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# A PRODUCT THEOREM FOR $I$-CONVEX-BAIRE SPACES

BY

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1. **Introduction.** M. Valdivia defines in [2], p. 22, §2.2(1) that a locally convex space $E$ is a convex-Baire space when given any sequence $\{A_n\}_{n=1}^{\infty}$ of closed convex subsets having void interior, then $\bigcup_{n=1}^{\infty} A_n$ has void interior. Also he proves that the product of convex-Baire spaces is a convex-Baire space. His proof is reproduced here with convenient adaptations, in order to help with reading, for showing that by weakening the closure condition it is obtained a product theorem with the same weakness in the number of factors.

2.1. **Convex-Baire-spaces.** Along the paper $I$ will denote an infinite set of indices and PF $(I)$ will be the set of the finite parts of $I$ directed by inclusion. Also $N$ will denote the set $\{1, 2, 3, \ldots\}$ of positive integers.

**Definition 2.1.** *A subset $A$ of a topological space $(X, T)$ is said to be I-closed when it contains the limit of every X-convergent net, directed by $PF(I)$, contained in $A$.*

Every $I$-closed set is $N$-closed and $A$ is $N$-closed if and only if it is sequentially closed. If $A \subset X$, we denote with $I$-cl (A) the intersection of all $I$-closed sets containing $A$; then $I$-cl $(A)$ is the smallest $I$-closed set containing $A$. We say that $A$ is $I$-dense in $X$ when $I\text{-cl}(A) = X$. Every $I$-dense set in $X$ is dense in $X$, and it is well known that if $A \subset Y \subset X$ then $I\text{-cl}_Y(A) \subset I\text{-cl}_X(A) \cap Y$; however, the equality does not hold as it is shown with the example of Engelking [1], p. 79 if we take $Y = X - \{1/n\}_{n=1}^{\infty}$ and $A = Y - \{0\}$.

**Definition 2.2.** *A locally convex space $E$ is said to be I-convex-Baire if given a countable family $\{A_n\}_{n=1}^{\infty}$ of I-closed convex subsets of $E$ with void interior, $\bigcup_{n=1}^{\infty} A_n$ has void interior.*

Obviously, $E$ is $I$-convex-Baire if and only if in every countable covering $E = \bigcup_{n=1}^{\infty} A_n$ by $I$-closed convex subsets there is an $A_n$ with non-void interior. It is easily proved that the Hausdorff quotient of an $I$-convex-Baire space is $I$-convex-Baire space.

Every Baire vector space is $I$-convex-Baire space. On the other hand, Valdivia [3] obtains a finite family of Baire vector spaces whose product is not Baire; then, in the light of the product theorem (section 4), there are $I$-convex-Baire spaces which are not Baire spaces, for each infinite $|I|$.

**Proposition 2.3.** *Let $F$ be an $I$-dense subspace of the locally convex space $E$. If $F$ is $I$-convex-Baire then $E$ is $I$-convex-Baire.*

*Proof.* If $E=\bigcup_{n=1}^{\infty} A_n$, being $A_n$ an $I$-closed convex subset of $E$, then $F=\bigcup_{n=1}^{\infty}(A_n\cap F)$ and, by hypothesis, there are an $A_m$ and an open set $B$ such that $B\cap F\subset A_m\cap F$.

Let $x\in B$ and $\{x_\alpha,\ \alpha\in \mathrm{PF}(I)\}$ a net in $F$ which converges to $x$. Then there is a finite set $\{i_1, i_2, \dots, i_k\}\subset I$ such that if $\{i_1, i_2, \dots, i_k\}\subset\alpha$ then $x_\alpha\in B$. The limit of a net in $B\cap F$ $\{y_\alpha,\ \alpha\in \mathrm{PF}(I)\}$ such that $y_\alpha=x_\alpha$ when $\{i_1, \dots, i_k\}\subset\alpha$ is $x$. This proves that

$$B\subset I\text{-}\mathrm{cl}_F(B\cap F)\subset I\text{-}\mathrm{cl}_E(A_m)=A_m$$

and therefore $E$ is $I$-convex-Baire.

Let $P$ be a set of natural numbers, $E$ a locally convex space, $F\neq\{0\}$ a space and let $T$ be a countable subset of $E$. Let $B=\{B_n : n\in P\}$ a family of $I$-closed convex subsets of $E$ such that for every $x\in T$ and $B_n\in B$ the set $(x+F)\cap B_n$ has void interior in $x+F$.

**Proposition 2.4.** *If $F$ is $I$-convex-Baire, there is a real one-dimensional subspace $L$ of $F$ such that $U\{(x+L)\cap B_n : x\in T,\ n\in P\}$ is a countable set.*

*Proof.* Let $\mathcal{F}$ be the family of pairs $(n, x)$ for which $(B_n-x)\cap F$ is met in more than one point by some ray coming from the origin. Then $(B_n-x)\cap F$ is contained in an union $C_{nx}$ of two opposite cones which are countable union of $I$-closed convex subsets without $F$-interior points (let $D_{nx}$ be an homotetic of $(B_n-x)\cap F$ such that $0\in D_{nx}$ and let $C_{nx}=\cup\{sD_{nx} : s\in Z\}$).

Since $F$ is $I$-convex-Baire, there is $z\in F-\cup\{C_{nx} : (n, x)\in\mathcal{F}\}$, then $L-\{0\}=<z>-\{0\}\subset F-\cup\{C_{nx} : (n, x)\in\mathcal{F}\}$.

Therefore $(B_n-x)\cap L\subset\{0\}$ for every $(n, x)\in\mathcal{F}$ which implies that $B_n\cap(x+L)\subset\{x\}$ for every $(n, x)\in\mathcal{F}$; then $\cup\{(x+L)\cap B_n; (n, x)\in\mathcal{F}\}$ is countable, because it is contained in $T$. Also $\cup\{(x+L)\cap B_n: (n, x)\in\mathcal{F}\}$ is countable by the definition of $\mathcal{F}$. This proves the proposition.

**3. Lemmas.** We shall need the three following lemmas. Lemma 3.1 is proved in [2], p. 26; we reproduce it, with a modified proof, to help with reading. Lemmas 3.2 and 3.3 are adaptation of results in [2], §2.

**Lemma 3.1.** *Let $E$ be a locally convex space, $B$ a convex subset and $F$ a subspace. If $x$ is an interior point of $B\cap(x+F)$ in $x+F$ and $u+F$ meets some homotetic of $B$ with centre $x$, then there is an homotetic of $B$ with centre $x$ and natural ratio which contains $u$.*

*Proof.* We can suppose $x=0$. If $v\in(u+F)\cap kB$ then, by convexity, there is a natural number $h_1$ such that $1/n\, v\in B$ for every $n>h_1$. Since $u-v\in F$ and $B\cap F$ is a $F$-neighbourhood of the origin, there is a natural $h_2$ such that $1/n(u-v)\in B$ for every $n>h_2$. Then for $n>\max(h_1, h_2)$ we have

$$\frac{1}{2n}u=\frac{1}{2}\cdot\frac{1}{n}v+\frac{1}{2}\cdot\frac{1}{n}(u-v)\in\frac{1}{2}B+\frac{1}{2}B=B$$

which proves the lemma.



**Lemma 3.2.** *Let $\{A_n\}_{n=1}^{\infty}$ be a family of convex subsets covering $E$, every $A_n$ being a convex subset of $E$. There is another countable covering of $E$ formed by homotetic sets of the $A_n$'s such that every set in the covering contains the origin.*

*Proof.* Let $F$ be the set of the natural numbers $n$ such that $A_n$ is met in more than one point by some ray coming from the origin. Then $A_n$ is contained in a convex cone $C_n=\bigcup_{m=1}^{\infty} C_{nm}$ which is the union of countably many homotetic images of $A_n$ each one of them containing the origin. In fact, let $C_{n1}$ be an homotetic image of $A_n$ which contains the origin and let $C_{nm}=mC_{n1}$.

Therefore $\bigcup_{n\in F}\bigcup_{m=1}^{\infty} C_{nm}=\bigcup_{n\in F} C_n$ covers $E$, because given $z\in E-\{0\}$, the ray $\{\lambda z : \lambda\geqslant 0\}$ cuts some $A_n$ in more than two points then $z\in C_n$.

Let $E=\pi\{E_i : i\in I\}$ be the product of a family of topological vector spaces and let $H\subset I$. Then $E(H)=\{(x_i)\in E : x_i=0$ for every $i\in I-H\}$.

**Lemma 3.3.** *Let $\{A_n\}_{n=1}^{\infty}$ be a covering of $E$ by $I$-closed convex subsets of $E$ which contain the origin. Then there is a finite subset $J\subset I$ and there is $s\in N$ such that $E(I-J)\subset A_s$.*

*Proof.* Let $M\subset I$ and let $A_n$ such that $E(\{i\})\subset A_n$ for every $i\in M$; then $E(M)\subset A_n$ because $A_n$ is $I$-closed, convex, $0\in A_n$, and every $(x_i)\in E(M)$ is the limit of the net $\{\sum_{j\in P} x(j) : P\in \mathrm{PF}(I)\}$, being $x(j)\in E(\{j\})$ and $x(j)_j=x_j$. Therefore if the lemma is not true, for every $A_n$ we can find infinitely many spaces $E(\{i\})$ not contained in $A_n$ and, consequently, we can determine a double sequence $\{E_{n_p} : (n, p)\in N\times N\}$ of pairwise different spaces such that $E_{n_p}\not\subset A_n$. Let $x_{n_p}\in E_{n_p}-A_n$. The closed linear hull $L$ of $\{x_{n_p} : (n, p)\in N\times N\}$ is a Fréchet space, therefore some $A_n\cap L$ has an $L$-interior point $x$. For this reason there is a finite set $P\subset I$ such that

$$[x+(\prod_{i\in P} U_i)\times(\prod_{i\notin P} E_i)]\cap L\subset A_n$$

being $U_i$ an $E_i$-neighbourhood of the origin. If $n_p\in I-P$ then $x+mx_{n_p}\in A_n$ and since the origin belongs to the sequentially closed convex set $A_n$ then $x_{n_p}=\lim_{m\to\infty}\left(\frac{1}{m}x+x_{n_p}\right)\in A_n$ which contradicts the choice of $x_{n_p}$.

**4. Product Theorem.** Let $\{E_i : i\in I\}$ a family of $I$-convex-Baire spaces and let $E=\Pi\{E_i : i\in I\}$ the product space. Let $\mathcal{P}$ be a countable family of convex $I$-closed subsets of $E$ and let $\mathcal{B}$ be the subfamily of $\mathcal{P}$ with $B\in\mathcal{B}$ if and only if there is an $i\in I$ such that $[x+E(\{i\})]\cap B$ has void interior in $x+E(\{i\})$ for each $x\in E$. We shall denote $\mathcal{B}=\{B_n : n\in M\}$, being $M=\{1, 2, \dots\}$ when $\mathcal{B}$ is infinite.

Following [2], p. 29, §2.4, if $\mathcal{B}$ is non-void there is an $i_1\in I$ such that $[x+E(\{i_1\})]\cap B_1$ has void interior in $x+E(\{i_1\})$ for every $x\in E$. Let $B_1$ be the family of all the sets in $\mathcal{B}$ with this property with respect to $i_1$. Let us suppose that, by recurrence, we have obtained indices $i_r\in I$ and families $B_r$, $i\leqslant r\leqslant n$. If $\mathcal{B}-\bigcup_{r=1}^{n} B_r\neq\varnothing$ there is an $i_{n+1}\in I$ such that $[x+E(\{i_{n+1}\})]\cap B_p$

has void interior in $x+E(\{i_{n+1}\})$ for every $x \in E$, being $B_p$ the first element
in $\mathcal{B}-\bigcup_{r=1}^{n} B_r$. Let $B_{n+1}$ be the family of all the sets in $\mathcal{B}-\bigcup_{r=1}^{n} B$, with this
property with respect to $i_{n+1}$.
We can write $\mathcal{B}=\cup\{B_p : p \in P\}$ where $P$ is finite or countable. Given
$n \in P$, let $M_n$ be the family of sets $B \in B_n$ such that $[x+E(\{i_p\})] \cap B$ has void
interior in $x+E(\{i_p\})$ for every $p \in P\{n+1, n+2, \ldots\}$ and every $x \in E$.
If $B \in B_n - M_n$ let $H(n, B)=\{s \in P \cap \{n+1, n+2, \ldots\}$ : some $x(s, B) \in B$
is interior point of $[x(s, B)+E(\{i_s\})] \cap B$ with respect to $x(s, B)+E(\{i_s\})\}$.
Let $N_n = M_n \cup \{m[B-x(s, B)]+x(s, B) : B \in B_n - M_n, s \in H(s, B), m =$
$=1, 2, \ldots\}$.

**Proposition 4.1.** *Given $u \in E$, for every $p \in P$ there is an one-dimensional real subspace $L_p$ of $E(\{i_p\})$ such that if $L$ is the closed linear hull of $\cup\{L_p, p \in P\}$ then $(u+L) \cap B_n$ has void interior in $u+L$ for every $n \in M$.*

*Proof.* Proposition 2.4 gives us a real one-dimensional subspace $L_1$ of
$E(\{i_1\})$ such that we can find a countable set $S_1$ contained in $(u+L_1)-\cup\{(u+$
$+L_1) B : B \in N_1\}$ which is dense in $u+L_1$.

By induction, let us suppose that we have obtained a real one-dimensional subspace $L_m$ of $E(\{i_m\})$, $1 \leqslant m \leqslant n$ and a countable subset $S_n$ of

$$(u+L_1+\ldots+L_n)-\cup\{(u+L_1+\ldots+L_n) \cap A : A \in \bigcup_{m=1}^{n} N_m\} \text{ which is dense in}$$

$$x+L_1+\ldots+L_n.$$

Let $v \in S_n$ and $A \in \bigcup_{m=1}^{n} N_m$. If $[v+E(\{i_{n+1}\})] \cap A$ has non-void interior
in $v+E(\{i_{n+1}\})$ then, since $A$ is homotetic to some $B \in \bigcup_{m=1}^{\infty} B_m$, there is $x(n+1, B)$
which is an interior point of $[x(n+1, B)+E(\{i_{n+1}\})] \cap B$ in $x(n+1, B)+$
$+E(\{i_{n+1}\})$ and, by lemma 3.1, $v$ belongs to some homotetic of $B$ with centre
in $x(n+1, B)$ and natural ratio, which contradicts the fact that $v \in S_n$.

By proposition 2.4 we obtain a real one-dimensional subspace $L_{n+1}$ of
$E(\{i_{n+1}\})$ such that $\cup\{(v+L_{n+1}) \cap A : v \in S_n, A \in \bigcup_{m=1}^{n+1} N_m\}$ is countable. There is a
countable subset $R_{n+1} \subset L_{n+1}$ which is dense in $L_{n+1}$, such that $S_{n+1}=S_n+R_{n+1}$
is disjoint with $\bigcup_{m=1}^{n+1} N_m$. Therefore we have obtained a real one-dimensional
subspace $L_{n+1}$ of $E(\{i_{n+1}\})$ and a countable subset $S_{n+1}$ of $(u+L_1+\ldots+L_n+$
$+L_{n+1})-\cup\{(x+L_1+\ldots+L_n+L_{n+1}) \cap A : A \in \bigcup_{m=1}^{n+1} N_m\}$ which is dense in $u+L_1+$
$+\ldots+L_n+L_{n+1}$.

Let $L$ be the closed linear hull of $\cup\{L_p, p \in P\}$. If $B \in \mathcal{B}$ and $z$ is an
interior point of $(u+L) \cap B$ in $u+L$, by definition of the product topology,
there is a $q \in P$ such that for $n \geqslant q$, $(u+L_1+\ldots+L_n) \cap B$ has an interior point
in $u+L_1+\ldots+L_n$, which is not possible if $B \in \bigcup_{m=1}^{n} N_m$ because $B$ is disjoint
with the $(u+L_1+\ldots+L_n)$-dense subset $S_n$.



**Proposition 4.2.** *With the previous construction, given $B \in \mathcal{P}-\mathcal{B}$ and a finite subset $J \subset I$ there is an $x \in B$ which is interior point in $(x+\Sigma\{E(\{i\}) : i \in J\}) \cap B$ with respect to $x+\Sigma\{E(\{i\}) : i \in J\}$.*

*Proof.* If $J$ has only one element, the proof is obvious. Let us suppose it is true when $J$ has $n$ elements. Then, if $J=\{j_1, \ldots, j_n, j_{n+1}\}$, $U$ is an $\sum_{m=1}^{n} E(\{j_m\})$-open set contained in $B$ and $V$ is an $E(\{j_{n+1}\})$-open set contained in $B$, then $1/_2 U+1/_2 V$ is an $\sum_{m=1}^{n+1} E(\{j_m\})$ — open set contained, by convexity, in $B$.

According with this proposition, for every $B \in \mathcal{P}-\mathcal{B}$ and every $n \in P$ there is an $x(n, B) \in E$ which is interior point of $(x(n, B)+\Sigma\{E(\{i_p\}), p=$ $=1, 2, \ldots, n\}) \cap B$ in $x(n, B)+\Sigma\{E(\{i_p\}), p=1, 2, \ldots, n\}$.

Let $\mathcal{R}=\cup\{m[B-x(n, B)]+x(n, B) : B \in \mathcal{P}-\mathcal{B}, n \in P, m=1, 2, \ldots\}$.

**Proposition 4.3.** *If $\mathcal{P}$ is a covering of E also $\mathcal{R}$ is a covering of E.*

*Proof.* Let $u \in E$. Proposition 4.1 gives us an one-dimensional subspace $L_p$ of $E(\{i_p\})$, $p \in P$, such that $(u+L) \cap B$ has void interior in $u+L$ for every $B \in \mathcal{B}$. Since $L$ is a Fréchet space, there is a $B \in \mathcal{P}$ such that $(u+L) \cap B$ has non-void interior in $u+L$. Therefore $B \in \mathcal{P}-\mathcal{B}$ and there is $n \in P$ such that $(u+\sum_{m=1}^{n} L_m) \cap B$ has non-void interior in $u+\sum_{m=1}^{n} L_m$. Obviously, $[u+\sum_{m=1}^{n} E(\{i_m\})] \cap B \neq \varnothing$ and by lemma 3.1 $u$ belongs to some homotetic set of $B$ with centre $x(n, B)$ and natural ratio. For this reason $u \in \cup\{R : R \in \mathcal{R}\}$.

**Theorem 4.4.** *If $\{E_i : i \in I\}$ is a family of I-convex-Baire spaces then $E=\pi\{E_i : i \in I\}$ is I-convex-Baire space.*

*Proof.* Let $\mathcal{P}'$ be a countable covering of $E$ composed of convex $I$-closed sets. By lemma 3.2 there is another countable covering $\mathcal{P}$ of $E$ composed of $I$-closed sets which are homotetic of some sets of $\mathcal{P}'$. Also the covering $\mathcal{R}$ of proposition 4.3 is composed of convex $I$-closed sets which contain the origin and are homotetics of some sets of $\mathcal{P}'$.

By lemma 3.3 there are a finite subset $J \subset I$ and a $B \in \mathcal{R}$ such that $E(I-J) \subset B$. By proposition 4.2 there is an $\Sigma\{E(\{i\}), i \in J\}$ - open subset $U$ contained in $B$. We have $U+E(I-J) \subset B$ because $u+v=\lim_{n \to \infty}[(n-1)/n]u+$ $+(1/n)nv \in B$ for each $u \in U$ and $v \in E(I-J)$. Since $u+E(I-J)$ is an $E$-open set and $B$ is an homotetic of some $C \in \mathcal{P}'$, then $C$ has non-void interior and $E$ is an $I$-convex-Baire space.

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