB^j_{a|\beta}=0. \tag{4.11}$$

Hence by (4.7), (4.10) and (4.11) we obtain

$$\bar{H}_a{}^\varphi{}_\beta=H_a{}^\varphi{}_\beta+g^{\alpha\varphi}g_{ij|\beta}B^i_\alpha B^j_a. \tag{4.12}$$

In a similar way we obtain

$$\bar{K}_a{}^\varphi{}_\beta=K_a{}^\varphi{}_\beta+g^{\alpha\varphi}g_{ij}|_\beta B^i_\alpha B^j_a. \tag{4.13}$$

Finally, taking account of (4.1), (4.3), (4.4), (4.5), (4.7) and (4.8) we obtain

$$B^i_{a|\alpha}=-\bar{H}_a{}^\varphi{}_\alpha B^i_\varphi+F_a{}^b{}_\alpha B^i_b \quad \text{and} \tag{4.14}$$

$$B^i_a|_\alpha=-\bar{K}_a{}^\varphi{}_\alpha B^i_\varphi+C_a{}^b{}_\alpha B^i_b\,. \tag{4.15}$$

§5. The structure equations of Gauss, Codazzi and Ricci for Finsler subspaces. Let F^n be a Finsler subspace of a Finsler space F^m endowed with a Finsler connection $FC=(\tilde{N},\tilde{\nabla})=(N^i_j, F_j{}^i{}_k, C_j{}^i{}_k)$. In the previous sections we obtain three vectorial Finsler connection $IVFC=(N,\overset{*}{\nabla})=$

$=(N^\beta_\alpha,\ F_j{}^i{}_\alpha,\ C_j{}^i{}_a)$, $IFC=(N,\nabla)=(N^\beta_\alpha,\ F_\alpha{}^\beta{}_\gamma,\ C_\alpha{}^\beta{}_\gamma)$ and $FNC=(N,\nabla^\perp)=(N^\beta_\alpha,\ F_a{}^b{}_\alpha,\ C_a{}^b{}_\alpha)$ on the vector bundles $VT\tilde{M}^*$, VTM and respectively $VTM^\perp$. Hence we have two types of relative differentiation given by the pairs $(IFC,\ IVFC)$ and $(IFC,\ FNC)$. In this section we need both these relative covariant derivatives. Thus, for a tensor field $T^{\beta j}_{\alpha i}$ we denote by $T^{\beta j}_{\alpha i|\gamma}$ and $T^{\beta j}_{\alpha i}|_\gamma$ its (h)-and respectively (v)-covariant derivatives with respect to the pair $(IFC,\ IVFC)$. Also, for a tensor field $T^{\beta b}_{\alpha a}$ we denote by $T^{\beta b}_{\alpha a\perp\gamma}$ and $T^{\beta b}_{\alpha a}\perp_\gamma$ its (h)- and respectively (v)-covariant derivatives with respect to the pair $(IFC,\ FNC)$.

By a direct computation, using (1.9), (1.10), (1.17), (1.18) (1.20) and (1.21) we obtain the following commutation formulas :

$$
(5.1)\quad \begin{cases}
B^i_{\alpha|\beta|\gamma}-B^i_{\alpha|\gamma|\beta}=B^j_\alpha R_j{}^i{}_{\beta\gamma}-B^i_\varphi R_\alpha{}^\varphi{}_{\beta\gamma}-(H^a{}_{\alpha\varphi}T_\beta{}^\varphi{}_\gamma+K^a{}_{\alpha\varphi}R^\varphi{}_{\beta\gamma})B^i_a\,, \\
B^i_{\alpha|\beta}|_\gamma-B^i_\alpha|_{\gamma|\beta}=B^j_\alpha P_j{}^i{}_{\beta\gamma}-B^i_\varphi P_\alpha{}^\varphi{}_{\beta\gamma}-(H^a{}_{\alpha\varphi}C_\beta{}^\varphi{}_\gamma+K^a{}_{\alpha\varphi}P^\varphi{}_{\beta\gamma})\,B^i_a\,, \\
B^i_\alpha|_\beta|_\gamma-B^i_\alpha|_\gamma|_\beta=B^j_\alpha S_j{}^i{}_{\beta\gamma}-B^i_\varphi S_\alpha{}^\varphi{}_{\beta\gamma}-K^a{}_{\alpha\varphi}S^\varphi{}_{\beta\gamma}B^i_a\,,
\end{cases}
$$

$$
(5.2)\quad \begin{cases}
B^i_{a|\alpha|\beta}-B^i_{a|\beta|\alpha}=B^j_a R_j{}^i{}_{\alpha\beta}+(R^\varphi{}_{\alpha\beta}\tilde{K}_a{}^\lambda{}_\varphi+T_\alpha{}^\varphi{}_\beta\tilde{H}_a{}^\lambda{}_\varphi)B^i_\lambda- \\
-(R^\varphi{}_{\alpha\beta}C_a{}^b{}_\varphi+T_\alpha{}^\varphi{}_\beta F_a{}^b{}_\varphi)\,B^i_b\,, \\
B_{a|\alpha}|_\beta-B^i_a|_{\beta|\alpha}=B^j_a P_j{}^i{}_{\alpha\beta}+(C_\alpha{}^\varphi{}_\beta\tilde{H}_a{}^\lambda{}_\varphi+P^\varphi{}_{\alpha\beta}\tilde{K}_a{}^\lambda{}_\varphi)\,B^i_\lambda- \\
-(C_\alpha{}^\varphi{}_\beta F_a{}^b{}_\varphi+P^\varphi{}_{\alpha\beta}C_a{}^b{}_\varphi)B^i_b, \\
B^i_a|_\alpha|_\beta-B^i_a|_\beta|_\alpha=B^j_\alpha S_j{}^i{}_{\alpha\beta}+S^\varphi{}_{\alpha\beta}\tilde{K}_a{}^\lambda{}_\varphi B^i_\lambda-S^\varphi{}_{\alpha\beta}C_a{}^b{}_\varphi B^i_b,
\end{cases}
$$

where $R_j{}^i{}_{\beta\gamma}$, $P_j{}^i{}_{\beta\gamma}$, $S_j{}^i{}_{\beta\gamma}$ and $R_\alpha{}^\varphi{}_{\beta\gamma}$, $P_\alpha{}^\varphi{}_{\beta\gamma}$, $S_\alpha{}^\varphi{}_{\beta\gamma}$ are the Finsler curvature tensors of $IVFC$ and respectively IFC.

Proposition 5.1. *The curvature tensors of $IVFC$ are given by*

$$
(5.3)\quad \begin{cases}
R_j{}^i{}_{\beta\gamma}=R_j{}^i{}_{kh}B^k_\beta B^h_\gamma+P_j{}^i{}_{kh}(H^a_\gamma B^k_\beta-H^a_\beta B^k_\gamma)B^h_a+S_j{}^i{}_{kh}B^k_a B^h_b H^a_\beta H^b_\gamma, \\
P_j{}^i{}_{\beta\gamma}=P_j{}^i{}_{kh}B^k_\beta B^h_\gamma+S_j{}^i{}_{kh}B^k_a H^a_\beta B^h_\gamma, \\
S_j{}^i{}_{\beta\gamma}=S_j{}^i{}_{kh}B^k_\beta B^h_\gamma.
\end{cases}
$$

Proof. By using (3.5), (3.8), (3.11) and (3.12) we obtain

$$
(5.4)\quad \begin{aligned}
&\frac{\delta}{\delta u^\gamma}(F_j{}^i{}_\beta)-\frac{\delta}{\delta u^\beta}(F_j{}^i{}_\gamma)+C_j{}^i{}_\varphi R^\varphi{}_{\beta\gamma}=\left(\frac{\delta F_j{}^i{}_k}{\delta x^h}-\frac{\delta F_j{}^i{}_h}{\delta x^k}+\right.\\
&\left.+C_j{}^i{}_s R^s{}_{kh}\right)B^k_\beta B^h_\gamma+\left(\frac{\partial C_j{}^i{}_k}{\partial y^h}-\frac{\partial C_j{}^i{}_h}{\partial y^k}\right)B^h_b B^k_a H^b_\gamma H^a_\beta+\\
&+\left(\frac{\partial F_j{}^i{}_k}{\partial y^h}-\frac{\delta F_j{}^i{}_h}{\delta x^k}+C_j{}^i{}_s\frac{\partial N^s{}_k}{\partial y^h}\right)(H^a_\gamma B^k_\beta-H^a_\beta B^k_\gamma)B^h_a, \quad \text{and}
\end{aligned}
$$

$$
(5.5)\quad \begin{aligned}
&F_j{}^k{}_\beta F_k{}^i{}_\gamma-F_j{}^k{}_\gamma F_k{}^i{}_\beta=(F_j{}^s{}_k F_s{}^i{}_h-F_j{}^s{}_h F_s{}^i{}_k)B^k_\beta B^h_\gamma+\\
&+(F_j{}^s{}_k C_s{}^i{}_h-C_j{}^s{}_h F_s{}^i{}_k)(B^k_\beta H^a_\gamma-B^k_\gamma H^a_\beta)B^h_a+\\
&+(C_j{}^s{}_k C_s{}^i{}_h-C_j{}^s{}_h C_s{}^i{}_k)B^k_a B^h_b H^a_\beta H^b_\gamma.
\end{aligned}
$$

Thus by adding (5.4) and (5.5) and taking account of (1.9) for vectorial Finsler connections $IVFC$ and FC we obtain the first line of equations in (5.3). In a similar way we get the remaining equations in (5.3).

Now we can state the main result of this section as follows

Theorem 5.1. *Let F^n be a Finsler subspace of the Finsler space F^m endowed with the Finsler connection $FC=(\tilde{N},\tilde{\nabla})$. Then the curvature tensors of vectorial Finsler connection $IVFC$, IFC and FNC are related by the following structure equations*

$$
(5.6)\quad \begin{cases}
R_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^\varphi_i=R_\alpha{}^\varphi{}_{\beta\gamma}+H^a{}_{\alpha\gamma}\tilde{H}_a{}^\varphi{}_\beta-H^a{}_{\alpha\beta}\tilde{H}_a{}^\varphi{}_\gamma, \\
P_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^\varphi_i=P_\alpha{}^\varphi{}_{\beta\gamma}+K^a{}_{\alpha\gamma}\tilde{H}_a{}^\varphi{}_\beta-H^a{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\gamma, \\
S_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^\varphi_i=S_\alpha{}^\varphi{}_{\beta\gamma}+K^a{}_{\alpha\gamma}\tilde{K}_a{}^\varphi{}_\beta-K^a{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\gamma,
\end{cases}
$$

$$
(5.7)\quad \begin{cases}
R_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^a_i=H^a{}_{\alpha\beta\perp\gamma}-H^a{}_{\alpha\gamma\perp\beta}+H^a{}_{\alpha\varphi}T_\beta{}^\varphi{}_\gamma+K^a{}_{\alpha\varphi}R^\varphi{}_{\beta\gamma}, \\
P_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^a_i=H^a{}_{\alpha\beta}\perp_\gamma-K^a{}_{\alpha\gamma\perp\beta}+C_\beta{}^\varphi{}_\gamma H^a{}_{\alpha\varphi}+P^\varphi{}_{\beta\gamma}K^a{}_{\alpha\varphi}, \\
S_j{}^i{}_{\beta\gamma}B^j_\alpha\tilde{B}^a_i=K^a{}_{\alpha\beta}\perp_\gamma-K^a{}_{\alpha\gamma}\perp_\beta+K^a{}_{\alpha\varphi}S^\varphi{}_{\beta\gamma},
\end{cases}
$$

$$
(5.8)\quad \begin{cases}
R_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^b_i=R_a{}^b{}_{\alpha\beta}+\tilde{H}_a{}^\varphi{}_\beta H^b{}_{\varphi\alpha}-\tilde{H}_a{}^\varphi{}_\alpha H^b{}_{\varphi\beta}, \\
P_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^b_i=P_a{}^b{}_{\alpha\beta}+\tilde{K}_a{}^\varphi{}_\beta H^b{}_{\varphi\alpha}-\tilde{H}_a{}^\varphi{}_\alpha K^b{}_{\varphi\beta}, \\
S_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^b_i=S_a{}^b{}_{\alpha\beta}+\tilde{K}_a{}^\varphi{}_\beta K^b{}_{\varphi\alpha}-\tilde{K}_a{}^\varphi{}_\alpha K^b{}_{\varphi\beta},
\end{cases}
$$

$$
(5.9)\quad \begin{cases}
R_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^\varphi_i=\tilde{H}_a{}^\varphi{}_{\gamma\perp\alpha}-\tilde{H}_a{}^\varphi{}_{\alpha\perp\beta}-T_\alpha{}^\lambda{}_\beta\tilde{H}_a{}^\varphi{}_\lambda-R^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda, \\
P_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^\varphi_i=\tilde{K}_a{}^\varphi{}_{\beta\perp\alpha}-\tilde{H}_a{}^\varphi{}_\alpha\perp_\beta-C_\alpha{}^\lambda{}_\beta\tilde{H}_a{}^\varphi{}_\lambda-P^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda, \\
S_j{}^i{}_{\alpha\beta}B^j_a\tilde{B}^\varphi_i=\tilde{K}_a{}^\varphi{}_\beta\perp_\alpha-\tilde{K}_a{}^\varphi{}_\alpha\perp_\beta-S^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda,
\end{cases}
$$

where $R_a{}^b{}_{\alpha\beta}$, $P_a{}^b{}_{\alpha\beta}$, $S_a{}^b{}_{\alpha\beta}$ are the Finsler curvature tensors of the Finsler normal connection.

Proof. By using (3.30), (4.14) and (4.15) in the first line of commutation formulas (5.1) we get

$$
(5.10)\quad \begin{aligned}
&B^j_\alpha R_j{}^i{}_{\beta\gamma}-B^i_\varphi R_\alpha{}^\varphi{}_{\beta\gamma}=(H^b{}_{\alpha\beta|\gamma}-H^b{}_{\alpha\gamma|\beta}+H^a{}_{\alpha\beta}F_a{}^b{}_\gamma-\\
&-H^a{}_{\alpha\gamma}F_a{}^b{}_\beta+H^b{}_{\alpha\varphi}T_\beta{}^\varphi{}_\gamma+K^b{}_{\alpha\varphi}R^\varphi{}_{\beta\gamma})B^i_b-(H^a{}_{\alpha\beta}\tilde{H}_a{}^\varphi{}_\gamma-H^a_{\alpha\gamma}\tilde{H}_a{}^\varphi{}_\beta)B^i_\varphi.
\end{aligned}
$$

By contracting (5.10) with $\tilde{B}^\lambda_i$ we obtain the first line of equations in (5.6) Also, by contracting (5.10) with $\tilde{B}^a_i$ we get

$$
(5.11)\quad \begin{aligned}
B^j_\alpha\tilde{B}^a_i R_j{}^i{}_{\beta\gamma}=(H^a{}_{\alpha\beta|\gamma}+H^c{}_{\alpha\beta}F_c{}^a{}_\gamma)-(H^a{}_{\alpha\gamma|\beta}+H^c{}_{\alpha\gamma}F_c{}^a{}_\beta)+\\
+T_\beta{}^\varphi{}_\gamma H^a{}_{\alpha\varphi}+R^\varphi{}_{\beta\gamma}K^a{}_{\alpha\varphi}.
\end{aligned}
$$

Next, by using both relative differentiations discussed in this section we obtain

$$
(5.12)\quad H^a{}_{\alpha\beta\perp\gamma}=H^a{}_{\alpha\beta|\gamma}+H^b{}_{\alpha\beta}F_b{}^a{}_\gamma.
$$

Thus by (5.12) we see that (5.11) becomes just the first line of equations in (5.7). In a similar way we get the second and the third line of equations in (5.6) and (5.7) from the second and respectively the third line of equations in (5.1). Finally, by using again (3.30), (4.14) and (4.15) in equations (5.2) we infer

$$(5.13)\quad \begin{cases} R_j{}^i{}_{\alpha\beta}B^j_a=(R_a{}^b{}_{\alpha\beta}+\tilde{H}_a{}^\varphi{}_\beta H^b{}_{\varphi\alpha}-\tilde{H}_a{}^\varphi{}_\alpha H^b{}_{\varphi\beta})B^i_b+(\tilde{H}_a{}^\varphi{}_{\beta\perp\alpha}-\tilde{H}_a{}^\varphi{}_{\alpha\perp\beta}- \\ \qquad -R^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda-T_\alpha{}^\lambda{}_\beta\tilde{H}_a{}^\varphi{}_\lambda)B^i_\varphi, \\ P_j{}^i{}_{\alpha\beta}B^j_a=(P_a{}^b{}_{\alpha\beta}+\tilde{K}_a{}^\varphi{}_\beta H^b{}_{\varphi\alpha}-\tilde{H}_a{}^\varphi{}_\alpha K^b{}_{\varphi\beta})B^i_b+(\tilde{K}_a{}^\varphi{}_{\beta\perp\alpha}-\tilde{H}_a{}^\varphi{}_\alpha\perp_\beta- \\ \qquad -C_\alpha{}^\lambda{}_\beta\tilde{H}_a{}^\varphi{}_\lambda-P^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda)B^i_\varphi, \\ S_j{}^i{}_{\alpha\beta}B^j_a=(S_a{}^b{}_{\alpha\beta}+\tilde{K}_a{}^\varphi{}_\beta K^b{}_{\varphi\alpha}-\tilde{K}_a{}^\varphi{}_\alpha K^b{}_{\varphi\beta})B^i_b+(\tilde{K}_a{}^\varphi{}_\beta\perp_\alpha-\tilde{K}_a{}^\varphi{}_\alpha\perp_\beta- \\ \qquad -S^\lambda{}_{\alpha\beta}\tilde{K}_a{}^\varphi{}_\lambda)B^i_\varphi. \end{cases}$$

Then the equations (5.8) and (5.9) are obtained by contracting (5.13) with $\tilde{B}^\varphi_i$ and respectively $\tilde{B}^\xi_i$.

The equations (5.6), (5.7) and (5.8) are the equations of Gauss, Codazzi and Ricci for the immersion of a Finsler space F^n in F^m. For a metric Finsler connection FC on F^m we have $g_{ij|k}=g_{ij}|_k=0$. Hence by using (3.31), (4.12) and (4.13) we obtain the equivalence of (5.7) and (5.9). However, for other Finsler connections of F^m the equations (5.9) represent something which shows how far from the geometry of Riemannian subspaces is the geometry of Finsler subspaces.

Now, let f be a Finsler scalar field on F^m, that is, f is a differentiable function on $T\tilde{M}$. Then by using (3.28) and (3.29) we get the following commutation formulas for IFC

$$(5.14)\quad \begin{cases} f_{|\beta|\gamma}-f_{|\gamma|\beta}=-f_{|\alpha}T_\beta{}^\alpha{}_\gamma-f|_\alpha R^\alpha{}_{\beta\gamma}, \\ f_{|\beta}|_\gamma-f|_{\gamma|\beta}=-f_{|\alpha}C_\beta{}^\alpha{}_\gamma-f|_\alpha P^\alpha{}_{\beta\gamma}, \\ f|_\beta|_\gamma-f|_\gamma|_\beta=-f|_\alpha S^\alpha{}_{\beta\gamma}. \end{cases}$$

By using (5.14) and similar formulas for FC on F^m we obtain

Theorem 5.2. *Let F^n be a Finsler subspace of the Finsler space F^m endowed with the Finsler connection $FC=(\tilde{N},\tilde{\nabla})$. Then the torsion tensors of IFC and FC are related by*

$$(5.15)\quad \begin{cases} R^\alpha{}_{\beta\gamma}=R^i{}_{jk}B^j_\beta B^k_\gamma\tilde{B}^\alpha_i+P^i{}_{jk}B^k_a\tilde{B}^\alpha_i(B^j_\beta H^a_\gamma-B^j_\gamma H^a_\beta)+H^a_\beta\tilde{H}_a{}^\alpha{}_\gamma-H^a_\gamma\tilde{H}_a{}^\alpha{}_\beta+ \\ \qquad +S^i_{jk}H^a_\beta H^b_\gamma B^j_a B^k_b\tilde{B}^\alpha_i \\ P^\alpha{}_{\beta\gamma}=H^a_\beta\tilde{K}_a{}^\alpha{}_\gamma+\tilde{B}^\alpha_i(S^i{}_{jk}H^a_\beta B^j_a B^k_\gamma+P^i{}_{jk}B^j_\beta B^k_\gamma), \\ T_\beta{}^\alpha{}_\gamma=\tilde{B}^\alpha_i\{T_j{}^i{}_k B^j_\beta B^k_\gamma+C_j{}^i{}_k(B^j_\beta H^a_\gamma-B^j_\gamma H^a_\beta)B^k_a\}, \\ S^\alpha{}_{\beta\gamma}=S^i{}_{jk}\tilde{B}^\alpha_i B^j_\beta B^k_\gamma;\quad C_\beta{}^\alpha{}_\gamma=C_j{}^i{}_k\tilde{B}^\alpha_i B^j_\beta B^k_\gamma, \\ D^\alpha{}_\beta=C_j{}^i{}_k\tilde{B}^\alpha_i B^k_a H^a_\beta y^j+\tilde{B}^\alpha_i B^j_\beta D^i{}_j, \end{cases}$$

where $D^\alpha{}_\beta$ and $D^i{}_j$ are the deflection tensor fields of IFC and respectively FC.

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Polytechnic Institute of Iaşi
Department of Mathematics
6600 Iaşi, R. S. Romania