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MAXIMUM TERM AND ITS RANK OF AN ENTIRE DIRICHLET SERIES

BY

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1. **Introduction.** Let $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s=\sigma+it,\ 0\leqslant\lambda_1<\lambda_n<\lambda_{n+1}\to\infty$ as $n\to\infty)$, be an entire Dirichlet series. Set $M(\sigma)=\max_{t\in R}|f(\sigma+it)|$ and $\lim_{n\to\infty}\log n/\lambda_n=D<\infty$. $M(\sigma)$ is called the maximum modulus of $f(s)$. The concept of (p,q)-order and lower (p,q)-order of an entire Dirichlet series $f(s)$ having index-pair (p,q), $p\geqslant q+1\geqslant 1$, has recently been introduced by Juneja et al. [2]. Thus $f(s)$ is said to be of (p,q)-order ρ and lower (p,q)-order λ if

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p]}M(\sigma)}{\log^{[q]}\sigma}=\begin{matrix}\rho(p,q)\equiv\rho,\\ \lambda(p,q)\equiv\lambda\end{matrix},\tag{1.1}$$

where $\log^{[p]}x$ stands for the p^{th} iterate of $\log x$.

Definition. *An entire function $f(s)$ with index-pair (p,q) is said to be of regular (p,q)-growth if its (p,q)-order and lower (p,q)-order are equal.*

Let $u(\sigma)\equiv u(\sigma,f)$ be the maximum term of the sequence $\{|a_n|\exp(\sigma\lambda_n)\}$ and $\nu(\sigma)\equiv\nu(\sigma,f)$ be its rank. Then, evidently $\lambda_{\nu(\sigma)}$ is a nondecreasing function and since the function $\nu(\sigma)$ is constant in intervals; consequently the functions $|a_{\nu(\sigma)}|$ and $\lambda_{\nu(\sigma)}$ are constant in intervals, have countable number of discontinuities only. Therefore $|a_{\nu(\sigma)}|$ and hence $u(\sigma)$ $u(\sigma)=|a_{\nu(\sigma)}|e^{\sigma\lambda_{\nu(\sigma)}}$ are differentiable almost everywhere.

The following notations have been frequently used in this paper:

(i) $\operatorname{Exp}^{[0]}x=\log^{[0]}x=x$; $\operatorname{Exp}^{[m]}x=\log^{[-m]}x=\exp(\exp^{[m-1]}x)=\log(\log^{[-m-1]}x)$, $m=\pm1,\ \pm2,\dots$ and $\Lambda_{[q]}(x)=\prod_{i=0}^{q}\log^{[i]}x$.

(ii) $P_1(\tau)\equiv P_1(\tau,p,q)=\tau$ if $q+1<p<\infty$, $=1+\tau$ if $p=q+1=2$, $=\max(1,\tau)$ if $3\leqslant q+1=p<\infty$, $=\infty$ if $p=q=\infty$,

where $0\leqslant\tau\leqslant\infty$. We shall write $P(\tau)$ for $P_1(\tau)$.

The following relations between $u(\sigma)$, $\nu(\sigma)$ and $M(\sigma)$ are well known:

$$\log u(\sigma)=\log u(\sigma_0)+\int_{\sigma_0}^{\sigma}\lambda_{\nu(t)}\,dt\tag{1.2}$$

and for every $\varepsilon>0$ and $\sigma>\sigma_0(\varepsilon)$,

$$u(\sigma)\leqslant M(\sigma)\leqslant ku(\sigma+D+\varepsilon), \tag{1.3}$$

where k is a constant depending on ε.

The object of the present paper is to study some properties of the maximum term of an entire function defined by Dirichlet series and its nth derivative. Certain inequalities involving maximum term, its derivative and maximum term of derivatives of an entire Dirichlet series have been obtained. In section 2, proximate order of an entire Dirichlet series with index-pair (p, q) has been defined and constructed under certain conditions.

2. We first give a few lemmas which will be used in sequel:

Let (p, q) be the index-pair of $f(s)$. Define

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\lambda_{\nu(\sigma)}}{\log^{[q]}\sigma}=\begin{matrix}\theta\\ \Phi\end{matrix} \tag{2.1}$$

and

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}u(\sigma)}{\log^{[q]}\sigma}=\begin{matrix}\psi\\ \xi\end{matrix}. \tag{2.2}$$

Lemma 1. *Let $f(s)=\sum_{n=1}^{\infty}a_n\exp(s\lambda_n)$ be an entire Dirichlet series having (p, q)-order ρ and lower (p, q)-order λ. If θ; Φ and Ψ; ξ are defined as in (2.1) and (2.2), then*

$$\rho=P(\theta)=\Psi,\qquad \lambda=P(\Phi)=\xi. \tag{2.3}$$

Proof. The lemma follows easily by using (1.2) and (1.3). Hence we omit the proof.

Lemma 2. *The (p, q)-order and lower (p, q)-order of an entire Dirichlet series $f(s)=\sum_{n=1}^{\infty}a_n\exp(s\lambda_n)$ and its nth derivative $f^{(n)}(s)$ are equal.*

Proof. The proof of this lemma is available as Theorem 2 in Kasana et al. [4].

Now we prove

Theorem 1. *Let $f(s)=\sum_{n=1}^{\infty}a_n\exp(s\lambda_n)$ be an entire Dirichlet series of (p, q)-order ρ and lower (p, q)-order λ. If $u(\sigma, f)$ and $u(\sigma, f^{(n)})$ denote the maximum term respectively of f and $f^{(n)}$, then*

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\{u(\sigma,f^{(n)})/u(\sigma,f)\}^{1/n}}{\log^{[q]}\sigma}=\begin{matrix}\theta\\ \Phi\end{matrix}, \tag{2.4}$$

where θ and Φ have the same meaning as in lemma 1.

Proof. We have

$$u(\sigma, f)=|a_{\nu(\sigma,f)}|e^{\sigma\lambda_{\nu(\sigma,f)}}\geqslant|a_{\nu(\sigma,f^{(1)})}|e^{\sigma\lambda_{\nu(\sigma,f^{(1)})}},$$

or

$$u(\sigma, f)\geqslant\frac{u(\sigma, f^{(1)})}{\lambda_{\nu(\sigma,f^{(1)})}}. \tag{2.5}$$

Since

$$u(\sigma, f^{(1)})=\lambda_{\nu(\sigma,f^{(1)})}|a_{\nu(\sigma,f^{(1)})}|e^{\sigma\lambda_{\nu(\sigma,f^{(1)})}}\geqslant\lambda_{\nu(\sigma,f)}|a_{\nu(\sigma,f)}|e^{\sigma\lambda_{\nu(\sigma,f)}},$$

it follows that

$$u(\sigma, f^{(1)})\geqslant\lambda_{\nu(\sigma,f)}u(\sigma, f). \tag{2.6}$$

(2.5) and (2.6) combine to give

$$\lambda_{\nu(\sigma,f)}\leqslant\frac{u(\sigma, f^{(1)})}{u(\sigma, f)}\leqslant\lambda_{\nu(\sigma,f^{(1)})}. \tag{2.7}$$

Writing (2.7) for the function $f(s)$ and its successive derivatives $f^{(1)}(s), f^{(2)}(s), \ldots, f^{(n-1)}(s)$ and multiplying the n inequalities thus obtained, we get

$$\lambda_{\nu(\sigma,f)}\ldots\lambda_{\nu(\sigma,f^{(n-1)})}\leqslant\frac{u(\sigma, f^{(n)})}{u(\sigma, f)}\leqslant\lambda_{\nu(\sigma,f^{(1)})}\ldots\lambda_{\nu(\sigma,f^{(n)})}. \tag{2.8}$$

In view of (2.7), (2.8) yields

$$\lambda_{\nu(\sigma,f)}<\left\{\frac{u(\sigma, f^{(n)})}{u(\sigma, f)}\right\}^{1/n}\leqslant\lambda_{\nu(\sigma,f^{(n)})}.$$

Therefore,

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\lambda_{\nu(\sigma,f)}}{\log^{[q]}\sigma}\leqslant\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\{u(\sigma,f^{(n)})/u(\sigma,f)\}^{1/n}}{\log^{[q]}\sigma}\leqslant$$
$$\leqslant\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p-1]}\lambda_{\nu(\sigma,f^{(n)})}}{\log^{[q]}\sigma}. \tag{2.9}$$

By lemma 2 and (2.3), it is easy to see that

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p]}u(\sigma, f)}{\log^{[q]}\sigma}=\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p]}u(\sigma, f^{(n)})}{\log^{[q]}\sigma}=\begin{matrix}\rho\\ \lambda\end{matrix}.$$

From above relations and (1.2), we conclude

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\lambda_{\nu(\sigma,f)}}{\log^{[q]}\sigma}=\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log^{[p-1]}\lambda_{\nu(\sigma,f^{(n)})}}{\log^{[q]}\sigma}=\begin{matrix}\theta\\ \Phi\end{matrix}. \tag{2.10}$$

Now, (2.9) and (2.10) coupled together prove the theorem.

Theorem 2. *Let $f(s)$ be an entire function with index-pair (p, q) such that $\Phi(p, q)>0$. Then, for almost all values of $\sigma>\sigma_0$*

$$u(\sigma, f)\leqslant u^{(1)}(\sigma, f)\leqslant u^{(1)}(\sigma, f^{(1)})\leqslant\ldots \tag{2.11}$$

and, for given $\varepsilon>0$ and $n=1, 2, \ldots$

$$u^{(1)}(\sigma, f^{(n)}) > u^{(1)}(\sigma, f)\left\{\frac{\log u^{(1)}(\sigma, f)}{(1+\varepsilon)\sigma}\right\}^n. \tag{2.12}$$

Proof. From (2.1) and a known result [5, p. 78], it is obvious that for $\Phi(p, q) > 0$ and almost all values of $\sigma > \sigma_0$, $u(\sigma, f) \leq u^{(1)}(\sigma, f)$. Rewriting above for $f^{(1)}$, we have $u(\sigma, f^{(1)}) \leq u^{(1)}(\sigma, f^{(1)})$.
Differentiating (1.2), we get

$$\frac{u^{(1)}(\sigma, f)}{u(\sigma, f)} = \lambda_{\nu(\sigma, f)}. \tag{2.13}$$

From (2.7) and (2.13), we have

$$u^{(1)}(\sigma, f) \leq u(\sigma, f^{(1)}) \tag{2.14}$$

for almost all $\sigma > \sigma_0$, hence

$$u^{(1)}(\sigma, f) \leq u^{(1)}(\sigma, f^{(1)}).$$

Similarly, we can prove for subsequent derivatives and hence (2.11) follows.
Srivastava [5] has shown that for every entire Dirichlet series

$$u^{(1)}(\sigma, f^{(n)}) > \frac{u^{(1)}(\sigma, f)}{\{(1+\varepsilon)\sigma\}^n}\prod_{k=1}^{n} \log u^{(1)}(\sigma, f^{(k-1)}). \tag{2.15}$$

Now (2.11) coupled with (2.15) gives the required result (2.12).

Corollary. *For an entire Dirichlet series of positive lower (p, q)-growth number defined by*

$$\liminf_{\sigma\to\infty} \frac{\log^{[p-2]}\lambda_{\nu(\sigma, f)}}{(\log^{[q-1]}\sigma)^{\rho-A}} = \beta > 0,$$

we find that for almost all values of $\sigma > \sigma_0$

(i) $u(\sigma, f) \leq u^{(1)}(\sigma, f) \leq u^{(2)}(\sigma, f) \leq \ldots,$

(ii) (2.11) *and* (2.12) *also hold,*

where $A = 1$ if $(p, q) = (2, 1)$ and $A = 0$ for all other index-pairs.

3. Recently, the author [3] has introduced the notion of proximate order for an entire Dirichlet series with index-pair (p, q). Here, we construct this comparison function with the help of maximum terms.

Definition. *A positive real valued function $\rho(\sigma)$ defined on* $[\exp^{[q]} 0, \infty)$ *is said to be a proximate order of an entire function $f(s)$ represented by Dirichlet series with index-pair (p, q) and (p, q)-order $\rho (b < \rho < \infty)$ if, for a given constant $\tau^* (0 < \tau^* < \infty)$, $\rho(\sigma)$ satisfies the following properties:*

(3.1) *$\rho(\sigma)$ is continuous and piecewise differentiable for $\sigma > \sigma_0$,*

(3.2) *$\rho(\sigma) \to \rho$ as $\sigma \to \infty$.*

(3.3) *$\Lambda_{[q]}(\sigma)\, \rho^{(1)}(\sigma) \to 0$ as $\sigma \to \infty$, where $\rho^{(1)}(\sigma)$ can be interpreted as either $\rho^{(1)}(\sigma+)$ or $\rho^{(1)}(\sigma-)$ when these are unequal, and*

$$\limsup_{\sigma\to\infty} \frac{\log^{[p-1]} M(\sigma)}{(\log^{[q-1]}\sigma)^{\rho(\sigma)}} = \tau^*. \tag{3.4}$$

Theorem 3. *For an entire function $f(s)$ of (p, q)-order ρ and lower (p, q)-order λ, we find*

$$\liminf_{\sigma\to\infty} \frac{u^{(1)}(\sigma)\,\Lambda_{[q-1]}(\sigma)}{\Lambda_{[p-1]}(u(\sigma))} \leq \lambda \leq \rho \leq \limsup_{\sigma\to\infty} \frac{u^{(1)}(\sigma)\,\Lambda_{[q-1]}(\sigma)}{\Lambda_{[p-1]}(u(\sigma))}. \tag{3.5}$$

Proof. For the proof of Theorem 3, we refer an identical result established by D a l a l [1].

For a given constant a $(\rho < a < \infty)$, we define the function $\xi(\sigma)$ as follows

$$\xi(\sigma) = \frac{\log\{a^{-1}\log^{[p-1]}u(\sigma)\}}{\log^{[q]}\sigma}. \tag{3.6}$$

Next, we prove

Theorem 4. *If $f(s)$ is an entire function of (p, q)-order $\rho (b < \rho < \infty)$ such that limit in (3.5) exists and finite, then $\xi(\sigma)$ is a proximate order of the function $f(s)$.*

Proof. It is obvious that $\xi(\sigma)$ is continuous for $\sigma > \sigma_0$. $u(\sigma)$ is differentiable almost everywhere with an increasing derivative (see result (i) of corollary) and the set of points where the left hand derivative is less than right hand derivative is of measure zero. Consequently $\xi(\sigma)$ is piecewise differentiable for $\sigma > \sigma_0$.

Secondly, we observe that in view of (3.5), the function $f(s)$ is of regular (p, q)-growth and moreover, $\lim_{\sigma\to\infty} \xi(\sigma) = \rho$.

$\xi(\sigma)$ satisfies (3.3) also, since

$$\xi(\sigma) = (\log a^{-1} + \log^{[p]}u(\sigma))/\log^{[q]}\sigma,$$

differentiating with respect to σ, it follows that

$$\xi^{(1)}(\sigma) = -\frac{\log a^{-1} + \log^{[p]}u(\sigma)}{(\log^{[q]}\sigma)^2\,\Lambda_{[q-1]}(\sigma)} + \frac{u^{(1)}(\sigma)}{\Lambda_{[p-1]}(u(\sigma))\,\log^{[q]}\sigma},$$

or

$$\Lambda_{[q]}(\sigma)\,\xi^{(1)}(\sigma) = \frac{u^{(1)}(\sigma)\,\Lambda_{[q-1]}(\sigma)}{\Lambda_{[p-1]}(u(\sigma))} - \frac{\log^{[p]}u(\sigma)}{\log^{[q]}\sigma} + 0(1). \tag{3.7}$$

Using the hypothesis of the theorem in (3.7), we at once have

$$\lim_{\sigma\to\infty} \Lambda_{[q]}(\sigma)\,\xi^{(1)}(\sigma) = 0.$$

From the definition of $\xi(\sigma)$, we again have

$$\limsup_{\sigma\to\infty} \frac{\log^{[p-1]}u(\sigma)}{(\log^{[q-1]}\sigma)^{\xi(\sigma)}} = a.$$

Thus $\xi(\sigma)$ satisfies (3.1) through (3.4) and hence the theorem.

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THE METHOD OF THE SMALL PARAMETER IN THE THEORY OF DIFFERENTIAL-DIFFERENCE EQUATIONS WITH IMPULSE EFFECT

BY

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1. **Introduction.** Systems with impulses are an adequate mathematical model of a number of phenomena and processes of the concern of physics, chemistry, control theory, radiotechnology, and so on. The first contributions on the mathematical theory of systems with impulses are the papers of V. D. Millman and A. D. Mishkis [1], [2]. The theory of systems of differential equations with impulses advances comparatively slowly in spite of the considerable interest displayed to it. This is due to a number of difficulties implied by some specific features of these systems.

One of the traditional problems considered by the qualitative theory of differential equations is related to the existence of periodic solutions.

The present paper proposes the Poincaré method for finding the periodic solutions of differential-difference systems with impulses.

2. **Statement of the problem. General assumptions.** Consider the system with impulse effect at fixed moments

$$\begin{aligned} &\dot{x}=f(t,\ x(t),\ x(t-h)), \qquad t\neq t_i, \\ &\Delta x/_{t=t_i}=I_i(x(t_i),\ x(t_i-h)), \end{aligned} \tag{1}$$

where $x\in R$, $f: R\times R\times R\to R$, $I_i: R\times R\to R$, $i\in Z$, $t_i\in R$ are fixed points such that $t_{i+1}>t_i$, $\lim\limits_{i\to\pm\infty} t_i=\pm\infty$, $\Delta x/_{t=t_i}=x(t_i+0)-x(t_i-0)$, Z is the set of integers.

Definition 1. *A solution of the system (1) will be called the piecewise continuous function $C: R\to R$ with first order discontinuity points as t_i, which, for $t\in R$, $t\neq t_i$, satisfies the equation*

$$\dot{x}=f(t,\ x(t),\ x(t-h))$$

and for $t=t_i$, fulfills the jump condition

$$x(t_i+0)-x(t_i-0)=I_i(x(t_i-0)),\ x(t_i-h-0)).$$

Definition 2. *Under value of the solution $C(t)$ at the point τ we will understand*

$$C(\tau)=C(\tau-0)=\lim_{\varepsilon\downarrow 0} C(\tau-\varepsilon).$$