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ALMOST SEMIQUATERNION STRUCTURES EXISTENCE AND CONNECTIONS

BY

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An almost semiquaternion structure is a degenerate, hipercomplex structure defined by the semiquaternion algebra [3]. The study of such a structure is similar to that of the quaternion and antiquaternion structure [1, 16, 17, 18, ...].

In the present paper we deal with two problems: examples of the semiquaternion structures and linear connections.

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§1. **Definitions.** *Let M be a differentiable manifold,* dim $M=4n$ *(the necessity of this assumation will be seen leter in the paper) and $\tau_q^p(M)$ the module of tensor fields of type (p, q) on M.*

Definition 1.1. *The structure $SQ=(F_1, F_2, F_3)$, $F_i\in\tau_1^1(M)$, $i=\overline{1, 3}$, satisfying:*

$$F_1^2=-I \quad , \quad F_2^2=F_3^2=0 \quad , \quad \text{rank } F_2=2n \tag{1.1}$$

$$F_1F_2=-F_2F_1=F_3\,; \quad F_1F_3=-F_3F_1=-F_2\,; \quad F_2F_3=F_3F_2=0$$

is called an almost semiquaternion structure, shorthly (SQ)-structure on M.

Let us consider structures defined by two linear complex tensor fields J_1, $J_2\in\tau_1^1(M)$, satisfying:

$$J_1 \cdot J_2+J_2 \cdot J_1=\alpha I \quad (\alpha\in R,\ I\in\tau_1^1(M) \text{ identity}). \tag{1.2}$$

Such structures appeared firstly in H. W a k a k u w a [15] and a study of them was made by F. T r i c e r r i in [13], [14].

If $\alpha^2<4$ we have almost quaternion structures, if $\alpha^2>4$ we have almost antiquaternion structures. Finally, the case $\alpha^2=4$ is strongly related to the almost semiquatern'on structures. However, the later case is not studied by Tricerri. Hence, once again, we see the importance of the present study.

An (SQ)-structure is a hipercomplex structure defined by the semiquaternion algebra ([3], p. 325). In fact, using left regular representation of semiquaternion algebra in a $4n$-dimensional space we obtain [10]:

$$S_0=I_{4n},\quad S_1=\begin{pmatrix}0 & -I & 0 & 0\\ I & 0 & 0 & 0\\ 0 & 0 & 0 & -I\\ 0 & 0 & I & 0\end{pmatrix},\quad S_2=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ I & 0 & 0 & 0\\ 0 & -I & 0 & 0\end{pmatrix},\quad S_3=S_1\cdot S_2.$$

The group of matrices which commute with S_α, $\alpha=\overline{0,3}$, is given by right regular representation, and it is described as follows:

$$(1.3)\qquad G_{SQ}=\left\{\begin{pmatrix} A & -B & 0 & 0\\ B & A & 0 & 0\\ C & -D & A & B\\ D & C & -B & A\end{pmatrix} \Big/\ 0,\ A,\ B,\ C,\ D\in gl(n,R)\right\}\subset GL(4n,R).$$

The reduction $P_{G_{SQ}}$ of the structural group $GL(4n, R)$, of principal frames bundle $P(M)$, to G_{SQ}, defines a G-structure on M, calld *almost semi-quaternion G-structure.*

Lemma 1.1. *If* $X_1\in\mathcal{X}(M)$, $X_1\notin \operatorname{Ker} F_2=\operatorname{Ker} F_3$ *then:*
$X_1, F_1X_1, F_2X_1, F_3X_1$ *are independent vector fields at every* $x\in M$.

Lemma 1.2. *If* $X_1, X_2\notin \operatorname{Ker} F_2$ *and* $X_1, F_1X_1, F_2X_1, F_3X_1, X_2$ *are independent, then* $X_1, F_1X_1, F_2X_1, F_3X_1, X_2, F_1X_2, F_2X_2, F_3X_2$ *are independent vector fields.*

By using these lemmas we obtain a set of $4n$ linear independent vector fields at every $x\in M$.

$$(1.4)\qquad \sigma=\{X_1, \ldots, X_n, F_1X_1, \ldots, F_1X_n, F_2X_1, \ldots, F_2X_n, F_3X_1, \ldots, F_3X_n\}.$$

Proposition 1.4. *The existence of an (SQ)-structure is equivalent to the existence of G-structure* $P_{G_{SQ}}$ *on differentiable manifold M.*

Proof. The tensor field F_i of structure $SQ=(F_1, F_2, F_3)$, given by (1.1), has in the frames σ the matrix S_i, $i=\overline{1,3}$. Considering at every $x\in M$ the frames obtained from σ by right translations with elements of G_{SQ} we get a subbundle of $P(M)$ with structural group G_{SQ}.

The converse is obvious.

Remark 1.1. From Prop. 1.4 it follows dim $M=4n$, compulsory.

§2. Examples of almost semiquaternion structures. We shall give two examples of (SQ)-structures.

1. The first example is the Lie group $\bar{Q}$ of non-zero quaternions (see [2]). Its Lie algebra is isomorphic to the algebra on the vector space of quaternions with multiplication defined by:

1. $i=j\cdot 1=k\cdot 1=0$, $i\cdot j=k$, $j\cdot k=i$, $k\cdot i=j$. If X_1, X_2, X_3, X_4 are the left invariant vector fields on $\bar{Q}$ corresponding to the elements i, j, k, respectivelly, then:

$$[X_1, X_4]=[X_2, X_4]=[X_3, X_4]=0;\ [X_1, X_2]=X_3,\ [X_2, X_3]=X_1, [X_3, X_1]=X_2.$$

Let us consider on $\bar{Q}$ the structure (F_1, F_2, F_3) given by:

$$\begin{array}{lll} F_1X_1=X_2 & F_2X_1=X_3 & F_3X_1=X_4\\ F_1X_2=-X_1 & F_2X_2=-X_4 & F_3X_2=X_3\\ F_1X_3=X_4 & F_2X_3=0 & F_3X_3=0\\ F_1X_4=-X_3 & F_2X_4=0 & F_3X_4=0.\end{array}$$

Then, (F_1, F_2, F_3) defines one (SQ)-structure.

2. The second example is given by the tangent bundle $T(M)$, where dim $M=2n$, and M carries an almost complex structure.

— It is known that on $T(M)$ there exists an almost tangent structure defined in local map (x^i, y^i), $i=\overline{1,2n}$, on $T(M)$ by

$$F_2\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial y^i};\qquad F_2\left(\frac{\partial}{\partial y^i}\right)=0.$$

— On $T(M)$ there exists a canonical complex structure F defined by means of both, the vertical and horizontal lifts (defined by a linear connection ∇ on M (see [11])) by: $FX^h=X^v$ and $FX^v=-X^h$.

— If J is an almost complex structure on M, then J can be lifted on $T(M)$ to an almost complex structure $\tilde{J}^h$ defined by $\tilde{J}^hX^v=(JX)^h$ and $\tilde{J}^hX^h=(JX)^v$.

The structure $F_1=F\tilde{J}^h$ is an almost complex structure (see [11]) and $SQ=(F_1, F_2, F_3=F_1F_2)$ is a (SQ)-structure.

§ 3. Linear semiquaternion connections. Let M be a $4n$-dimensional manifold endowed with a (SQ)-structure given by (1.1). Denote by $\mathcal{V}=\operatorname{Ker} F_2$, the $2n$-distribution called the vertical distribution. If $X\in\mathcal{V}$ then $F_1X\in\mathcal{V}$.

Let $\mathcal{H}$ be a fixed chosen distribution, called horizontal, complementary to $\mathcal{V}$ (i.e. $T_x(M)=\mathcal{H}_x\oplus\mathcal{V}_x=\ \forall x\in M$.) We can suppose that $\mathcal{H}$ is chosen so that it is invariant by F_1 (this is possible, since if g is a hermitian metric with respect to F_1, then $\mathcal{H}$ is the ortogonal complement in $T(M)$ to $\mathcal{V}$ with respect to g). Denote by h, v the corresponding projectors on these distributions. We have:

$$(3.1)\qquad hF_1h=F_1h \ \text{ and } \ vF_1v=F_1v.$$

The following Theorem is know from [7] (see also [19]) and was suggested to us by V. Oproiu:

Theorem 3.1. *There exists a unique tensor field* $F_2^*\in\mathcal{T}_1^1(M)$ *such that:*

$$(3.2)\qquad F_2^*F_2=h,\qquad vF_2^*=0,\qquad F_2^*h=0.$$

Remark 3.1. F_2^* depends on the choice of the distribution $\mathcal{H}$ and is the generalised inverse of tensor field F_2 in every $x\in M$.

Denote $F_3^*=F_1\cdot F_2^*$

Proposition 3.2. *The following relations hold:*

(i) $F_\alpha v=0$, $F_\alpha h=F_\alpha$, $vF_\alpha=F_\alpha$, $hF_\alpha=0$, $hF_\alpha^*=F_\alpha^*$, $F_\alpha^*v=F_\alpha^*$,

(ii) $F_\alpha^*F_\alpha=h$, $F_\alpha F_\alpha^*=v$, $(F_\alpha^*)^2=0$, $F_2^*F_3^*=F_3^*F_2^*=0$, $\alpha=2, 3$

(iii) $F_1F_2^*=-F_2^*F_1=F_3^*$, $F_1F_3^*=-F_3^*F_1=-F_2^*$

The proof follows from (3.2) and (3.1).

From Prop. 3.2 we see that $SQ^*=(F_1, F_2^*, F_3^*)$ defines an almost semiquaternion structure, called the adjoint of (SQ).

Next, we are interested in finding of linear (SQ)-connections, i.e. linear connection ∇ such that: $\nabla F_i=0$, $i=\overline{1,3}$. From [5] it is known the set of all almost complex connections with respect to F_1:

$$\overset{1}{\nabla}_X Y=\overset{\circ}{\nabla}_X Y+\frac{1}{2}(\overset{\circ}{\nabla}_X F_1)(F_1 Y)+\Psi_{1Y}Q_X, \tag{3.3}$$

and from [7] is known the set of all almost tangent connections with respect to F_2, or F_3:

$$\overset{\alpha}{\nabla}_X Y=\overset{\circ}{\nabla}_X Y+F^*_\alpha(\overset{\circ}{\nabla}_X F_\alpha)\,Y+\overset{\alpha}{O}_{1Y}Q_X,\quad \alpha=2,\ 3, \tag{3.4}$$

where $\overset{\circ}{\nabla}$ is an arbitrary fixed connection, $X,\ Y\in\mathcal{X}(M)$, $Q\in\tau^1_2(M)$, $Q_X Y=Q(X,\ Y)$, and:

$$\begin{aligned}\Psi_1&=\frac{1}{2}(I\otimes I-F_1\otimes F_1) &\qquad O^\alpha_1&=v\otimes I+F^*_\alpha\otimes F_\alpha\\ \Psi_2&=\frac{1}{2}(I\otimes I+F_1\otimes F_1) &\qquad O^\alpha_2&=h\otimes I-F^*_\alpha\otimes F_\alpha,\end{aligned} \tag{3.5}$$

are pairs of complementary projection operators on $\tau^1_1(M)$.

Now, we consider the transformations of connections:

$$\begin{aligned}&\overset{\circ}{\nabla}_X Y\xrightarrow{\mathcal{T}_i}\overset{1}{\mathcal{D}}_X Y=\overset{\circ}{\nabla}_X Y+\frac{1}{2}(\overset{\circ}{\nabla}_X F_1)(F_1 Y), && i=1,\ 2,\ 3,\\ &\overset{\alpha}{\mathcal{D}}_X Y=\overset{\circ}{\nabla}_X Y+F^*_\alpha(\overset{\circ}{\nabla}_X F_\alpha)\,Y, && \alpha=2,\ 3.\end{aligned} \tag{3.6}$$

Obviously, $\overset{i}{\mathcal{D}}F_i=0$, $i=1,\ 2,\ 3$.

Let us consider the connection $\overset{ij}{\mathcal{D}}$ obtained from connection $\overset{j}{\mathcal{D}}$ by means of transformation $\mathcal{T}_i$. ($\overset{ij}{\mathcal{D}}$ is obtained by replacing arbitrary linear connection $\overset{\circ}{\nabla}$ by $\overset{j}{\mathcal{D}}$ in expression of connection $\overset{i}{\mathcal{D}}$.)

Proposition 3.3. *The relations:*

$$\overset{1\alpha}{\mathcal{D}}=\overset{\alpha 1}{\mathcal{D}},\qquad \alpha=2,3\ , \tag{3.7}$$

hold good.

Proof. After the necessary computations we get:

$$\overset{1}{\mathcal{D}}_X Y=\frac{1}{2}\{\overset{\circ}{\nabla}_X Y-F_1\overset{\circ}{\nabla}_X(F_1 Y)\},\quad \overset{2}{\mathcal{D}}_X Y=v\overset{\circ}{\nabla}_X Y+F^*_2\overset{\circ}{\nabla}_X(F_2 Y).$$

Thus we have:

$$\overset{12}{\mathcal{D}}_X Y=\frac{1}{2}\{v\overset{\circ}{\nabla}_X Y-vF_1\overset{\circ}{\nabla}_X(F_1 Y)+F^*_2\overset{\circ}{\nabla}_X(F_2 Y)+F^*_3\overset{\circ}{\nabla}_X(F_3 Y)\}=\overset{21}{\mathcal{D}}_X Y.$$

In a similar way we infer $\overset{13}{\mathcal{D}}=\overset{31}{\mathcal{D}}$.

Theorem 3.4. *A linear (SQ)-connection is either:*

$$\mathcal{D}=\overset{12}{\mathcal{D}}\ \text{ or:} \tag{3.8}$$

$$\mathcal{D}_X Y=\overset{\circ}{\nabla}_X Y+\frac{1}{2}\{v\overset{\circ}{\nabla}_X Y-vF_1(\overset{\circ}{\nabla}_X F_1)\,Y+F^*_2(\overset{\circ}{\nabla}_X F_2)\,Y+F^*_3(\overset{\circ}{\nabla}_X F_3)\,Y\}; \tag{3.8'}$$

where, $\overset{\circ}{\nabla}$ is an arbitrary linear connection, $X,\ Y\in\mathcal{X}(M)$.

Proof. $\overset{12}{\mathcal{D}}F_1=0$, since $\overset{12}{\mathcal{D}}$ is obtained from $\overset{1}{\mathcal{D}}$ by replacing $\overset{\circ}{\nabla}$ by $\overset{2}{\mathcal{D}}$. Using (3.7) we get $\overset{12}{\mathcal{D}}F_2=\overset{21}{\mathcal{D}}F_2=0$.

Next we are going to determine the set of all (SQ)-connections. We look for all (SQ)-connection of the form: $\nabla=\tilde{\nabla}+T$, where $\tilde{\nabla}$ is a fixed but arbitrary linear (SQ)-connection and $T\in\tau^1_2(M)$ is an unknown tensor field to be determined.

Proposition 3.5. *∇ is a linear (SQ)-connection if and only if T_X commute with F_i, $i=1,\ 2,\ 3$, $\forall X\in\mathcal{X}(M)$.*

Proof. $\nabla_X F_i=0\Leftrightarrow T_X F_i-F_i T_X=-\tilde{\nabla}_X F_i=0\quad \forall X\in\mathcal{X}(M)$.

Let us remark that $T_X F_1=F_1 T_X\Leftrightarrow\Psi_2 T_X=0$ and $T_X F_\alpha=F_\alpha T_X\Leftrightarrow O^\alpha_2 T_X=0$, $\alpha=2,3$ (see [7]), where Ψ_2, O^α_2 are given by (3.5).

From the general theory of projection operators we have.

Proposition 3.6.

T_X commutes with $F_1\Leftrightarrow\exists Q\in\tau^1_2(M)$ such that $T_X=\Psi_1 Q_X$

T_X commutes with $F_\alpha\Leftrightarrow\exists Q\in\tau^1_2(M)$ such that $T_X=O^\alpha_1 Q_X$ $\alpha=2,3$.

By a direct computation we get:

Proposition 3.7. *The following relations hold good:*

$$\Psi_1\overset{\alpha}{O}_1=\overset{\alpha}{O}_1\Psi_1;\quad \Psi_1\overset{2}{O}_1=\Psi_1\overset{3}{O}_1;\quad \overset{\alpha}{O}_1\overset{\beta}{O}_1=\overset{\alpha}{O}_1,\qquad \alpha,\ \beta=2,3.$$

Proposition 3.8. *T_X commutes with all F_i, ($i=1,\ 2,\ 3$) if and only if there exist $Q\in\tau^1_2(M)$ such that $T_X=\Psi_1\overset{2}{O}_1 Q_X$, where $Q^Y_X=Q(X,\ Y)$.*

The proof follows from Prop. 3.5, Prop. 3.7 and the general theory of projection operators.

Therefore, from Prop. 3.5 and $\nabla=\tilde{\nabla}+T$, taking $\tilde{\nabla}=\mathcal{D}$ given by (3.8), we obtain:

Theorem 3.9. *The set of all linear (SQ)-connection is given by:*

$$\nabla_X Y=\mathcal{D}_X Y+(\Psi_1 O^2_1)_Y Q_X, \tag{3.9}$$

where $\mathcal{D}$ is given by (3.8), $Q\in\tau^1_2(M)$, $Q_X Y=Q(X,\ Y)$ arbitrary. Or,

$$\begin{aligned}\nabla_X Y=\overset{\circ}{\nabla}_X Y&+\frac{1}{2}\{v\overset{\circ}{\nabla}_X Y-vF_1(\overset{\circ}{\nabla}_X F_1)\,Y+F^*_2(\overset{\circ}{\nabla}_X F_2)Y+F^*_3(\overset{\circ}{\nabla}_X F_3)Y\}+\\ &+\frac{1}{2}\{vQ_X Y-vF_1Q_XF_1Y+F^*_2Q_XF_2Y+F^*_3Q_XF_3Y\},\end{aligned} \tag{3.10}$$

where $\overset{\circ}{\nabla}$ is a fixed but arbitrary linear connection, $Q\in\tau^1_2(M)$.

Remark 3.1. The set of (SQ)-connections (3.9) or (3.10), doesn't depend on the distribution $\mathcal{H}$ since the connection (3.4) not depend [7] on $\mathcal{H}$.

Remark 3.2. After calculating covariant derivative from (3.10) we infer:

$v\nabla_X Y = \frac{1}{2} v\{(\overset{\circ}{\nabla}_X + Q_X)Y - F_1(\overset{\circ}{\nabla}_X + Q_X)F_1 Y\}$, so that $v\nabla$ is the vertical part of set of complex connections with respect to F_1.

Remark 3.3. Analogous computations give the set of all linear connections with respect to $(SQ)^* = (F_1, F_2^*, F_3^*)$ structure:

$$(3.11)\quad {}^*\nabla_X Y = \frac{1}{2} h(\overset{\circ}{\nabla}_X + Q_X) - hF_1(\overset{\circ}{\nabla}_X + Q_X)F_1 Y + F_2(\overset{\circ}{\nabla}_X + Q_X)F_2^* Y + F_3(\overset{\circ}{\nabla}_X Q_X)F_3^* Y.$$

§ 4. **Symmetric (SQ)-connections.** For the (SQ)-structure, given by (1.1) we determined the following set of connections:

— The connections $\overset{1}{\nabla}$, given by (3.3), such that $\overset{1}{\nabla} F_1 = 0$, or:

$$(3.3')\qquad \overset{1}{\nabla}_X Y = \Psi_{1Y}(\overset{\circ}{\nabla}_X + Q_X) = \frac{1}{2}\{(\overset{\circ}{\nabla}_X + Q_X)Y - F_1(\overset{\circ}{\nabla}_X + Q_X)(F_1 Y)\}.$$

— The connections $\overset{\alpha}{\nabla}$, $\alpha = 2,3$, such that $\overset{\alpha}{\nabla} F_\alpha = 0$, or:

$$(3.4')\qquad \overset{\alpha}{\nabla}_X Y = \overset{\alpha}{O}_{1Y}(\overset{\circ}{\nabla}_X + Q_X) = v(\overset{\circ}{\nabla}_X + Q_X) + F_\alpha^*(\overset{\circ}{\nabla}_X + Q_X)(F_\alpha Y).$$

— The (SQ)-connections ∇, given by (3.10), or:

$$(3.10')\qquad \nabla_X Y = (\overset{2}{O}_1 \Psi_1)_Y (\overset{\circ}{\nabla}_X + Q_X) = \overset{2}{O}_{1Y} \overset{1}{\nabla}_X.$$

(The action of Ψ_1 or $\overset{\alpha}{O}_1$ on $\overset{\circ}{\nabla}_X$ and $\overset{1}{\nabla}_X$ are formally defined.)

Proposition 4.1. *There exists an (SQ)-connection ∇ with $vT = 0$ if and only if $vN_{F_1} = 0$, where T is torsion of ∇ and N_{F_1} is Nijenhuis tensor associated to F_1.*

Proof. If ∇ is an (SQ)-connection with $vT = 0$, then calculating N_{F_1} by means of T we obtain $vN_{F_1} = 0$. Conversely, considering the almost complex connection $\overset{1}{\nabla}$ with respect to F_1 and with torsion $\overset{1}{T} = \frac{1}{4} N_{F_1}$ (see [5], p. 170) then, based on Remark 3.2, the connection $\nabla = \overset{2}{O}_1 \overset{1}{\nabla}$ is an (SQ)-connection with $vT = v\overset{1}{T} = (1/4)\, vN_{F_1}$.

Theorem 4.2. *Let the connection $\overset{1}{\nabla}$ be defined by:*

$$(4.1)\qquad \begin{aligned} \overset{1}{\nabla}_X Y = \frac{1}{2}\{&[vX, Y] - F_1[vX, F_1 Y] - [F_2 X, F_2^* Y] - F_1[F_2 X, F_3^* Y] + \\ &+ F_2[hX, F_2^* Y] + F_3[hX, F_3^* Y] - F_2[F_2^* X, Y] + F_3[F_3^* X, F_1 Y] + \\ &+ F_2^*[hX, F_2 Y] + F_3^*[hX, F_3 Y]\}. \end{aligned}$$

Then $\overset{1}{\nabla}$ is an almost complex connection with respect to F_1.

The proof follows by a straightforward computation.

Calculating $v\,\overset{1}{T}(X, Y) = v\overset{1}{\nabla}_X Y - v\overset{1}{\nabla}_Y X - v\,[X, Y]$ and using the Nijenhuis tensors:

$$N_F(X, Y) = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y]$$

$$\begin{aligned} N_{F_2 F_2^*}(X, Y) = &[F_2 X, F_2^* Y] - F_2[F_2^* X, Y] - F_2^*[X, F_2 Y] + \\ &+ [F_2^* X, F_2 Y] - F_2^*[F_2 X, Y] - F_2[X, F_2^* Y] + [X, Y], \end{aligned}$$

we obtain:

Proposition 4.3. *If $N_{F_2^*} = N_{F_2 F_2^*} = 0$ then $v\overset{1}{T} = 0$, where $\overset{1}{T}$ is the torsion of connection given by* (4.1).

Let us consider the (SQ)-connection:

$$(4.2)\qquad \nabla_X Y = \overset{2}{O}_{1Y} \overset{1}{\nabla}_X, \qquad X, Y \in \mathcal{X}(M).$$

If $N_{F_2^*} = N_{F_2 F_2^*} = 0$ then the connection (4.2) has $vT = v\overset{1}{T} = 0$.

Theorem 4.4. *If $N_{F_2} = N_{F_2^*} = N_{F_2 F_2^*} = 0$ then a symmetric (torsion free) (SQ)-connection is given by:*

$$(4.3)\qquad \tilde{\nabla}_X Y = \nabla_X Y + \overset{2}{O}_{1Y}(T_{vX} - F_2 T_{F_2^* X} + F_2 T_X F_2^*).$$

where, ∇ is the (SQ)-connection given by (4.2), T is its torsion, $T_X Y = T(X, Y)$, $X, Y \in \mathcal{X}(M)$.

Proof. Let us remark that $\tilde{\nabla}$ is an almost tangent connection with respect to F_2 (since belongs to set (3.4')).

From $N_{F_2^*} = N_{F_2 F_2^*} = 0$ it follows $v\tilde{T} = vT = v\overset{1}{T} = 0$, and using the results from [7] it follows:

$$\tilde{h}T(X, Y) = -hN_{F_2}(F_2^* X, F_2^* Y) + \frac{1}{2}\{F_2^* N_{F_2}(F_2^* X, hY) + F_2^* N_{F_2}(hX, F_2^* Y)\}.$$

Thus, if $N_{F_2} = 0$, then $h\tilde{T} = 0$. It follows that $\tilde{\nabla}$ is an almost tangent connection, with $\tilde{T} = 0$. The proof of the theorem is completed if $\tilde{\nabla}$ becomes an (SQ)-connection. In base of relation (3.10'), $\tilde{\nabla}$ will be an (SQ)-connection if and only if there exists $P \in \tau_1^2(M)$ so that:

$$(4.4)\qquad \overset{2}{O}_{1Y} Q_X = (\overset{2}{O}_1 \Psi_1)_Y P_X, \qquad X, Y \in \mathcal{X}(M),$$

where $Q_X = T_{vX} - F_2 T_{F_2^* X} + F_2 T_X F_2^*$.

In particular, if there exists $P \in \tau_2^1(M)$ so that:

$$(4.5)\qquad Q_X = \Psi_1 P_X$$

then condition (4.4) is verified.

The existence of $P\in\tau^1_2(M)$ is equivalent to Obata compatibility of equation (4.5), namely:

$$\Psi_{21}Q_X=0,\ \text{or}\ Q_XY+F_1Q_XF_1Y=0,\qquad X,\ Y\in\mathscr{X}(M). \tag{4.5'}$$

Taking into account that $v\tilde{T}=vT=0$, the condition (4.5') can be written as:

$$hT_{rX}Y-F_2T_XF_2^*Y+F_2T_XF_2^*Y+hF_1T_{rX}F_1Y-F_3T_{F_2^*X}F_1Y-F_3T_XF_3^*Y=0. \tag{4.5''}$$

Calculating the horizontal and vertical parts and using the torsion expression from (4.2), the condition (4.5'') is verified by a direct calculus. So $\tilde{\nabla}$ is an (SQ)-connection.

Remark 4.1. The existence of a symmetric (SQ)-connection is a strong condition, this being equivalent to the integrability of the (SQ)-structure [8].

§5. **Symmetric (SQ)-connections and pair of connections.** Considering the (SQ)-structure, given by (1.1), and $(SQ^*)=(F_1,\ F_2^*,\ F_3^*)$ obtained from a chosen distribution $\mathscr{H}$, then:

$$\begin{aligned}J&=(J_1,\ J_2,\ J_3)=(F_1,\ F_2-F_2^*,\ F_3-F_3^*),\quad P=(P_1,\ P_2,\ P_3)=\\&=(F_1,\ F_2+F_2^*,\ F_3+F_3^*),\end{aligned} \tag{5.1}$$

define an almost quaternion and, respectively, antiquaternion structure. Considering the Nijenhuis tensor fields associated to (J) and (P), by a direct computation, we have:

Proposition 5.1. *The equalities:*

$$N_{J_1}=N_{P_1}=N_{F_1},$$

$$N_{J_2}=N_{F_2}+N_{F_2^*}-N_{F_2F_2^*},\quad N_{P_2}=N_{F_2}+N_{F_2^*}+N_{F_2F_2^*}$$

Proposition 5.2. *In conditions of Theorem 4.4 there exists symmetric connections adapted to the structures* (J) *and* (P).

Proof. If $N_{F_2}=N_{F_2^*}=N_{F_2F_2^*}=0$ then there exists a symmetric (SQ)-connection given by (4.3) and expressing N_{F_1} only by torsion of (SQ)-connection we obtain $N_{F_1}=0$. From Prop. 5.1 it follows there exists (J) and (P) symmetric connections (see [16], [17]).

The converse problem is also interesting. Let:

$$\overset{J}{N}=N_{J_1}+N_{J_2}+N_{J_3}\ \text{and}\ \overset{P}{N}=N_{P_1}+N_{P_2}+N_{P_3}, \tag{5.2}$$

be the Chern-Bernard invariants of (J) and (P)-structures. Then the connection (see [12]):

$$\begin{aligned}\overset{J}{\nabla}_XY&=\frac{1}{12}\sum_{\text{cicl}}\{J_a[J_bX,\ J_cY]-J_a[J_cX,\ J_bY]\}+\\&+\frac{1}{6}\sum_{a=1}^{3}\{J_a[J_aX,\ Y]-J_a[X,\ J_aY]\}+\frac{1}{2}[X,\ Y],\end{aligned} \tag{5.3}$$

has torsion $\overset{J}{T}=0$ and $(\overset{J}{\nabla}_XJ_a)\,Y=\frac{1}{12}\{J_a\overset{J}{N}(X,\ Y)-\overset{J}{N}(X,\ J_aY)\}\quad a=1,\ 2,\ 3.$

Thus, the vanishing of $\overset{J}{N}$ implies the existence of symmetric connection $\overset{J}{\nabla}$.

The connection:

$$\begin{aligned}\overset{P}{\nabla}_XY&=\frac{1}{12}\{-P_1[P_2X,\ P_3Y]+P_1[P_3X,\ P_2Y]-P_2[P_3X,\ P_1Y]+\\&+P_2[P_1X,\ P_3Y]-P_3[P_1X,\ P_2Y]+P_3[P_2X,\ P_1Y]\}+\frac{1}{6}\{P_1[P_1X,\ Y]-\\&-P_1[X,\ P_1Y]-P_2[P_2X,\ Y]+P_2[X,\ P_2Y]-P_3[P_3X,\ Y]+\\&+P_3[X,\ P_3Y]\}+\frac{1}{2}[X,\ Y]\end{aligned} \tag{5.4}$$

has $\overset{P}{T}=0$ and $(\overset{P}{\nabla}_XP_a)\,Y=(1/12)\{P_a\overset{P}{N}(X,\ Y)-\overset{P}{N}(X,\ P_aY)\}\ a=1,\ 2,\ 3.$

The vanishing of $\overset{P}{N}$ gives a symmetric antiquaternion connection $\overset{P}{\nabla}$. By considering the pair of connections $(\overset{J}{\nabla},\ \overset{P}{\nabla})$ [4] and:

$$\overset{m}{\nabla}=\frac{1}{2}(\overset{J}{\nabla}+\overset{P}{\nabla})\ \text{and}\ \tau=\overset{J}{\nabla}-\overset{P}{\nabla} \tag{5.5}$$

the mean connection and deformation tensor of pair $(\overset{J}{\nabla},\ \overset{P}{\nabla})$ it follows $\overset{m}{T}=0$ and:

$$(\overset{m}{\nabla}_XF_1)\,Y=\frac{1}{12}\{F_1(\overset{J}{N}+\overset{P}{N})(X,\ Y)-(\overset{J}{N}+\overset{P}{N})(X,\ F_1Y)\},$$

$$\begin{aligned}2(\overset{m}{\nabla}_XF_2)\,Y-\tau_X(F_2^*Y)+F_2^*\tau_XY&=\frac{1}{12}\{J_2\overset{J}{N}(X,\ Y)+P_2\overset{P}{N}(X,\ Y)-\overset{J}{N}(X,\ J_2Y)-\\&-\overset{P}{N}(X,\ P_2Y)\}.\end{aligned}$$

Hence, we have:

Theorem 5.3. *If* $\overset{J}{\nabla},\ \overset{P}{\nabla}$ *are symmetric connections for* (J) *and* (P) *structures, given by (5.1), then the mean connection (5.5) is a symmetric (SQ)-connection if and only if:*

$$\tau_X(F_2^*Y)=F_2^*\tau_XY,\qquad X,\ Y\in\mathscr{X}(M). \tag{5.6}$$

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SULLE PROPRIETA' DELLA JACOBIANA DI UN SISTEMA LINEARE DI QUADRICHE

LANDO DEGOLI

1. Nello spazio complesso lineare S_r riferito a coordinate proiettive omogenee $x_i (i=0, 1, \ldots, r)$ scegliamo $d+1$ quadriche linearmente indipendenti:

$$f_0=0,\ f_1=0,\ \ldots,\ f_d=0,$$

con:

$$f_q=\sum_{i,k=0}^{r} a_q^{ik} x_i x_k \qquad (a_q^{ik}=a_q^{ki}).$$

Il sistema lineare L_d di dimensione d, che ne risulta, è espresso dall'equazione:

$$\sum_{q=0}^{d} \lambda_q f_q = 0. \tag{1}$$

Supponiamo che la matrice Jacobiana ad $r+1$ righe e $d+1$ colonne:

$$J=\| \partial f_q / \partial x_i \|, \qquad q=0, 1, \ldots, d,\ i=0, 1, \ldots, r,$$

sia di caratteristica $r \leqslant d$.

La varietà Jacobiana data dalla matrice precedente uguagliata a zero rappresenta il luogo dei vertici dei coni del sistema (1). Supponiamo che il sistema lineare L_d sia *irriducibile di prima specie* (vedi 10). Ciò significa che L_d non possiede *sistemi subordinati essenziali*, ossia sistemi $L_{g/c}$ di dimensione g e Jacobiana di caratteristica c con:

$$2 \leqslant g \leqslant d-1, \quad 2 \leqslant c \leqslant r-1, \quad c \leqslant g.$$

In tal caso abbiamo dimostrato il teorema (vedi 6): „*Le quadriche di L_d che passano per un punto hanno in comune una retta.*"

Poichè la Jacobiana ha caratteristica r, tutti i minori di ordine $r+1$ estratti da J sono identicamente nulli. Ricordando che è: $r \leqslant d$, consideriamo il determinante individuato da $r+1$ quadriche qualsiasi lin. ind. del sistema (1). Potremo scegliere, senza nuocere alla generalità, le prime $r+1$ quadriche: $f_0, f_1, \ldots, f_r$ e si avrà:

$$D=| \partial f_i / \partial x_j | = 0. \qquad (i, j=0, 1, \ldots, r).$$