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SOME PROPERTIES OF $(B-M)$-REGULAR MULTIMEASURES

BY

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In an earlier paper [13] we introduced the notion of regular multimeasure and we studied some properties of that; in [14], as application of this kind of regularity, we established some theorems of Nikodym-type for families of regular multimeasures and some results concerning the locally measurable sets with respect to such multimeasures.

Taking as starting point the work of J. Belley and P. Morales [1] concerning an abstract type of regularity of multimeasures with values in a topological group, we point out a new type of regularity of multimeasures, more general than that introduced by us in [13]; this kind of regularity will be called $(B-M)$-regularity.

Then, as application, we shall obtain some results concerning sequences of $(B-M)$-regular multimeasures, some theorems of Nikodym, Brooks-Jewett and Vitali-Hahn-Saks type for sequences of multimeasures.

§1. Terminology and notations. Let Ω be a nonvoid set, $\mathcal{R}$ a ring of subsets of Ω and $\mathcal{P}(\Omega)$ the family of all subsets of Ω.

Let $\mathcal{D}$, $\mathcal{K}$, be two nonvoid classes of subsets of Ω and $\alpha_1 : \mathcal{D} \to \mathcal{P}(\Omega)$, $\alpha_2 : \mathcal{K} \to \mathcal{P}(\Omega)$ two functions with the following properties:

$$(\alpha)\quad \begin{cases} 1)\ \mathcal{K} \subset \mathcal{R}; \\ 2)\ K_1,\ K_2 \in \mathcal{K} \Rightarrow K_1 \cap K_2 \in \mathcal{K}; \\ 3)\ \text{for every } D \in \mathcal{D} \Rightarrow D \subset \alpha_1(D); \\ 4)\ \text{for every } K \in \mathcal{K} \Rightarrow \alpha_2(K) \subset K; \\ 5)\ K_1,\ K_2 \in \mathcal{K} \Rightarrow \alpha_2(K_1) \cap \alpha_2(K_2) \subset \alpha_2(K_1 \cap K_2). \end{cases}$$

Let X denote a Hausdorff locally convex topological vector space, τ its topology and $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin in X. Using notations as in [5], $\mathcal{A}(X)$ is the family of all nonempty subsets of X, $\mathcal{C}(X)$ the family of closed sets of X and $\mathcal{B}(X)$ the family of bounded sets of X.

On $\mathcal{A}(X)$ we define the following equivalence relation „ρ":

$$(1)\qquad A \rho B \Leftrightarrow \bar{A} = \bar{B},$$

where $\bar{A}$ denotes the closure of A in X.

It is easy to see that the addition $(A, B) \to A + B$ in $\mathcal{A}(X)$ is compatible with „ρ".

On $\mathcal{A}(X)$ we also introduce another addition, denoted by symbol $\dot{+}$, defined by

$$A \dot{+} B = \overline{A + B}, \text{ for every } A,\ B \in \mathcal{A}(X). \tag{2}$$

In this way, $(\mathcal{A}(X), \dot{+})$ admits a factorization by „ρ" and the quotient semigroup can be identified with $(\mathcal{C}(X), \dot{+})$.

If $\mathcal{U}$ is the invariant uniformity on X compatible with τ, then the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{A}(X)$ defined by the following base of vicinities:

$$\mathcal{W}(U) = \{(A, B) \in \mathcal{A}(X) \times \mathcal{A}(X)\ ;\ A \subset B \dot{+} U,\ B \subset A \dot{+} U\},\ U \in \mathcal{U}$$

is said to be the *exponential extension of* $\mathcal{U}$ *to* $\mathcal{A}(X)$.

The topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$ will be denoted by $\tilde{\tau}$.

Using the equivalence relation (1) we may indentify $(\mathcal{C}(X), \tilde{\tau})$ with $(\mathcal{A}(X), \tilde{\tau})/\rho$ and $(\mathcal{C}(X), \tilde{\mathcal{U}})$ with the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})$.

If X is metrizable, then $(\mathcal{A}(X), \tilde{\mathcal{U}})$ becomes a semimetrizable space for the Hausdorff distance between sets. If in addition X is complete, then so is $\mathcal{C}(X)$. If $(X, \mathcal{U})$ is a complete uniform space, then so is $(\mathcal{C}(X), \tilde{\mathcal{U}})$.

It is easy to see that $(\mathcal{C}(X), \dot{+}; \tilde{\mathcal{U}})$ is a Hausdorff uniform commutative semigroup with the unit $\{0\}$. Furthermore the semigroup $\mathcal{C}(X)$ has the important property that the law of cancellation

$$A \dot{+} C = B \dot{+} C \Rightarrow A = B, \tag{3}$$

holds in it.

According to a result of W e b e r [17] there exists a family $\mathcal{P} = \{p\}$ of semiinvariant pseudometrics, taking values in $[0, 1]$, which generates the uniformity $\tilde{\mathcal{U}}$. The pseudometric p is semiinvariant if

$$p(A \dot{+} C,\ B \dot{+} C) \leqslant p(A, B),\ (\forall)\ A,\ B,\ C \in \mathcal{C}(X). \tag{4}$$

Without any loss of generality, we can suppose that the family $\mathcal{P}$ is directed, that is for every finite family $(p_i)_{i \in I} \subset \mathcal{P}$, there exists a pseudometric $p \in \mathcal{P}$ such that

$$\{(x, y)\ ;\ p(x, y) \leqslant \varepsilon\} \subset \bigcap_{i=1}^{n} \{(x, y)\ ;\ p_i(x, y) \leqslant \varepsilon\}.$$

If $p \in \mathcal{P}$ is an arbitrary pseudometric we denote by

$$|A|_p = p(A, 0), \tag{5}$$

where 0 represents the set $\{0\}$ from $\mathcal{C}(X)$.

Since every p is semiinvariant it is easy to verify the following properties:

$$|A \dot{+} B|_p \leqslant |A|_p + |B|_p, \text{ for every } A,\ B \in \mathcal{C}(X) \text{ and} \tag{6}$$

$$|A \dot{+} B|_p \geqslant |A|_p - |B|_p, \text{ for every } A,\ B \in \mathcal{C}(X). \tag{7}$$

In the following we shall consider correspondences from $\mathcal{R}$ to X, that is set-functions defined on $\mathcal{R}$ taking values in $(\mathcal{A}(X), +)$ or, if sets with the same closure are identified, in the semigroup $(\mathcal{C}(X), \dot{+})$, with the property $\mu(\varnothing) = 0$.

1.1. **Definition.** *A correspondence* μ *from* $\mathcal{R}$ *to* X *is said to be:*

I) *an additive multimeasure, if*

$$\mu(A \cup B) = \mu(A) \dot{+} \mu(B),\ (\forall)\ A,\ B \in \mathcal{R} \text{ with } A \cap B = \varnothing. \tag{8}$$

II) *a* σ*-additive multimeasure, if for every sequence* $(A_n)_{n \in N} \subset \mathcal{R}$ *with* $A_i \cap A_j = \varnothing$ $(i \neq j)$ *and* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n), \text{ holds} \tag{9}$$

(the sum from the right side of the equality is considered with respect to the topology $\tilde{\tau}$).

In what follows, by a multimeasure we shall mean an additive multimeasure such that $\mu(\varnothing) = 0$.

1.2. **Definition.** *A correspondence* μ *from* $\mathcal{R}$ *to* X *is said to be exhausting (locally exhausting) if* $\lim_{n \to \infty} \mu(A_n) = 0$ *for every disjoint sequence* $(A_n) \subset \mathcal{R}$ *(for every* $T \in \mathcal{R}$ *and every disjoint sequence* $(A_n) \subset \mathcal{R}$ *such that* $A_n \subset T$*).*

Here the limit is also considered with respect to the topology $\tilde{\tau}$ induced by the uniformity $\tilde{\mathcal{U}}$.

1.3. **Definition.** *A family* $(\mu_i)_{i \in I}$ *of correspondences from* $\mathcal{R}$ *to* X *is said to be uniformly exhausting if for every disjoint sequence* $\lim_{n \to \infty} \mu_i(A_n) = 0$, $(A_n)_{n \in N} \subset \mathcal{R}$, *uniformly with respect to* $i \in I$.

1.4. **Definition.** *Let* μ *be a multimeasure from* $\mathcal{R}$ *to* X *and* $p \in \mathcal{P}$ *an arbitrary pseudometric.*

The function $\tilde{\mu}_p : \mathcal{R} \to R$, *defined by*

$$\tilde{\mu}_p(A) = \sup\{|\mu(T)|_p\ ;\ T \subset A,\ T \in \mathcal{R}\},\ (\forall) A \in \mathcal{R}, \tag{10}$$

is said to be the p-semivariation of μ.

It is easy to remark that: a) $\tilde{\mu}_p(\varnothing) = 0$; b) if $A \subset B \Rightarrow \tilde{\mu}_p(A) \leqslant \tilde{\mu}_p(B)$; c) $\tilde{\mu}_p(A \cup B) \leqslant \tilde{\mu}_p(A) + \tilde{\mu}_p(B)$, for every $A,\ B \in \mathcal{R}$.

Hence the p-semivariation $\tilde{\mu}_p$ of μ is a submeasure in Drewnowski sense [3].

We shall also use the following auxiliary result [7]:

1.5. **Proposition.** *Let* $(\mu_n)_{n \in N}$ *be a sequence of multimeasures from* $\mathcal{R}$ *to* X. *Then* $(\mu_n)_{n \in N}$ *is uniformly exhausting if and only if for every* $p \in \mathcal{P}$ *the corresponding sequence of the p-semivariations* $\{(\tilde{\mu}_n)_p\}_{n \in N}$ *is uniformly exhausting.*

In particular, a multimeasure μ from $\mathcal{R}$ to X is exhausting if and only if for every $p \in \mathcal{P}$ the p-semivariation $\tilde{\mu}_p$ is exhausting.

In what follows we shall need two important conditions, which we shall call the conditions (β) and (γ) respectively.

These conditions appear for the first time in the works of Jimenez-Guerra and Rodrigues-Salinas [15] and J. Belley and P. Morales [1] respectively.

The condition (β) says:

if μ is a multimeasure from $\mathcal{R}$ to X and $\mathcal{P}$ is the family of pseudometrics which generates the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{C}(X)$, then

(β)

1) for every $D\in\mathcal{D} \Rightarrow \alpha_1(D)\in\mathcal{R}$;

2) if $D\in\mathcal{D}$ and $(K_n)_{n\in N}\subset\mathcal{K}$ are such that $\alpha_1(D)\subset\bigcup_{n=1}^{\infty}\alpha_2(K_n)$, for every $p\in\mathcal{P}$ and every $\varepsilon>0$ there is a positive integer $n_0\in N$ such that for every $H\in\mathcal{R}$ with $H\subset\alpha_1(D)-\bigcup_{n=1}^{n_0}\alpha_2(K_n)\Rightarrow|\mu(H)|_p<\varepsilon$.

The condition (γ) is the following:

(γ) If $D\in\mathcal{D}$ and $(K_n)_{n\in N}\subset\mathcal{K}$ are such that $\alpha_1(D)\subset\bigcup_{n=1}^{\infty}\alpha_2(K_n)$, then there is a positive integer n_0 for which

$$\alpha_1(D)\subset\bigcup_{n=1}^{n_0}K_n.$$

§2. $(B-M)$ - **regular multimeasures.** 2.1. **Definition.** *A multimeasure μ from the ring $\mathcal{R}$ to X is said to be $(B-M)$-regular if for every $E\in\mathcal{R}$, every $p\in\mathcal{P}$ and every $\varepsilon>0$, there exist $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that*

$$\alpha_1(D)\subset E\subset\alpha_2(K) \quad \text{and} \quad |\mu(H)|_p<\varepsilon,$$

for every $H\in\mathcal{R}$, with $H\subset K\setminus D$.

Examples. 1) Let Ω be a topological space and $\mathcal{R}$ a ring of its subsets. We say that a multimeasure μ from $\mathcal{R}$ to X is a *multicharge* if for every $E\in\mathcal{R}$, every $p\in\mathcal{P}$ and every $\varepsilon>0$, there exist the sets $A,B\in\mathcal{R}$ such that

$$\bar{A}\subset E\subset\mathring{B} \quad \text{and} \quad |\mu(H)|_p<\varepsilon,$$

for every $H\in\mathcal{R}$ with $H\subset B\setminus A$. Here $\bar{A}$ denotes the closure of the set A and $\mathring{B}$ the interior of the set B.

It is easy to see that if we take $\mathcal{D}=\mathcal{K}=\mathcal{R}$ and $\alpha_1(A)=\bar{A}$ and $\alpha_2(B)=\mathring{B}$ for every A, $B\in\mathcal{R}$, then every multicharge on Ω is a $(B-M)$-regular multimeasure.

2) If Ω is a Hausdorff locally compact space, $\mathcal{R}$ a ring of subsets of Ω, $\mathcal{D}$ the set of open relatively compact sets from $\mathcal{R}$, $\mathcal{K}=\mathcal{R}$ and α_1, α_2 as in example 1, we easily see that every regular multimeasure considered by the author in [13] and [14] is a $(B-M)$-regular multimeasure.

3) If Ω is a Hausdorff locally compact space, $\mathcal{K}_0$ the family of compact G_δ subsets of Ω, $\mathcal{R}=\delta(\mathcal{K}_0)$, the δ-ring generated by $\mathcal{K}_0$, $\mathcal{D}$, $\mathcal{K}$, α_1, α_2 as in the example 2 and μ a σ-additive and locally exhausting multimeasure from $\delta(\mathcal{K}_0)$ to $\mathcal{C}(X)$, then by virtue of [11] (lemma 3.2) it follows that μ is a $(B-M)$-regular multimeasure.

4) Let Ω, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{K}$, α_1, α_2 be as in example 2. Following Dinculeanu-Kluvanek [2], we shall say that the multimeasure μ from Ω to X is (R_2)-*regular*, if for every $E\in\mathcal{R}$, every $p\in\mathcal{P}$ and every $\varepsilon>0$ there are a relatively compact set $A\in\mathcal{R}$ and a set $B\in\mathcal{R}$ such that

$$\bar{A}\subset E\subset\mathring{B} \quad \text{and} \quad |\mu(H)|_p<\varepsilon,$$

for every $H\in\mathcal{R}$ with $H\subset B\setminus A$.

Thus we easily see that every (R_2)-regular multimeasure is a $(B-M)$-regular multimeasure.

Remark. If X in definition 2.1 is a real normed space with the norm $\|\ \|$ and $\mathcal{C}_b(X)$ denotes the class of all closed bounded subsets of X endowed with Hausdorff metric d, then the family of pseudometrics which generates the uniformity $\tilde{\mathcal{U}}$ reduces to the single element d.

In this case, we observe that a multimeasure μ from Ω to X is $(B-M)$-regular if for every $E\in\mathcal{R}$ and every $\varepsilon>0$, there are the sets $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that

$$\alpha_1(D)\subset E\subset\alpha_2(K)$$

and

$$\|\mu(H)\|=|\mu(H)|_d=d(\mu(H),\ 0)<\varepsilon$$

for every $H\in\mathcal{R}$ with $H\subset K\setminus D$.

2.2. **Theorem.** *If μ is a $(B-M)$-regular multimeasure from Ω to X for which (β) holds, then μ is a σ-additive multimeasure.*

Proof. Let $(E_n)_{n\in N}$ be a disjoint sequence of sets from $\mathcal{R}$ with $E=\bigcup_{n=1}^{\infty}E_n\in\mathcal{R}$. From the $(B-M)$-regularity of μ it follows that for every $p\in\mathcal{P}$ and every $\varepsilon>0$ there are the sets $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that

$$(1)\quad \begin{cases} \alpha_1(D)\subset E\subset\alpha_2(K) \text{ and} \\ \text{for every } H\in\mathcal{R} \text{ with } H\subset K\setminus D\Rightarrow|\mu(H)|_p<\varepsilon/6 \end{cases}$$

and the sets $D_n\in\mathcal{D}$ and $K_n\in\mathcal{K}$ such that

$$(2)\quad \begin{cases} \alpha_1(D_n)\subset E_n\subset\alpha_2(K_n) \text{ and} \\ \text{for every } H\in\mathcal{R} \text{ with } H\subset K_n\setminus D_n\ |\mu(H)|_p<\varepsilon/6.2^n, \end{cases}$$

for every $n\in N$. By virtue of the hypothesis (α)-3 and from (2) we obtain

$$D_n\subset E_n\subset\alpha_2(K_n),$$

from which it follows that the second condition from (2) is also satisfied for every $H\in\mathcal{R}$, with $H\subset K_n\setminus E_n$.

Let us denote $K_n'=K_n\cap K$. According to (α)-2 we have that $K_n'\in\mathcal{K}$. But $E_n\subset E$ and by virtue of (1), we obtain

$$(3)\quad E_n\subset E\subset\alpha_2(K),$$

and then

$$(4)\quad |\mu(H)|_p<\varepsilon/6.2^n,$$

for every $H\in\mathcal{R}$ with $H\subset K'_n\setminus E_n$.

From (3), taking into account that $E_n\subset\alpha_2(K_n)$ it follows, according to (α)-5 and (α)-4,

$$E_n\subset\alpha_2(K)\cap\alpha_2(K_n)\subset\alpha_2(K_n\cap K)=\alpha_2(K'_n)\subset K'_n. \tag{5}$$

From (1) and (5) we have:

$$\alpha_1(D)\subset E=\bigcup_{n=1}^{\infty}E_n\subset\bigcup_{n=1}^{\infty}\alpha_2(K'_n). \tag{6}$$

Now, we remark that we can apply the hypothesis (β) for $D\in\mathcal{D}$ and the sequence $(K'_n)\subset\mathcal{K}$. Thus for every $p\in\mathcal{P}$ and every $\varepsilon>0$, there exists a positive integer $n_0\in N$ such that for every $H\in\mathcal{R}$

$$H\subset\alpha_1(D)-\bigcup_{n=1}^{n_0}\alpha_2(K'_n)\Rightarrow|\mu(H)|_p<\varepsilon/6. \tag{7}$$

Let be $m>n_0$. Taking into account that p is semiinvariant, we obtain

$$\begin{aligned}&p(\mu(E),\ \mu(\bigcup_{n=1}^{m}K'_n))=p(\mu(E\setminus(\bigcup_{n=1}^{m}K'_n\cap E))+\mu(\bigcup_{n=1}^{m}K'_n\cap E),\\ &\mu(\bigcup_{n=1}^{m}K'_n\setminus E)+\mu(\bigcup_{n=1}^{m}K'_n\cap E))\leqslant|\mu(E\setminus\bigcup_{n=1}^{m}(K'_n\cap E)|_p+|\mu(\bigcup_{n=1}^{m}K'_n\setminus E)|_p\end{aligned} \tag{8}$$

From (α)-3, (5) and (6) we have

$$D\subset D\cup\bigcup_{n=1}^{m}K'_n\subset\alpha_1(D)\cup\bigcup_{n=1}^{m}K'_n\subset\bigcup_{n=1}^{\infty}K'_n,$$

from which it follows

$$E\setminus\bigcup_{n=1}^{\infty}K'_n\subset E\setminus(D\cup\bigcup_{n=1}^{m}K'_n)\subset E\setminus D\subset K\setminus D$$

and then, applying (1) and (7), we obtain

$$\begin{aligned}&|\mu(E\setminus(\bigcup_{n=1}^{m}K'_n\cap E))|_p\leqslant|\mu(E\setminus(\bigcup_{n=1}^{\infty}K'_n\cup\alpha_1(D))|_p+\\ &+|\mu(\alpha_1(D)\setminus\bigcup_{n=1}^{m_0}K'_n)|_p<\varepsilon/6+\varepsilon/6=\varepsilon/3.\end{aligned} \tag{9}$$

On the other hand, since $\bigcup_{n=1}^{m}K'_n\setminus E\subset K\setminus D$, by virtue of the same regularity condition (1) we have

$$|\mu(\bigcup_{n=1}^{m}K'_n\setminus E)|_p<\varepsilon/6. \tag{10}$$

From (8), taking into account (9) and (10), we obtain

$$p(\mu(E),\ \mu(\bigcup_{n=1}^{m}K'_n))<\varepsilon/6+\varepsilon/3. \tag{11}$$

Using again the semi-invariance of p, from (4) we have

$$\begin{aligned}&p(\mu(\bigcup_{n=1}^{m}K'_n),\ \mu(\bigcup_{n=1}^{m}E_n))\leqslant|\mu(\bigcup_{n=1}^{m}K'_n\setminus\bigcup_{n=1}^{m}E_n)_p|\leqslant\\ &\leqslant\sum_{i=1}^{m}|\mu(K'_i)\setminus(\bigcup_{j=1}^{m}E_j\cup\bigcup_{n=1}^{i-1}K'_n)|_p<\varepsilon/3,\end{aligned} \tag{12}$$

since the sets $K'_i\setminus(\bigcup_{j=1}^{m}E_j\cup\bigcup_{n=1}^{i-1}K'_n)\in\mathcal{R}$, are mutually disjoint and $K'_i\setminus(\bigcup_{j=1}^{m}E_j\cup\bigcup_{n=1}^{i-1}K'_i)\subset K'_i\setminus E_i$.

Now, from (1) and (12) we obtain

$$p(\mu(\bigcup_{n=1}^{m}E_n),\ \mu(E))<\varepsilon/6+\varepsilon/3+\varepsilon/3<\varepsilon,$$

for every $m>n_0$, which says that μ is σ-additive.

The theorem 2.2 remains valid if the hypothesis (β) is replaced by the hypothesis (γ), the proof being similar.

Thus we can state:

2.2′. **Theorem.** *If μ is a $(B-M)$-regular multimeasure from Ω to X and the property (γ) holds, then μ is a σ-additive multimeasure.*

2.3. **Corollary.** *If Ω is a semi-compact space, then every multicharge on Ω is a σ-additive multimeasure.*

Proof. In the example 3 we have seen that every multicharge on Ω is $(B-M)$-regular. Taking $\mathcal{D}=\mathcal{R}=\mathcal{K}$ and defining $\alpha_1(A)=\overline{A}$, where $\overline{A}$ is the closure of A, and $\alpha_2(B)=\mathring{B}$, where $\mathring{B}$ is the interior of B, for every A, $B\in\mathcal{R}$, we observe that for Ω the property (γ) holds.

Hence if $D\in\mathcal{D}$ and $(K_n)\subset K$ are such that $\overline{D}\subset\bigcup_{n=1}^{\infty}\mathring{K}_n$, remarking, at the same time, that the semi-compactness property is hereditary with respect to the closed sets, it follows that there is a finite number of sets $K_1,\ldots,K_{n_0}$ such that

$$\overline{D}\subset\bigcup_{i=1}^{n_0}\mathring{K}_i\subset\bigcup_{i=1}^{n_0}K_i.$$

Therefore, according to theorem 2.2′ the multimeasure μ is σ-additive.

2.4. Corollary. *If Ω is a Hausdorff locally compact space and μ is a (R_2)-regular multimeasure, then μ is σ-additive.*

Proof. Let $\mathcal{R}$ be a ring of subsets of Ω, $\mathcal{D}$ the class of relatively compact sets belonging to $\mathcal{R}$ and let $\mathcal{K}=\mathcal{R}$. Defining the functions α_1 and α_2 as in the corollary 2.3 we note that μ is $(B-M)$-regular and in addition holds the condition (β). By virtue of theorem 2.2 it follows that μ is σ-additive.

§3. **Sequences of (B—M)-regular multimeasures.** As an application of the $(B-M)$-regularity of multimeasures, in this section we shall establish some theorems of N i k o d y m, B r o o k s - J e w e t t and V i t a l i-H a h n-S a k s type for multimeasures and we shall point out some sufficient conditions under which the limit of a sequence of $(B-M)$-regular multimeasures is a $(B-M)$-regular multimeasure.

3.1. **Theorem.** *If $(\mu_i)_{i\in I}$ is a family of $(B-M)$-regular multimeasures for which the condition (α) holds uniformly, then $(\mu_i)_{i\in I}$ is uniformly σ-additive.*

Proof. Let us consider the class $\tilde{S}_I=\{(A_i)_{i\in I};\ A_i\in\mathcal{C}(X)\}$, which is evidently a commutative semigroup with unit. With the aid of the family of pseudometrics $\mathcal{P}$ which generates the uniformity of $\mathcal{C}(X)$, we can define another family of pseudometrics $\{\tilde{p}\}$ on $\tilde{S}_I$, as follows: for every $p\in\mathcal{P}$, every $(A_i)_{i\in I}$, $(B_i)_{i\in I}\in\tilde{S}_I$ we define

$$\tilde{p}((A_i)_{i\in I},\ (B_i)_{i\in I})=\sup_{i\in I}\ (p(A_i,\ B_i)).$$

This family of pseudometrics $\tilde{\mathcal{P}}=\{\tilde{p}\}$ generates an uniformity on $\tilde{S}_I$. Thus $\tilde{S}_I$ becomes a commutative uniform semigroup with unit.

Let us define now the function $m:\mathcal{R}\to\tilde{S}_I$ taking

$$m(A)=(\mu_i(A))_{i\in I},\qquad (\forall)\ A\in\mathcal{R}\ .$$

It is easy to see that m is a semigroup valued measure with the property that for every $E\in\mathcal{R}$, every $p\in\mathcal{P}$ and every $\varepsilon>0$, there exist the sets $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that

$$\alpha_1(D)\subset E\subset\alpha_2(K)\ \text{ and }\ |\ m(H)\ |_{\tilde{p}}<\varepsilon,$$

for every $H\in\mathcal{R}$, $H\subset K\setminus D$.

Since the condition (α) is uniformly satisfied, it follows that m is σ-additive, which involves that $(\mu_i)_{i\in I}$ is uniformly σ-additive.

In what follows we shall assume that $\mathcal{R}$ is a σ-algebra.

3.2. **Theorem.** *If $\mathcal{D}\subset\mathcal{R}$ and $(\mu_n)_{n\in N}$ is a sequence of $(B-M)$-regular multimeasures, such that for every sequence $E_k\downarrow E$ and for every $p\in\mathcal{P}$, $(\tilde{\mu}_n)_p(E_k)\to$ $\to(\tilde{\mu}_n)_p(E)$ holds uniformly with respect to n, then $(\mu_n)_{n\in N}$ is uniformly $(B-M)$-regular.*

Proof. From the $(B-M)$-regularity of every multimeasure μ_n, we obtain that for every $p\in\mathcal{P}$, every $\varepsilon>0$ and every $E\in\mathcal{R}$, there exist $D_n\in\mathcal{D}$ and $K_n\in\mathcal{K}$ such that

$$\alpha_1(D_1)\subset E_n\subset\alpha_2(K_n)\quad\text{and}\quad(\tilde{\mu}_n)_p\,(K_n\setminus D_n)<\varepsilon,$$

for every $n\in N$. If $D=\bigcup_{n\in N}D_n$ and $K=\bigcap_{n=1}^{\infty}K_n$, we have

$$(1)\qquad (\tilde{\mu}_n)_p(K\setminus D)<\varepsilon,\ \text{ for every } n\in N.$$

But the sequence $C_n=K_n\setminus D_n\downarrow K\setminus D$ and then, by virtue of hypothesis $(\tilde{\mu}_n)_p\,(K_n\setminus D_n)\to(\tilde{\mu}_n)_p\,(K\setminus D)$, uniformly with respect to n. Hence

$$\lim_{m\to\infty}(\tilde{\mu}_n)_p(\bigcap_{i=1}^{m}K_i\setminus\bigcup_{i=1}^{m}D_i)=(\tilde{\mu}_n)_p(K\setminus D)<\varepsilon,$$

uniformly with respect to n.

Thus there exists $m_0\in N$ such that

$$\alpha_1(\bigcup_{j=1}^{m_0}D_j)\subset E\subset\alpha_2(\bigcap_{j=1}^{m_0}K_j)\ \text{ and}$$

$$(\tilde{\mu}_n)_p\,(\bigcap_{j=1}^{m_0}K_j\setminus\bigcup_{j=1}^{m_0}D_j)<2\varepsilon,\ \text{ for every } n\in N,$$

which says that $(\mu_n)_{n\in N}$ is uniformly $(B-M)$-regular.

Let $(\mu_n)_{n\in N}$ be a sequence of multimeasures. For every $p\in\mathcal{P}$ we define a set-function

$$(2)\qquad \nu_p:\mathcal{R}\to\bar{R}_+,\quad \nu_p(A)=\sum_{n=1}^{\infty}2^{-n}\,(\tilde{\mu}_n)_p(A),\quad (\forall)A\in\mathcal{R}.$$

It is easy to see that ν_p is a submeasure in D r e w n o w s k i 's sense [3]. Also it is easy to verify using the $(B-M)$-regularity of every μ_n the following property:

3.3. **Proposition.** *If $(\mu_n)_{n\in N}$ is a sequence of multimeasures, then, for every $p\in\mathcal{P}$, the function ν_p associated to (μ_n) by (2) has the property that for every $\varepsilon>0$ and every $E\in\mathcal{R}$ there exist two sets $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that $\alpha_1(D)\subset E\subset\alpha_2(K)$ and $\nu_p(K\setminus D)<\varepsilon$.*

3.4. **Definition.** *A multimeasure μ is said to be ν_p-continuous, denoted $\mu\ll\nu_p$, if*

$$\lim_{\nu_p(A)\to 0}\mu(A)=0.$$

3.5. **Definition.** *We say that the submeasure ν_p controls the family (μ_n) if (μ_n) is ν_p-equicontinuous, that is $\lim\limits_{\nu_p(A)\to 0}\mu_n(A)=0$, for every $n\in N$.*

3.6. **Theorem.** (V i t a l i - H a h n - S a k s). *If (μ_n) is an uniformly exhausting sequence of multimeasures, then for every $p\in\mathcal{P}$, the submeasure ν_p, defined by (2), controls the family $(\mu_n)_{n\in N}$.*

Proof. Let $p\in\mathcal{P}$ a fixed pseudometric. According to the proposition 3.3, for every $\varepsilon>0$ and every $E\in\mathcal{R}$, there exist two sets $D\in\mathcal{D}$ and $K\in\mathcal{K}$ such that $\alpha_1(D)\subset E<\alpha_2(K)$ and $\nu_p(K\setminus D)<\varepsilon$. From here we obtain that $\lim\limits_{\nu_p(A)\to 0}\mu_n(A)=0$ for every $n\in N$.

But ν_p is an upper complete submeasure, that is for every sequence $(A_n)\subset\mathcal{R}$ for which $\sum\limits_n\nu_p(A_n)<\infty$, there exists a set $B\in\mathcal{R}$ such that $A_n\subset B$, $\forall n\in N$, which says that every such as sequence (A_n) is dominated.

Since ν_p is upper complete, the family of their p-semivariations $(\tilde{\mu}_n)_p$ is uniformly R i c k a r t on $\mathcal{R}$ (see [16]), that is $\lim\limits_{k\to\infty}(\tilde{\mu}_n)_p(A_k)=0$ for every dominated disjoint sequence $(A_k)\subset\mathcal{R}$. Furthermore $(\tilde{\mu}_n)_p\ll\nu_p$ for every $n\in N$, and then, by virtue of lemma 2 [12], we obtain that the family $(\tilde{\mu}_n)_p$ is ν_p-equicontinuous. Consequently, ν_p controls the family $(\mu_n)_{n\in N}$.

In what follows we shall assume that the space X is such that the pseudometrics of the family $\mathcal{P}$ satisfy the properties :

$$(3)\quad \begin{cases} p(A\dotplus C,\ B\dotplus C)=p(A,\ B),\ (\forall)\ A,\ B,\ C\in\mathcal{A}(X). \\ |\,p(A,\ B)-p(A',\ B')\,|\leqslant p(A\dotplus A',\ B\dotplus B'),\ (\forall)\ A,\ B,\ A',\ B'\in\mathcal{A}(X)\end{cases}$$

for every $p\in\mathcal{P}$.

3.7. **Theorem.** (Brooks-Jewett). *If (μ_n) is a sequence of exhausting multimeasures from $\mathcal{R}$ to X pointwise convergent to μ, that is $\mu(A)=\lim\limits_{n\to\infty}\mu_n(A)$ for every $A\in\mathcal{R}$, then (μ_n) is uniformly exhausting.*

Proof. We reason by contradiction. Let us assume that under our hypotheses the family (μ_n) is not uniformly exhausting. Then there exist a pseudometric $p\in\mathcal{P}$, a disjoint sequence $(E_n)\subset\mathcal{R}$ and a subsequnce (μ_{n_k}) of (μ_n) such that

$$|\,\mu_{n_k}(E_k)\,|_p>\varepsilon \ \text{ but } \ |\,\mu_{n_k}(E_{k+1})\,|_p<\varepsilon/2.$$

Without any loss of generality we shall write $n_k=k$ and then we have

$$(4)\qquad |\,\mu_k(E_k)\,|_p>\varepsilon,\quad |\,\mu_k(E_{k+1})\,|_p<\varepsilon/2.$$

On the other hand, we associate to the semigroup $(\mathcal{C}(X),\ \dotplus)$ another semigroup $\tilde{S}$ which is uniform and with unit element.

Indeed, first we define an equivalence relation on $\mathcal{C}(X)\times\mathcal{C}(X)$ by

$$(A,\ B)\sim(A',\ B')\quad A\dotplus B'=B\dotplus A',$$

and let $\tilde{S}$ be the quotient semigroup $\mathcal{C}(X)\times\mathcal{C}(X)/\sim$. Denoting $[A,\ B]$ a class with respect to the equivalence relation $\sim$, which has as the representative the pair $(A,\ B)$, and taking into account (3), we see that the function $\tilde{p}:\tilde{S}\times\tilde{S}\to R$ defined by

$$(5)\qquad \tilde{p}([A,\ B],\ [A',\ B'])=p(A\dotplus B',\ B\dotplus A'),$$

where $p\in\mathcal{P}$, is a pseudometric.

Therefore the family $\tilde{\mathcal{P}}$ of all such as pseudometrics generates a uniformity on the semigroup $\tilde{S}$.

We observe that $\tilde{S}$ is commutative and it has the unit $\tilde{e}=[0,0]$. Let m_n be a sequence of measures from $\mathcal{R}$ to $\tilde{S}$ defined by

$$(6)\qquad m_n(E)=[\mu_n(E),\ \mu_{n+1}(E)],\ \ \forall E\in\mathcal{R}.$$

It is easy to see that every measure m_n is exhausting and

$$\lim_{n\to\infty} m_n(E)=\tilde{e}.$$

From (4) we have

$$\tilde{p}(m_k(E_{k+1}),\ \tilde{e})=p(\mu_k(E_{k+1}),\ \mu_{k+1}(E_{k+1}))\geqslant|\,|\,\mu_k(E_{k+1})\,|_p-|\,\mu_{k+1}(E_{k+1})\,|_p\,|>\varepsilon/2.$$

which means that the sequence of the measures (m_n) is not uniformly exhausting. But according to the lemma 2.4 from [7] this is not possible. Consequently the sequence of multimeasures (μ_n) is uniformly exhausting.

3.8. **Theorem.** *If (μ_n) is a sequence of $(B-M)$-regular multimeasures pointwise convergent on $\mathcal{R}$ to μ, then μ is also a $(B-M)$-regular multimeasure.*

Proof. Since (μ_n) are $(B-M)$-regular, by virtue of theorem 3.1 (μ_n) is uniformly σ-additive. Since the semi-group $(\mathcal{C}(X),\ \dotplus)$ satisfies the properties (4) and the cancellation law, it follows that μ_n is exhausting. By virtue of theorem 3.7 the family (μ_n) is uniformly exhausting and then applying the theorem 3.6, it follows that ν_p controls the family $((\tilde{\mu}_n)_p)$.

According to the proposition 3.3 we obtain that ν_p is $(B-M)$-regular, which involves that the sequnce (μ_n) is uniformly $(B-M)$-regular. From here it follows that its pointwise limit μ is also $(B-M)$-regular.

REFERENCES

1. Belley J. et Morales P. — *Régularité d'une fonction d'ensemble à valeurs dans un groupe topologique*. Ann. Sc. Math. Québec, 3 (1979), 185—197.
2. Dinculeanu N. and Kluwanek I. — *On vector measures*, Proc. London Math. Soc., 17 (1967), 605—612.
3. Drewnowski L. — *Topological rings of sets, continuous set functions, Integration, I, II, III*. Bull. Acad. Polon. Sci., 20 (1972), 269—276, 277—287, 439—445.
4. Drewnowski L. — *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym Theorems*, Bull. Acad. Polon. Sci., 20(1972), 725—731.
5. Drewnowski L. — *Additive and countably additive correspondences*, Comment. Math. 19 (1976—1977), 25—54.
6. Faires B. — *On Vitali-Hahn-Saks type theorems*, Bull. Am. Math. Soc. 80(1974), 670—674.
7. Fox G. et Morales P. — *Théorèmes de Nikodym et de Vitali-Hahn-Saks pour les mesures à valeurs dans un semigroupe uniforme*, in „Measure Theory and its Applications", Lect. Notes Math., 1033 (1983), 199—208.
8. Huneycutt J. E., Jr. — *Regularity of set functions of bounded variation on the line*, Rev. Roum. Math. Pures Appl. 14 (1969), 1113—1119.
9. Landers D. and Rogge L. — *The Hahn-Vitali-Saks and the Uniform Boundedness Theorem in topological groups*, Manuscripta Math., 4 (1971), 351—359.
10. de Maria Gonzalez J. L. — *Sur la régularité d'une mesure à valeurs dans un semigroupe*, in „Measure theory and its Applications", Lecture Notes in Math., 1033(1983), 230—239.
11. Morales P. — *Regularity and extension of semigroup valued Baire Measures*, in „Proc. Conf. Oberwolfach, 1979", Lecture Notes in Math. 794, Springer-Verlag, Berlin-Heidelberg-New York, (1980), 318—323.
12. Oberle R. A. — *Generalizations of the Vitali-Hahn-Saks theorem on Vector measures*, Proc. Amer. Math. Soc., 67(1977), 227—235.
13. Precupanu A. M. — *On the set-valued additive and subadditive functions*, An. şt. Univ. Iaşi, 1, I a Mat. 29 (1983), 41—48.
14. Precupanu A. M. — *Some applications of regular multimeasures*, An. şt. Univ. Iaşi, s. I a Mat. 31 (1985), p. 5—15.
15. Rodriguez-Salinas y Jimenez Guerra P. — *Medidas de Radon de tipo en espacios topologicos arbitrarios*, Real Acad. Ciencias, Madrid, 1979.
16. Rickart C.E. — *Decompositions of additive set functions*, Duke Math. J. 10(1943), 653—665.
17. Weber H. — *Fortsetzung von massen unit Werten in Uniformen Halbgruppen*, Arch. Math., 27(1976), 412—423.

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