$$b/a = a/b = d/c = c/d = f/e = e/f.$$

Ora i piani base: $a=\pm b$, $d=\pm c$, $e=\pm f$ conducono all'identità: $\pm 1 = \pm 1 = \pm 1 = \pm 1 = \pm 1 = \pm 1$, che soddisfa tutte le precedenti proporzioni.

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FUNDAMENTAL SOLUTION AND RETARDED POTENTIAL FOR A RIGID THERMAL CONDUCTOR OF CATTANEO TYPE

BY

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1. **Introduction.** It is well-known that Fourier's model of a rigid heat conductor gives unbounded velocity of propagation for thermal disturbances. In order to avoid this fact, Cattaneo introduced in 1948 [1] a modified version of the Fourier's law for the heat flux by taking into account a kind of „thermal inertia". This model, obtained also by Vernotte in 1958 [2] and by Kaliski in 1965 [3], gives a finite speed for the propagation of thermal effects (for other details see [4], Cap. I and [5]).

In the following I shall obtain the fundamental solution and the retarded potential for Cattaneo's model, because the similar results given by Kaliski and Nowacki in 1969 [6] are erroneus (see further).

The fundamental equations of this model are:

— The local form of the energy balance:

$$\rho_0 \dot{\varepsilon} = -\mathrm{div}\ \mathbf{q} + \rho_0 r, \tag{1.1}$$

where ρ_0 is the constant mass density, ε represents the specific internal energy per unit mass, $\mathbf{q}$ is the heat flux vector and r represents the external heat source density per unit mass of the rigid heat conducting body.

— The constitutive equation for the specific internal energy:

$$\varepsilon = \varepsilon_0 + c_0 T, \qquad c_0 > 0, \tag{1.2}$$

where ε_0 is the constant specific internal energy corresponding to the constant refference absolute temperature θ_0 of the conductor, c_0 represents the constant specific heat per unit mass corresponding to the refference temperature θ_0 and $T = \theta - \theta_0$ is the variation of the absolute temperature θ due to external thermal actions.

— The evolution equation of Cattaneo type for the heat flux $\mathbf{q}$:

$$\tau \dot{\mathbf{q}} + \mathbf{q} = -\varkappa_0\ \mathrm{grad}\ T, \quad \tau > 0, \quad \varkappa_0 > 0, \tag{1.3}$$

where τ is the relaxation time of the heat flux and the material constant $\varkappa_0$ represents the coefficient of heat conductivity (of Fourier type) for the considered rigid conductor.

In the given relations ε, $\mathbf{q}$, r and T are functions on the space variable $\mathbf{x}$ and on the time variable t; the superposed dot stands for the time derivative with respect to t.

As it is clear from the given relations, we have a linear model for an isotropic and homogeneous rigid heat conductor.

In all what follows I assume that all operations are meaningfull in usual or in generalized sense.

In order to obtain the field equation satisfied by T we can eliminate q by taking the time derivative of (1.1) and by using (1.2) and (1.3); after some simple calculus it results

$$\rho_0 c_0(\tau\ddot{T}+\dot{T})=\varkappa_0\Delta T+\rho_0(\tau\dot{\gamma}+\gamma). \tag{1.4}$$

The term $\rho_0\tau\dot{\gamma}$ is dismissed by Kaliski and Nowacki (see Eq. (2.1) in [6]). It is clear that this term cannot be neglegted when the heat source r has important variations in small time intervals, even if τ is of order $10^{-11}-10^{-6}$ sec. (see for instance [4], Cap. I and [5]). In order to obtain the fundamental solution of Eq. (1.4) we must take $r(\mathbf{x}, t)=\text{const}\ \delta(\mathbf{x})\ \delta(t)$ exactly in the same manner as Kaliski and Nowacki done in [6]. Here δ represents Dirac's distribution. As it is well-known (see [7] §1.2.3.2), the time derivative $\dot{\delta}(t)$ as well as the time derivative of usual functions approximating $\delta(t)$ and $\dot{\delta}(t)$, respectively, have important variations in their magnitude for very small variations of the argument t. Consequently, $\tau\dot{\gamma}$ cannot be neglected in (1.4) in comparison with r when our aim is the search of the fundamental solution corresponding to the generalized heat propagation law (1.4).

In the sequel I assume

$$T(\mathbf{x},\ t)=0,\ \gamma(\mathbf{x},\ t)=0 \text{ for } t<0, \tag{1.5}$$

2. **The fundamental solution.** By introducing

$$\lambda=\varkappa_0/\rho_0 c_0,\ q=\gamma/c_0, \tag{2.1}$$

from (1.4) it results

$$\tau\ddot{T}+\dot{T}=\lambda\Delta T+\tau\dot{q}+q, \tag{2.2}$$

$q=q(\mathbf{x},\ t)$ being a given function of $\mathbf{x}$ and t.

This equation must be supplemented by appropriate initial and boundary conditions.

In this section I assume that the rigid conductor is unbounded. Also, I consider an external heat impuls placed in $\mathbf{x}=0$; namely

$$q(\mathbf{x},\ t)=\delta(\mathbf{x})\ \delta(t). \tag{2.3}$$

In this case from (2.2) it results

$$\tau\ddot{T}+\dot{T}=\lambda\Delta T+(\tau\dot{\delta}(t)+\delta(t))\ \delta(\mathbf{x}). \tag{2.4}$$

The assumed initial conditions are taken to be homogeneous:

$$T(\mathbf{x},\ 0)=0,\ \dot{T}(\mathbf{x},\ 0)=0, \tag{2.5}$$

and I consider the following boundary conditions

$$T(\mathbf{x},\ t)=O(|\mathbf{x}|^{-1}),\ \text{grad}\ T(\mathbf{x},\ t)=O(|\mathbf{x}|^{-2}) \quad \text{for} \quad |\mathbf{x}|\to\infty. \tag{2.6}$$

By definition *the fundamental solution* (the Green's function) is the thermal field $T=T(\mathbf{x},\ t)$ satisfying (2.4)–(2.6).

In order to obtain this field I use Laplace and Fourier integral transforms. Denoting by $\overline{T}=\overline{T}(\mathbf{x},\ p)$ the Laplace transform of $T=T(\mathbf{x},\ t)$, taking into account the homogeneous initial conditions (2.5) and using the relations*)

$$\mathfrak{L}[\ddot{f}(t)]=p^2\bar{f}(p)-pf(0)-\dot{f}(0),\ \mathfrak{L}[\dot{f}(t)]=p\bar{f}(p)-f(0), \tag{2.7}$$

$$\mathfrak{L}[\delta(t)]=1,\ \mathfrak{L}[\dot{\delta}(t)]=p, \tag{2.8}$$

from (2.4) I obtain

$$(\tau p^2+p)\ \overline{T}=\lambda\Delta\overline{T}+(\tau p+1)\ \delta(\mathbf{x}). \tag{2.9}$$

Let me denote by

$$\hat{T}(\mathbf{k},\ p)=\mathfrak{F}[\overline{T}(\mathbf{x},\ p)]=(1/2\pi)^{3/2}\iiint_{-\infty}^{+\infty}\overline{T}(\mathbf{x},\ p)\,e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x},$$

$$\mathbf{k}\cdot\mathbf{x}=k_1x_1+k_2x_2+k_3x_3,\ d\mathbf{x}=dx_1dx_2dx_3$$

the three-dimensional Fourier transform of $\overline{T}(x,\ p)$ (see [8], Cap. I). Using

$$\mathfrak{F}(\Delta\overline{T})=-k^2\hat{T},\ k^2=k_1^2+k_2^2+k_3^2,\quad \mathfrak{F}(\delta(\mathbf{x}))=(1/2\pi)^{3/2},$$

after some simple algebra, from (2.9) it results

$$\hat{T}(\mathbf{k},\ p)=(1/2\pi)^{3/2}\frac{\tau}{\lambda}\,\frac{p+1/\tau}{k^2+(\tau/\lambda)p(p+1/\tau)}. \tag{2.10}$$

By introducing the quantity

$$U=\sqrt{\lambda/\tau}, \tag{2.11}$$

having the physical dimension of a velocity, (2.10) can be written in the following form

$$\hat{T}(\mathbf{k},\ p)=\frac{(1/2\pi)^{3/2}}{U^2}\,\frac{p+1/\tau}{k^2+U^{-2}\,p(p+1/\tau)}. \tag{2.12}$$

According to the inversion theorem

$$\overline{T}(\mathbf{x},\ p)=\mathfrak{F}^{-1}[\hat{T}(\mathbf{k},\ p)]=(1/2\pi)^{3/2}\iiint_{-\infty}^{+\infty}\hat{T}(\mathbf{k},\ p)e^{-i\mathbf{x}\cdot\mathbf{k}}d\mathbf{k},\ d\mathbf{k}=dk_1dk_2dk_3,$$

and, consequently, from (2.12) I get

$$\overline{T}(\mathbf{x},\ p)=\frac{(1/2\pi)^3}{U^2}(p+1/\tau)\iiint_{-\infty}^{\infty}\frac{e^{-\mathbf{x}\cdot\mathbf{k}}}{k^2+U^2p(p+1/\tau)}\,d\mathbf{k},$$

*) Concerning the notations, see Appendix.

or, equivalently,

$$\bar{T}(\mathrm{x},\ p)=\frac{p+1/\tau}{\pi^3 U^2}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos x_1k_1\ \cos x_2k_2\ \cos x_3k_3}{k_1^2+k_2^2+k_3^2+U^{-2}p(p+1/\tau)}\,\mathrm{d}k_1,\ \mathrm{d}k_2,\ \mathrm{d}k_3. \tag{2.13}$$

Taking into account the relation ([9], Cap. XI, §3)

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos x_1k_1\ \cos x_2k_2\ \cos x_3k_3}{k_1^2+k_2^2+k_3^2+\alpha^2}=\frac{\pi^2}{4}\frac{1}{R}\,\mathrm{e}^{-\alpha R},\ \alpha=\mathrm{const}>0,\ R=|\mathrm{x}|=\sqrt{x_1^2+x_2^2+x_3^2},$$

from (2.13) it results for $\mathcal{L}=U^{-1}\sqrt{p(p+1/\tau)}$

$$\bar{T}(\mathrm{x},\ p)=\bar{T}(R,\ p)=\frac{1}{4\pi U^2R}\,(p+1/\tau)\,\mathrm{e}^{-(R/U)\sqrt{p(p+1/\tau)}} \tag{2.14}$$

Again, according to the inversion theorem

$$T(\mathrm{x},\ t)=\mathfrak{L}^{-1}\ [\bar{T}(\mathrm{x},\ p)]=\frac{1}{4\pi U^2R}\,\mathfrak{L}^{-1}\ [(p+1/\tau)\,\mathrm{e}^{-R/U\sqrt{p(p+1/\tau)}}]\,. \tag{2.15}$$

Making now use on the relation (A.25) for $b=1/\tau$, $c=N/U$ and taking into account the equation (2.11) for U it results *the following final expression for the fundamental solution* denoted by $G(\mathrm{x},\ 0,\ t)$:

$$\begin{aligned} G(\mathrm{x},\ 0,\ t)=G(R,\ t)=\gamma(\mathrm{x},\ 0,\ t)\ H(t-R/U)+\\ +\frac{1}{4\pi\lambda R}\,\mathrm{e}^{-R/(2\tau U)}\left\{\left(1+\frac{R}{8\tau U}\right)\delta(t-R/U)+\tau\dot{\delta}(t-R/U)\right\},\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}\gamma(\mathrm{x},\ 0,\ t)=\gamma(R,\ t)=\frac{1}{8\pi\lambda U}\frac{\mathrm{e}^{-t/2\tau}}{\sqrt{t^2-R^2/U^2}}\left\{\left(\frac{1}{2\tau}-\frac{2t}{t^2-R^2/U^2}\right)I_1\left(\frac{1}{2\tau}\sqrt{t^2-R^2/U^2}\right)+\right.\\ \left.+\frac{1}{2\tau}\frac{t}{\sqrt{t^2-R^2/U^2}}I_0\left(\frac{1}{2\tau}\sqrt{t^2-R^2/U^2}\right)\right\},\quad t\geqslant R/U.\end{aligned} \tag{2.17}$$

This solution obviously satisfies the boundary conditions (2.6) for all $t>0$.

Concerning the obtained result it can be done some important observations :

(1) From (2.16) it results that the heat impulse applied at $t=0$ in $\mathrm{x}=0$ propagates in space and time at finite velocity U; this is a characteristic property of Cattaneo's model in comparison with the corresponding property of Fourier's model, predicting infinite velocity for the propagation of thermal disturbances.

(2) The value (2.11) obtained for the velocity of propagation coincides with that given by L u c a [5] in studying the acceleration waves of Hadamard type in rigid heat conductors of integral type.

(3) The term in (2.16) containing the Heaviside function represents a singular surface of Hadamard type. Across this surface the thermal field has a finite jump. The singular surface is spherical, centered in $\mathrm{x}=0$, its radius R being increasing in time according to the law $R=Ut$.

By using

$$I_0(x)=1+x^2/4+O(x^4),\ \ I_1(x)=x/2+x^3/16+O(x^5)$$

from (2.17) for the magnitude of the jump it results

$$\begin{aligned}[T]\,(R,\ t)=\lim_{t\to(R/U)^+}\gamma(R,\ t)\ H(t-R/U)-\lim_{t\to(R/U)^-}\gamma(R,\ t)\ H(t-R/U)=\\ =\frac{1}{64\pi(\lambda\tau)^{3/2}}\left(1+\frac{1}{8\tau}\frac{R}{U}\right)\mathrm{e}^{-R/(2\tau U)}=\frac{1}{64\pi(\lambda\tau)^{3/2}}\left(1+\frac{1}{8}\frac{t}{\tau}\right)\mathrm{e}^{-t/(2\tau)}\text{for } R=Ut.\end{aligned} \tag{2.18}$$

The first equation gives the jump as a space function, the second as a time function.

(4) From (2.18) it is clear that the magnitude of the jump is a decreasing function of time, or, equivalently, of space, the damping in time being completely dermined by the relaxation time τ. This result is in perfect agreement with that obtained by L u c a in [5], by investigating the attenuation of acceleration waves in a rigid conductor of integral type.

(5) It is clear that for $\tau=0$, the heat propagation equation (2.2) reduces to that of Fourier type. Consequently, it is interesting to etablish the connexion between the fundamental solution of Cattaneo type (2.16), (2.17) and the fundamental solution corresponding to Fourier's model. Denoting by G_F (x,0, t) the last field, I remember that (see [10], Cap. III)

$$G_F(\mathrm{x},\ 0,\ t)=G_F(R,\ t)=(4\pi\lambda t)^{-3/2}\ \mathrm{e}^{-R/(4\lambda t)}. \tag{2.19}$$

It is clear from (2.19) that according to the classical model the heat pulse applied at $t=0$ in $x=0$ propagates instantaneously ; namely

$$G_F(R,\ t)>0\ \ \forall R>0\ \text{ for } t>0.$$

It is clear from (2.11) that

$$\lim_{\tau\to 0}\ U(\tau)=\infty, \tag{2.20}$$

a fact which agrees perfectly with the properties of the classical model. Consequently, taking into account the properties of Dirac's distribution, from (2.16) it results

$$\lim_{\tau\to 0}\ G(R,\ t,\ \tau)=\lim_{\tau\to 0}\ \gamma(R,\ t,\ \tau)\qquad \textit{for } t>0.$$

In order to obtain this value I make use on the assymptotic expansion formula ([11], Cap. C) :

$$I_n(x)=\sqrt{1/(2\pi x)}\ \mathrm{e}^x\{1+O(x^{-1})\}\qquad \text{for } x\to\infty. \tag{2.21}$$

Also, according to the relations (2.11) and (2.20), we have

$$(2.22)\quad \frac{1}{2\tau}\sqrt{t^2-R^2/U^2}=\frac{t}{2\tau}-\frac{R^2}{4\lambda t}+O(U^{-2}_{(\tau)})\quad \text{and}\quad U(\tau)\to\infty\quad\text{for}\quad \tau\to 0.$$

By using (2.22) and (2.21), after some calculus, it results

$$(2.23)\qquad \lim_{\tau\to 0}\gamma(R,\,t,\,\tau)=(4\pi\lambda t)^{-3/2}\,e^{R^2/(4\lambda t)}\qquad\text{for}\qquad t>0.$$

Comparing (2.19) and (2.23) I obtain the principal result:

$$(2.24)\qquad \lim_{\tau\to 0}G(R,\,t,\,\tau)=G_F(R,\,t)\qquad\text{for}\qquad t>0.$$

(6) In view of the differences in the heat propagation equation it is clear that the fundamental solution obtained in this paper does not coincide with that given by Kaliski and Novacki (see Eq. (3.4) in [6]).

Closing this section I observe that in the case in which the instantaneous heat impulse acts in a point $\xi\neq 0$, the fundamental solution denoted by $G(\mathbf{x},\,\xi,\,t)$ has the following form

$$(2.25)\qquad G(\mathbf{x},\,\xi,\,t)=\gamma(\mathbf{x},\,\xi,\,t)\,H\left(t-\frac{R(\mathbf{x},\,\xi)}{U}\right)+$$
$$+\frac{1}{4\pi\lambda R(\mathbf{x},\,\xi)}e^{-R(\mathbf{x},\,\xi)/U}\left\{\left(1+\frac{R(\mathbf{x},\,\xi)}{8\tau U}\right)\delta\left(t-\frac{R(\mathbf{x},\,\xi)}{U}\right)+\tau\dot{\delta}\left(t-\frac{R(\mathbf{x},\,\xi)}{U}\right)\right\},$$

$$(2.26)\qquad R(\mathbf{x},\,\xi)=|\,\mathbf{x}-\xi\,|=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2},$$

and $\gamma(\mathbf{x},\,\xi,\,t)$ can be obtained from (2.17) by replacing R with $R(x,\,\xi)$.

In the following, for conciseness, $R(\mathbf{x},\,\xi)$ given in (2.26) will be denoted simply by R.

3. **The reciprocity theorem.** Let V be the spatial domain occupied by the rigid conductor and let S be the boundary of V. I consider now two sets of causes and effects. The causes will include the action of heat sources q and q_1 given in V, the action of external temperatures $\tilde{T}$ and $\tilde{T}_1$ given on S. To the effects belong the thermal fields T and T_1 in V at any instant t. The first set of causes and effects is described by the propagation equation (2.2), reproduced here for commodity as Eq. (3.1):

$$(3.1)\qquad \tau\ddot{T}+\dot{T}=\lambda\Delta T+\tau\dot{q}+q\quad\text{in}\quad V.$$

To this equation I add the boundary and initial conditions

$$(3.2)\quad T(\mathbf{x},\,t)=\tilde{T}(\mathbf{x},\,t)\ \text{on}\ S,\ t\geq 0,\ T(\mathbf{x},\,0)=T_0(\mathbf{x}),\ \dot{T}(\mathbf{x},\,0)=\dot{T}_0(\mathbf{x})\ \text{in}\ V.$$

The second set of causes and effects satisfies the propagation equation

$$(3.3)\qquad \tau\ddot{T}_1+\dot{T}_1=\lambda\Delta T_1+\tau\dot{q}_1+q_1\quad\text{in}\quad V,$$

with boundary and initial conditions

$$(3.4)\quad T_1(\mathbf{x},\,t)=\tilde{T}_1(\mathbf{x},\,t)\ \text{on}\ S,\ t\geq 0,\ T_1(\mathbf{x},\,0)=T_{10}(\mathbf{x}),\ \dot{T}_1(\mathbf{x},\,0)=\dot{T}_{10}(\mathbf{x})\ \text{in}\ V.$$

By applying the Laplace transform and taking into account (2.7) and the initial conditions (3.2), (3.4), from (3.1) and (3.3) it results

$$(3.5)\quad \tau(p^2\bar{T}-pT_0-\dot{T}_0)+(p\bar{T}-T_0)=\lambda\Delta\bar{T}+\tau(p\bar{q}-q_0)+\bar{q},\qquad q_0(\mathbf{x})=q(\mathbf{x},\,0),$$

$$(3.6)\quad \begin{aligned}&\tau(p^2\bar{T}_1-pT_{10}-\dot{T}_{10})+(p\bar{T}_1-T_{10})=\lambda\Delta\bar{T}_1+\tau(p\bar{q}_1-q_{10})+\bar{q}_1,\\ &q_{10}(\mathbf{x})=q_1(\mathbf{x},\,0).\end{aligned}$$

Let us multiply Eq. (3.5) by $\bar{T}_1$ and Eq. (3.6) by $\bar{T}$ and substract one of the obtained result from the other; in this way it results:

$$(3.7)\quad \begin{aligned}&-\tau\{p(T_0\bar{T}_1-T_{10}\bar{T})+(\dot{T}_0\bar{T}_1-\dot{T}_{10}\bar{T})\}-(T_0\bar{T}_1-T_{10}\bar{T})=\\ &=\lambda(\bar{T}_1\Delta\bar{T}-\bar{T}\Delta\bar{T}_1)+\tau\{p(\bar{q}\bar{T}_1-\bar{q}_1\bar{T})-(q_0\bar{T}_1-q_{10}\bar{T})\}+(\bar{q}\bar{T}_1-\bar{q}_1\bar{T}).\end{aligned}$$

By using $\mathfrak{L}^{-1}\,[1]=\delta(t)$, from (2.7) we get

$$(3.8)\qquad \mathfrak{L}^{-1}[p\bar{f}(p)]=\dot{f}(t)+f(0)\,\delta(t).$$

Consequently, making use of the convolution theorem and taking into account the well-known properties of Dirac's δ, it results

$$\mathfrak{L}^{-1}[p(\bar{q}\bar{T}_1-\bar{q}_1\bar{T})-(q_0\bar{T}_1-q_{10}\bar{T})]=\int_0^t\{\dot{q}(t-s)\,T_1(s)-\dot{q}_1(t-s)\,T(s)\}\,\mathrm{d}s.$$

Applying this formula in (3.7) and taking into account again (3.8), we get

$$(3.9)\quad \begin{aligned}&-\tau\{(T_0\dot{T}_1-T_{10}\dot{T})+(\dot{T}_0T_1-\dot{T}_{10}T)\}-(T_0T_1-T_{10}T)=\\ &=\lambda\int_0^t\{T_1(t-s)\,\Delta T(s)-T(t-s)\,\Delta T_1(s)\}\,\mathrm{d}s+\int_0^t\{w(t-s)\,T_1(s)-\\ &\qquad -w_1(t-s)\,T(s)\}\,\mathrm{d}s,\end{aligned}$$

$$(3.10)\qquad w=\tau\dot{q}+q,\qquad w_1=\tau\dot{q}_1+q_1.$$

Let us now integrate Eq. (3.9) over the region V. Applying Green's transformation it results *the* following *reciprocity relation of Betti type*:

$$(3.11)\quad \begin{aligned}&-\tau\int_V\{T_0\dot{T}_1(t)-T_{10}\dot{T}(t)+\dot{T}_0T_1(t)-\dot{T}_{10}T(t)\}\mathrm{d}V-\int_V\{T_0T_1(t)-T_{10}T(t)\}\mathrm{d}V=\\ &=\lambda\int_S\mathrm{d}S\int_0^t\left\{\tilde{T}_1(t-s)\frac{\partial T(s)}{\partial n}-\tilde{T}(t-s)\frac{\partial T_1(s)}{\partial n}\right\}\mathrm{d}s+\\ &+\int_V\mathrm{d}V\int_0^t\{w(t-s)\,T_1(s)-w_1(t-s)\,T(s)\}\,\mathrm{d}s.\end{aligned}$$

For conscieness the spatial argument $\mathbf{x}$ was omitted.

In the following I assume that the conductor is unbounded and the thermal fields satisfy boundary conditions of the type (2.6). Also I assume that the heat sources and the initial conditions concern a bounded region of V. In this conditions, in the limit the surface integral in (3.11) vanishes and (3.11) becomes

$$
(3.12)\qquad \begin{aligned} &-\tau\int\{T_0\dot{T}_1(t)-T_{10}\dot{T}(t)+\dot{T}_0T_1(t)-\dot{T}_{10}T(t)\}\,\mathrm{d}V-\int\{T_0T_1(t)-\\ &-T_{10}T(t)\}\mathrm{d}V=\int\mathrm{d}V\int_0^t\{w(t-s)\;T_1(s)-w_1(t-s)\;T(s)\}\,\mathrm{d}s, \end{aligned}
$$

the volum integral being extended on the whole space.

In the special case of homogeneous initial conditions of type (2.5) the reciprocity relation (3.11) reduces to the following simple form

$$
(3.13)\qquad \int\mathrm{d}V\int_0^t w(\mathbf{x},\ t-s)\;T_1(\mathbf{x},\ s)\,\mathrm{d}s=\int\mathrm{d}V\int_0^t w_1(\mathbf{x},\ t-s)\;T(\mathbf{x},s)\,\mathrm{d}s,
$$

where w and w_1 are given by (3.10).

The relations (3.11)—(3.13) must take the place of the erroneous reciprocity formulas given by Kaliski and Novacki (see especially §2 in [6]).

4. The retarded potential. Let me consider now an infinite body, wherein some source $q(\mathbf{x},\ t)$ located within a bounded region is acting. The *homogeneous initial conditions* are assumed. The thermal field $T(\mathbf{x},\ t)$ produced by the given source $q(\mathbf{x},\ t)$ satisfies Eq. (3.1). It can be determined from the reciprocity relation (3.13) assuming $q_1(\mathbf{x},\ \xi,\ t)=\delta(\mathbf{x}-\xi)\ \delta(t)$. Consequently in (3.12) $T_1=G(\mathbf{x},\ \xi,\ t)$, where $G(\mathbf{x},\ \xi,\ t)$ denotes the fundamental solution given by (2.25).

Applying well-known properties of Dirac's δ, from (3.13) it results the following differential equation which must be satisfied by the thermal field $T(\mathbf{x},\ t)$:

$$
(4.1)\qquad \tau\dot{T}(\mathbf{x},\ t)+T(\mathbf{x},\ t)=F(\mathbf{x},\ t),
$$

the known function $F(\mathbf{x},\ t)$ having the following expression

$$
(4.2)\qquad F(\mathbf{x},\ t)=\int\mathrm{d}V(\xi)\int_0^t w(\xi,\ t-s)\;G(\mathbf{x},\ \xi,\ s)\,\mathrm{d}s.
$$

Taking into account the assumed homogeneous initial condition (2.5), from (4.1) it results

$$
(4.3)\qquad T\,(\mathbf{x},\ t)=\frac{1}{\tau}\,e^{-t/\tau}\int_0^t F(\mathbf{x},\ s)\;e^{s/\tau}\,\mathrm{d}s.
$$

Consequently,

$$
(4.4)\qquad T(\mathbf{x},\ t)=\frac{1}{\tau}\,e^{-t/\tau}\int\mathrm{d}V(\xi)\int_0^t e^{s/\tau}\mathrm{d}s\int_0^s w(\xi,\ s-\alpha)\;G(\mathbf{x},\ \xi,\ \alpha)\,\mathrm{d}\alpha.
$$

By using Fubini's theorem the last result can be written in the equivalent form

$$
T(\mathbf{x},\ t)=\frac{1}{\tau}\,e^{-t/\tau}\int\mathrm{d}V(\xi)\int_0^t G(\mathbf{x},\ \xi,\ \alpha)\,\mathrm{d}\alpha\int_\alpha^t e^{s/\tau}w(\xi,\ s-\alpha)\,\mathrm{d}s.
$$

Taking into account (3.10) it is easy to see that

$$
\frac{\mathrm{d}}{\mathrm{d}s}\,(e^{s/\tau}q(\xi,\ s-\alpha))=\frac{1}{\tau}\,e^{s/\tau}w(\xi,\ s-\alpha).
$$

Consequently, it results the following expression for the thermal field produced by the given source distribution

$$
(4.5)\qquad T(\mathbf{x},\ t)=\int\mathrm{d}V(\xi)\int_0^t G(\mathbf{x},\ \xi,\ s)\;Q(\xi,\ t-s)\,\mathrm{d}s,
$$

$$
(4.6)\qquad Q(\mathbf{x},\ t)=q(\mathbf{x},\ t)-e^{-t/\tau}q(\mathbf{x},\ o).
$$

Applying again in (4.5) the well-known properties of Dirac's δ and using (2.25), after some calculus it results *the final form of the field*:

$$
(4.7)\qquad \begin{aligned} T(\mathbf{x},\ t)=&\int\mathrm{d}V(\xi)\int_0^t\gamma(\mathbf{x},\ \xi,\ s)\;Q(\xi,\ t-s)\;H(s-R/U)\,\mathrm{d}s+\\ &+\frac{1}{4\pi\lambda}\int\frac{1}{R}\,e^{-R/(2\tau U)}\,w(\xi,t-R/U)\,\mathrm{d}V(\xi)+\frac{1}{32\lambda\tau U}\int e^{-R/(2\tau U)}Q(\xi,\ t-\\ &-R/U)\,\mathrm{d}V(\xi), \end{aligned}
$$

where $R=R(\mathbf{x},\ \xi)=|\ \mathbf{x}-\xi\ |$ and $\gamma,\ w,\ Q$ are given by (2 .17), (2.25) and (4.6), respectively.

The relation (4.7) must take the place of the incorrect formula given by Kaliski and Nowacki for the field produced by given sources (see especially Eq. (3.6) in [4]).

Concerning the obtained result it can be done some remarks

(1) the expression (4.7) containes the retarded argument $t-R/U$. It occurs directly in the second and the third integral. Thus the solution is given by a retarded potential.

(2) The occurence of the Heaviside function in the first integral shows that the thermal field in a place $\mathbf{x}$ at a moment t is entirelly determined by sources acting in a sphere centered in $\mathbf{x}$ and having the radius $R=Ut$.

(3) Consequently, the Cattaneo model indeed provides finite speed of propagation for thermal disturbances, the magnitude of the velocity being $U=\sqrt{\lambda/\tau}$.

Finally I note that in the special case $q(\mathbf{x},\ 0)=0$ in V, the solution (4.7) gets the following more simple form:

$$
(4.8)\qquad \begin{aligned} T(\mathbf{x},\ t)=&\int\mathrm{d}V(\xi)\int_0^t\gamma(\mathbf{x},\ \xi,\ s)\;q(\xi,\ t-s)\;H(s-R(U)\,\mathrm{d}s+\\ &+\frac{1}{4\pi\lambda}\int\frac{1}{R}\,e^{-R/(\tau U)}W(\xi,\ t-R/U)\,\mathrm{d}V/\xi)+\frac{1}{32\lambda\tau U}\int e^{-R/(2\tau U)}\\ &q(\xi,\ t-R/U)\,\mathrm{d}V/\xi) \end{aligned}
$$

5. Application: The concentred heat source of Heaviside type. For a concentrated heat source of Heaviside type $q(\mathbf{x},\ t)=\delta(\mathbf{x})\ H(t)$ $(H(t)=0$ for

$t \leqslant 0$, $H(t)=1$ for $t>0$) acting in an infinite body, the thermal field $T(\mathbf{x}, t)$ must satisfy the propagation equation

$$\tau \ddot{T} + \dot{T} = \lambda \Delta T + w, \qquad w(\mathbf{x}, t) = \{\tau\delta(t) + H(t)\}\,\delta(\mathbf{x}). \tag{5.1}$$

I assume that $T(\mathbf{x}, t)$ satisfies the homogeneous initial and boundary conditions (2.5) and (2.6), respectively.

The temperature field produced by the given source can be obtained by two methods: (*a*) using the Eq. (4.8) containing the retarded potentials; (*b*) using Laplace and Fourier integral transforms.

(*a*) By using (4.8) for $q(\xi, t) = \delta(\xi)\,H(t)$ and $w(\xi, t) = \delta(\xi)\,(\tau\delta(t)+H(t))$, it results

$$T(\mathbf{x}, t) = \int_{R/U}^{t} \gamma(R, s)\,\mathrm{d}s H(t-R/U) + \frac{1}{4\pi\lambda R}\, e^{-R/(2\tau U)} \left\{\left(1+\frac{R}{8\tau U}\right)\cdot \right.$$
$$\left. \cdot H(t-R/U) + \tau\delta(t-R/U)\right\}, \tag{5.2}$$

where $R = R(\mathbf{x}, 0) = |\,\mathbf{x}\,|$.

Taking into account (A.18) and (A.23) it is easy to see that $\gamma(R, t)$ given by (2.17) can be written in the following equivalent form:

$$\gamma(R, t) = \frac{1}{8\pi\lambda U}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^{-t/2\tau}}{\sqrt{t^2-R^2/U^2}}\, I_1\left(\frac{1}{2\tau}\sqrt{t^2-R^2/U^2}\right)\right) + \right.$$
$$\left. + \frac{1}{\tau}\,\frac{e^{-t/2\tau}}{\sqrt{t^2-R^2/U^2}}\, I_1\left(\frac{1}{2\tau}\sqrt{t^2-R^2/U^2}\right)\right\}. \tag{5.3}$$

Finally, by introducing (5.3) in (5.2), after some elementary calculus, it results

$$T(\mathbf{x}, t) = \frac{1}{8\pi\lambda U}\left\{\frac{e^{-t/2\tau}}{\sqrt{t^2-R^2/U^2}}\, I_1\left(\frac{1}{2\tau}\sqrt{t^2-R^2/U^2}\right) + \right.$$
$$\left. + \frac{1}{\tau}\int_{R/U}^{t} \frac{e^{-s/2\tau}}{\sqrt{s^2-R^2/U^2}}\, I_1\left(\frac{1}{2\tau}\sqrt{s^2-R^2/U^2}\right)\mathrm{d}s\right\} H(t-R/U) + \tag{5.4}$$
$$+ \frac{1}{4\pi\lambda R}\, e^{-R/(2\tau U)}\{H(t-R/U)+\tau\delta(t-R/U)\}.$$

(*b*) By using Laplace and Fourier integral transforms and proceeding as in section 2, for the Laplace transform $\overline{T}(\mathbf{x}, p)$ of the produced field it results

$$\overline{T}(\mathbf{x}, p) = \overline{T}(R, p) = \frac{1}{4\pi U^2 R}\left\{\left(1+\frac{1}{\tau p}\right) e^{-(R/U)\sqrt{p(p+1/\tau)}}\right\}, \qquad R = |\,\mathbf{x}\,|. \tag{5.5}$$

Now, by taking into account the relation

$$\mathfrak{L}^{-1}\left[\frac{1}{p}\,\mathfrak{L}\,[f(t)]\right] = \int_0^t f(s)\,\mathrm{d}s$$

and using (A.14), we get again the expression (5.4) for the temperature field $T(x, t)$ produced by the concentrated source of the Heaviside type.

Closing this section, I note that for a concentrated source $q(\mathbf{x}, t) = \delta(\mathbf{x})\,tH(t)$ the expression giving the produced thermal field will be an ordinary function having a nonvanishing jump for $t = R/U$. This and similar situations will be investigated elsewhere.

Appendix. I assume that all operations are meaningfull in ordinary or in generalized sense and I use classical notations if generalized functions (distributions) appear. I denote by

$$\bar{f}(p) = \mathfrak{L}[f(t)] = \int_0^\infty f(t)\, e^{-pt}\mathrm{d}t \tag{A.1}$$

the Laplace transform of a function $f(t)$ and by $\mathfrak{L}^{-1}$ the inverse Laplace operator.

In what follows I shall obtain

$$\mathfrak{L}^{-1}\,[(p+b)\, e^{-c\sqrt{p(p+b)}}], \quad b = \text{const.} > 0, \quad c = \text{const.} > 0. \tag{A.2}$$

In order to solve this problem, let me consider the Laplace image

$$\bar{f}(p) = \mathfrak{L}[f(t)] = \frac{1}{\sqrt{p(p+d)}}\, e^{-c\sqrt{p(p+b)}}. \tag{A.3}$$

It is known that (see [12], rel. 5.138)

$$f(t) = \varphi(t)\,H(t-c), \quad \varphi(t) = e^{bt/2}\, I_0\left(\frac{b}{2}\sqrt{t^2-c^2}\right), \quad t \geqslant c, \tag{A.4}$$

where

$$H(t) = 0 \text{ for } t \leqslant 0, \; H(t) = 1 \text{ for } t > 0 \tag{A.5}$$

is the Heaviside step function and $I_n(t)$ represents the n^{th} order Bessel function having modified argument.

Let me consider now the function

$$\bar{g}(p) = e^{-c\sqrt{p(p+b)}}; \tag{A.6}$$

by taking the derivative $\bar{g}'(p)$ of $\bar{g}(p)$ I get

$$\bar{g}'(p) = c - \left(p + \frac{b}{2}\right)\bar{f}(p),$$

consequently I have

$$\mathfrak{L}^{-1}\,[\bar{g}'(p)] = -c\mathfrak{L}^{-1}\left[\left(p+\frac{b}{2}\right)\bar{f}(p)\right]. \tag{A.7}$$

Taking into account the well-known relation

$$\mathfrak{L}^{-1}[\bar{g}'(p)] = -t\mathfrak{L}^{-1}[\bar{g}(p)] = -tg(t), \tag{A.8}$$

from (A.7) I get

$$tg(t) = c\mathfrak{L}^{-1}\left[\left(p+\frac{b}{2}\right)\bar{f}(p)\right], \tag{A.9}$$

or, by using $(A.3)_1$,

$$tg(t)=c\left\{\mathfrak{L}^{-1}\,[p\bar{f}(p)]+\frac{b}{2}f(t)\right\}.$$

Making now use on the relation

$$\mathfrak{L}[\dot{f}(t)]=p\bar{f}(p)-f(0), \tag{A.10}$$

I obtain

$$\mathfrak{L}^{-1}\,[p\bar{f}(p)]=\dot{f}(t)+f(0)\delta(t), \tag{A.11}$$

where δ represents the Dirac distribution having the properties

$$\mathfrak{L}[\delta(t)]=1, \qquad \mathfrak{L}^{-1}[1]=\delta(t). \tag{A.12}$$

By using (A.11) in (A.10) it results

$$tg(t)=c\left\{\dot{f}(t)+f(0)\,\delta(t)+\frac{b}{2}f(t)\right\}. \tag{A.13}$$

According to (A.4) $f(0)=0$, c representing a strictely positive real number; consequently, from (A.13) I obtain

$$g(t)=\mathfrak{L}^{-1}[e^{-c\sqrt{p(p+b)}}]=\frac{c}{t}\left\{\dot{f}(t)+\frac{b}{2}f(t)\right\}, \qquad t>0. \tag{A.14}$$

In the following it must be obtained the time derivative $\dot{f}(t)$. To solve this problem I shall make use on the well-known relations

$$\dot{H}(t-c)=\delta(t-c), \qquad \alpha(t)\delta(t-c)=\alpha(c)\,\delta(t-c). \tag{A.15}$$

By using (A.4) I get

$$\dot{f}(t)=\dot{\varphi}(t)\,H(t-c)+\varphi(c)\delta(t-c).$$

But (see [11], Cap. III)

$$I_n(t)=\sum_{s=0}^{\infty}\frac{1}{s\,!\,(s+n)\,!}\left(\frac{t}{2}\right)^{2s+n}, \tag{A.16}$$

hence $\varphi(c)=e^{-cb/2}$ and thus for $\dot{f}(t)$ it results

$$\dot{f}(t)=\dot{\varphi}(t)\,H(t-c)+e^{-bc/2}\delta(t-c).$$

By introducing this expression in (A.14) I obtain

$$g(t)=\frac{c}{t}\left\{\dot{\varphi}(t)+\frac{b}{2}\varphi(t)\right\}H(t-c)+e^{-bc/2}\delta(t-c). \tag{A.17}$$

In order to obtain $\dot{\varphi}(t)$ I shall use the relation

$$\dot{I}_0(t)=I_1(t). \tag{A.18}$$

After some elementary operations, from (A.4) and (A.17) it results

$$g(t)=\mathfrak{L}^{-1}[e^{-c\sqrt{p(p+b)}}]=\Psi(t)\,H(t-c)+e^{-bc/2}\delta(t-c),$$

$$\Psi(t)=\frac{bc}{2}\,\frac{e^{-bt/2}}{\sqrt{t^2-c^2}}\,I_1\left(\frac{b}{2}\sqrt{t^2-c^2}\right), \qquad t\geqslant c. \tag{A.19}$$

In order to obtain

$$h(t)=\mathfrak{L}^{-1}\,[p\bar{g}(p)]=\mathfrak{L}^{-1}\,[pe^{-c\sqrt{p(p+b)}}], \tag{A.20}$$

I shall use (A.19) and (A.11). In this way it results

$$h(t)=\dot{g}(t), \tag{A.21}$$

because $c>0$ and supp $\delta(t-c)=\{c\}$. Now, by using (A.19) I obtain

$$\dot{g}(t)=\dot{\Psi}(t)\,H(t-c)+\Psi(c)\delta(t-c)+e^{-bc/2}\dot{\delta}(t-c).$$

Taking int account again (A.16) it results $\Psi(c)=(b^2c/8)\,e^{-bc/2}$ and, consequently,

$$\dot{g}(t)=\dot{\Psi}(t)\,H(t-c)+\frac{b^2c}{8}e^{-bc/2}\delta(t-c)+e^{-bc/2}\dot{\delta}(t-c). \tag{A.22}$$

By using the well-known relation

$$\dot{I}_1(t)=-t^{-1}\,I_1(t)+I_0(t) \tag{A.23}$$

and on account of (A.19) it results

$$h(t)=\mathfrak{L}^{-1}\,[pe^{-c\sqrt{p(p+b)}}]=\eta(t)\,H(t-c)+\frac{b^2c}{8}\,e^{-bc/2}\delta(t-c)+e^{-bc/2}\dot{\delta}(t-c),$$

$$\eta(t)=\dot{\Psi}(t)=\frac{bc}{2}\,\frac{e^{-bt/2}}{\sqrt{t^2-c^2}}\left\{-\left(\frac{b}{2}+\frac{2t}{t^2-c^2}\right)I_1\left(\frac{b}{2}\sqrt{t^2-c^2}\right)+\right.$$
$$\left.+\frac{b}{2}\,\frac{t}{\sqrt{t^2-c^2}}\,I_0\left(\frac{b}{2}\sqrt{t^2-c^2}\right)\right\}. \tag{A.24}$$

Now, in order to obtain the final result, I shall use the relations (A.19) and (A.24); after some elementary operations it results

$$l(t)=\mathfrak{L}^{-1}[(p+b)\,e^{-c\sqrt{p(p+b)}}]=\beta(t)\,H(t-c)+$$
$$+b\left(1+\frac{bc}{8}\right)e^{-bc/2}\delta(t-c)+e^{-bc/2}\dot{\delta}(t-c).$$

$$\beta(t)=\frac{bc}{2}\,\frac{e^{-bt/2}}{\sqrt{t^2-c^2}}\left\{\left(\frac{b}{2}-\frac{2t}{t^2-c^2}\right)I_1\left(\frac{b}{2}\sqrt{t^2-c^2}\right)+\right.$$
$$\left.+\frac{b}{2}\,\frac{t}{\sqrt{t^2-c^2}}\,I_0\left(\frac{b}{2}\sqrt{t^2-c^2}\right)\right\}, \qquad t\geqslant c. \tag{A.25}$$

It is clear that $g(t)=h(t)=l(t)=0$ for $t<c$.

The relations (A.19), (A.21) and (A.25) constitute useful formulas which are not given in tables of Laplace transforms.

In the first part of this paper it will be used (A.25) in order to obtain the fundamental solution for a rigid heat conducting body of Cattaneo type.

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RECENZII

S. R. GIVANT, R. N. McKENZIE (Editors): *ALFRED TARSKI — Collected Papers*, vol. I—IV, Contemporary Mathematicians, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1986, vol. I, 659 p., vol. II, 699 p., vol. III, 682 p., vol. IV, 757 p., SFr 1280

Publicarea operelor complete ale matematicienilor remarcabili constituie o tradiţie valoroasă a literaturii matematice mondiale. De cele mai multe ori lucrările unui astfel de matematician sînt răspîndite în timp şi spaţiu, scrise într-o formă uneori greu accesibilă datorită referirilor la lucrări anterioare. Adunînd împreună şi organizînd toate aceste lucrări, opera se încheagă şi se cristalizează, oferind cititorului o viziune clarificatoare asupra contribuţiilor esenţiale aduse matematicii şi, nu de puţine ori, noi valenţe. În cazul de faţă avem de-a face cu patru volume dedicate lucrărilor unuia dintre pilonii de bază, alături de K. Gödel, ai logicii secolului XX: Alfred Tarski. Cele 4 volume adună împreună toate textele publicate de Tarski, cu excepţia cărţilor. Criteriul de organizare a acestora este cel cronologic, criteriu care e posibil să fi fost impus şi de marea varietate a articolelor lui Tarski, lucru care făcea dificilă o catalogare pe domenii. Prin urmare, volumul 1 cuprinde 32 de articole publicate în perioada 1921—1934, articole care vădesc orientarea spre probleme de fundamentare a matematicii. Volumul 2 cuprinde articolele publicate între anii 1935 şi 1944 (31 la număr). Aceste articole sînt de referinţă şi reprezintă contribuţii importante ale autorului. Majoritatea articolelor din volumele 1 şi 2 sînt scrise în limba germană sau poloneză şi oglindesc activitatea ştiinţifică desfăşurată în Polonia. În timpul celui de-al doilea război mondial, Tarski emigrează în Statele Unite şi fondează o faimoasă şcoală de logică la Universitatea Berkeley — California. Volumele 3 şi 4 cuprind articolele publicate în perioada 1945—1957, respectiv 1958—1979, majoritatea fiind scrise în engleză. Ultimul volum conţine şi un număr însemnat de rezumate, probleme propuse de Tarski, recenzii şi contribuţii la discuţiile provocate în cadrul unor congrese; volumul se încheie cu o bibliografie exhaustivă. Conţinutul acestor volume convinge că A. Tarski a fost prolific. El a publicat mai bine de 100 de lucrări şi a fost autor sau coautor a peste 10 monografii. Dar ceea ce impresionează este calitatea şi deschiderea articolelor; Tarski are lucrări fundamentale, de mare importanţă, care acoperă o uluitoare varietate de subiecte: teoria adevărului în sisteme formale, teoria decidabilităţii, fundamentele geometriei, teoria mulţimilor şi teoria modelelor, logica algebrică şi algebra universală. Timpul, judecător aspru, a selectat şi impus mai ales contribuţiile din domeniul logicii matematice. Meditînd asupra impactului pe care îl provoacă întîlnirea cu opera unui mare matematician, putem conchide că publicarea acestor volume reprezintă o lecţie şi o provocare pentru generaţii prezente şi viitoare de matematicieni, precum şi un suport şi un ghid valoros pentru cei preocupaţi de fundamentele matematicii.

Gabriel Ciobanu

B. STEPHENSON: *Kepler's Physical Astronomy, Studies in the History of Mathematics and Physical Sciences*, Vol. 13, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1987, VII + 217 p., DM 118

În studiul său asupra operei lui Johannes Kepler (1571—1630), autorul propune o reconsiderare a celor două contribuţii majore aduse de marele astronom la crearea astronomiei moderne: descoperirea pe cale empirică a legilor mişcării