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ON PERIODIC SOLUTIONS OF A FIRST ORDER DIFFERENCE EQUATION

BY

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The periodicity of solutions of linear difference equations was treated in [1] and [2]. The aim of our paper is to study the periodic solutions of a first order difference equation of the form $\Delta x_n = f(n, x_n)$. Similar problems for differential equations were discussed in [3].

Let us consider the nonlinear difference equation

(E) $$\Delta x_n = f(n, x_n),$$

where the function $f: N \times R \to R$ is t-periodic, $t \in N$, $t > 0$, with respect to the first variable. That is, for all $a \in R$ and for all $n \in N$, $f(n+t, a) = f(n, a)$. Also, $f(i, .)$ is a continuous function on R for $i \in \overline{0, t-1}$.

We denote: N — the set of nonnegative integers, R — the set of real numbers, R_+ — the set of positive reals, and $\overline{i, j} = \{i, i+1, \ldots, j\}$ for $i, j \in N$, $i < j$.

For the function $x: N \to R$ the forward difference operator Δ is defined as

$$\Delta x_n = x_{n+1} - x_n.$$

It is supposed that

$$\sum_{i=k}^{k-1} x_i = 0.$$

Definition. *Consider the following equation*

(E1) $$\Delta x_n = f(n, x_n) - p.$$

Suppose equation (E1) has a t-periodic solution x_n with initial value x_0. Suppose that p is unique in the sense that there is not another t-periodic solution of (E1) for the same x_0 but a different p. Then we say that p is p-constant for the point x_0 for equation (E).

Let $y = \{y_0, y_1, \ldots, y_t\}$. We denote $\| y \| = \max\limits_{n \in \overline{0, t}} | y_n |$.

Lemma 1. *For all $y \in R^{t+1}$, and $n \in \overline{0, t}$, $t > 0$,*

$$\left| \sum_{i=0}^{n-1} \left[y_i - \frac{1}{t} \sum_{j=0}^{t-1} y_j \right] \right| \leqslant 2 \, \| y \| \, n \left(1 - \frac{n}{t} \right).$$

Proof. Let us take into consideration the equality

$$\sum_{i=0}^{n-1}\left[y_i-\frac{1}{t}\sum_{i=0}^{t-1}y_i\right]=\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}y_i-\frac{n}{t}\sum_{i=n}^{t-1}y_i.$$

Thus we have

$$\left|\sum_{i=0}^{n-1}\left[y_i-\frac{1}{t}\sum_{i=0}^{t-1}y_i\right]\right|\leqslant\left(1-\frac{n}{t}\right)\|y\|\sum_{i=0}^{n-1}1+\frac{n}{t}\|y\|\sum_{i=n}^{t-1}1=$$

$$=(1-n/t)\,n\,\|y\|+(n/t)\,(t-n)\,\|y\|=2\,\|y\|\,n(1-n/t),$$

and so the Lemma holds.

Lemma 2. *Let* $y_{m,\cdot}:\overline{0,\ t}\to R_+$, $t>0$, $y_{0,\cdot}\equiv 1$, $m=1, 2, \ldots$ *If for every* $n\in\overline{0,\ t}$,

$$y_{m+1,\,n}\leqslant\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}y_{m,\,i}+\frac{n}{t}\sum_{i=n}^{t-1}y_{m,\,i},$$

then for every $m\in N$, $n\in\overline{0,\ t}$

(1) $$y_{m+1,\,n}\leqslant 2^{-1}\,.\,3^{-m}\,t^{m+1}.$$

Proof. Let us consider $z_{m,\cdot}:\overline{0,\ t}\to R_+$ which is defined by

$$z_{0,\cdot}\equiv 1,\qquad z_{m+1,\,n}=\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}z_{m,\,i}+\frac{n}{t}\sum_{i=n}^{t-1}z_{m,\,i},\quad\text{for}\quad m\geqslant 1.$$

Succesive calculations give:

$$z_{1,\,n}=\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}1+\frac{n}{t}\sum_{i=n}^{t-1}1=2n\left(1-\frac{n}{t}\right),$$

$$z_{2,\,n}=\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}2i\left(1-\frac{i}{t}\right)+\frac{n}{t}\sum_{i=n}^{t-1}2i\left(1-\frac{i}{t}\right)\leqslant\frac{2n}{6t}\,[-(t-2n)^2-$$

$$-3(t-2n)-2+2t^2]\left(1-\frac{n}{t}\right)$$

and since there exist no integers n, t such that $(t-2n)\in(-2,\ -1)$. We get

$$z_{2,\,n}\leqslant 2n(1-n/t)\,.\,t/3\,.$$

Repeating the same calculation we can prove that

$$z_{m+1,\,n}\leqslant 2n(1-n/t)\,.\,3^{-m}t^m.$$

As a result of $z_0\equiv y_0$ and $y_1\leqslant z_1$, we obtain

$$y_{m+1,\,n}\leqslant z_{m+1,\,n}\leqslant 2n\left(1-\frac{n}{t}\right).\,3^{-m}\,t^m\leqslant 2^{-1}\,.\,3^{-m}\,t^{m+1}.$$

The lemma is proved.

Lemma 3. *Let us assume:*

(2) *the function* $f: N\times R\to R$ *is* t*-periodic for some* $t>1$ *with respect to the first variable,*

(3) *the functions* $f(n,\,.)$ *are continuous on* R *for every* $n\in\overline{0,\ t-1}$,

(4) *for* $(n,\,a)\in\overline{0,\ t-1}\times[c,\ d]=\overline{0,\ t-1}\times D$ *there exists* $M>0$ *such that* $|f(n,\,a)|\leqslant M$, *and* $d-c\geqslant Mt$, *and*

(5) *for* $a_1,\ a_2\in D$, $n\in\overline{0,\ t-1}$ *there exists* $K>0$, *such that the inequality* $|f(n,\ a_1)-f(n,\ a_2)|\leqslant K|a_1-a_2|$ *is satisfied, and* $Kt<3$.

Then, the recurrent sequence $\{y_m\}_{m=0}^{\infty}$ *defined by*

(6) $$y_{0,\,n}=x_0,\ y_{m+1,\,n}=x_0+\sum_{i=0}^{n-1}\left[f(i,\ y_{m,i})-\frac{1}{t}\sum_{j=1}^{t-1}f(j,\ y_{m,j})\right],$$

for $m\geqslant 0$, $n\in N$, *tends to a* t*-periodic solution of the equation*

(7) $$x_n=x_0+\sum_{i=0}^{n-1}\left[f(i,\ x_i)-\frac{1}{t}\sum_{i=0}^{t-1}f(i,\ x_i)\right].$$

Proof. Let us take into consideration the recurrent sequence (6) where $x_0\in R$. Obviously, y_0 is t-periodic, since it is a constant function. Let y_k be a t-periodic function. Then we get

$$y_{k+1,\,n+t}=x_0+\sum_{i=0}^{n+t-1}\left[f(i,\ y_{k,\,i})-\frac{1}{t}\sum_{j=0}^{t-1}f(j,\ y_{k,\,j})\right]=y_{k+1,\,n}+$$

$$+\sum_{i=0}^{t-1}f(i,\ y_{k,\,i})-\sum_{i=n}^{n+t-1}\frac{1}{t}\sum_{j=0}^{t-1}f(j,\ y_{k,\,j})=y_{k+1,\,n},$$

i.e. $y_{k+1,\,\cdot}$ is t-periodic too.

Choose any $x_0\in[c+Mt/2,\ d-Mt/2]$. By virtue of (4),

$$|y_{1,\,n}-x_0|=\left|\sum_{i=0}^{n-1}f(i,\ x_0)-\frac{n}{t}\sum_{j=0}^{t-1}f(j,\ x_0)\right|=$$

(8) $$=\left|\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}f(i,\ x_0)-\frac{n}{t}\sum_{j=n}^{t-1}f(j,\ x_0)\right|\leqslant\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}|f(i,\ x_0)|+$$

$$+\frac{n}{t}\sum_{j=n}^{t-1}|f(j,\ x_0)|\leqslant\frac{Mt}{2}.$$

Then $y_{1,\,n}\in D$ for every $n\in\overline{0,\ t}$.

Similarly, if $y_{m,\cdot}:\overline{0,\ t-1}\to D$ then

$$|y_{m+1,\,n}-x_0|=\left|\sum_{i=0}^{n-1}\left[f(i,\ y_{m,\,i})-\frac{1}{t}\sum_{j=0}^{t-1}f(j,\ y_{m,\,j})\right]\right|.$$

Therefore by Lemma 1

$$|y_{m+1,n}-x_0| \leqslant 2n\left(1-\frac{n}{t}\right)\max_{n\in\overline{0,t}}|f(n, y_{m,n})| \leqslant \frac{Mt}{2}.$$

Hence $y_{m+1,\cdot}: \overline{0, t} \to D$.
Now, we examine the difference $y_{m+1,n}-y_{m,n}$. By (5) we get

$$|y_{m+1,n}-y_{m,n}| = \Bigg|\sum_{i=0}^{n-1}[f(i, y_{m,i})-f(i, y_{m-1,i})]-$$

$$(9) \qquad -\frac{n}{t}\sum_{j=0}^{t-1}[f(j, y_{m,j})-f(j, y_{m-1,j})]\Bigg| \leqslant$$

$$\leqslant K\left\{\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}|y_{m,i}-y_{m-1,i}|+\frac{n}{t}\sum_{j=n}^{t-1}|y_{m,j}-y_{m-1,j}|\right\},$$

for $m\geqslant 1$. Let $r_{m,n}$ be defined by

$$r_{m,n}=|y_{m,n}-y_{m-1,n}|, \quad m\geqslant 1, \quad n\in\overline{0, t}.$$

By virtue of (9) we obtain

$$(10) \qquad r_{m+1,n}\leqslant K\left[\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}r_{m,i}+\frac{n}{t}\sum_{i=n}^{t-1}r_{m,i}\right], \quad m\geqslant 1.$$

Furthermore, given $r_{m,n}$ we define $s_{m,n}$ as follows

$$(11) \qquad r_{m,n}=K^{m-1}Ms_{m,n}, \quad m\geqslant 1.$$

By (10) we get

$$s_{m+1,n}\leqslant\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}s_{m,i}+\frac{n}{t}\sum_{i=n}^{t-1}s_{m,i}, \quad m\geqslant 1.$$

We see from (8) that $r_{1,n}\leqslant 2Mn(1-n/t)$, and $s_{1,n}\leqslant 2n(1-n/t)$.
Taking $s_{0,n}=1$, in view of Lemma 2 we get

$$s_{m+1,n}\leqslant 2^{-1}\cdot 3^{-m}\cdot t^{m+1},$$

and by (11)

$$r_{m+1,n}\leqslant K^m M 2^{-1}\cdot 3^{-m}t^{m+1}=(Mt/2)\ (Kt/3)^m.$$

Consequently

$$|y_{m+k,n}-y_{m,n}|\leqslant\frac{Mt}{2}\sum_{i=0}^{k-1}\left(\frac{Kt}{3}\right)^{m+i}.$$

By using (5), $(Kt/3)^m\to 0$ as $m\to\infty$ and so we obtain:

$$(12) \qquad |y_{m+1,n}-y_{m,n}|\leqslant(Mt/2)\ (1-Kt/3)^{-1}\cdot(Kt/3)^m.$$

It follows that the sequences $\{y_{m,n}\}_{m=0}^{\infty}$, $n\in N$, are convergent Cauchy sequences. Let $y_{\infty,n}$, $n\in N$ be the limits of the above sequences. From (6), since the functions $f(n, \cdot)$ are continuous, we get

$$y_{\infty,n}=x_0+\sum_{i=0}^{n-1}\left[f(i, y_{\infty,i})-\frac{1}{t}\sum_{i=0}^{t-1}f(i, y_{\infty,i})\right].$$

Clearly $\{y_{\infty,n}\}_{n=0}^{\infty}$ is t-periodic and a function with finite values.

Remark 1. Let us remark that if $y_{m,n}\in D$ then $y_{\infty,n}\in D$, and so

$$|y_{\infty,n}-x_0|\leqslant\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}|f(i, y_{\infty,i})|+\frac{n}{t}\sum_{i=n}^{t-1}|f(i, y_{\infty,i})|\leqslant\frac{Mt}{2}.$$

Thus, by (12), for $m>1$

$$(13) \qquad |y_{\infty,n}-y_{m,n}|\leqslant(Mt/2)\ (1-Kt/3)^{-1}\ (Kt/3)^m,$$

Theorem 1. *Let the function f satisfy the conditions* (2)—(5). *Then for every* $x_0\in\left[c+\frac{Mt}{2},\ d-\frac{Mt}{2}\right]$ *there exists a p-constant for equation (E).*

Proof. Since the function f satisfies the assumptions of Lemma 3, the sequence of t-periodic functions $\{y_{m,\cdot}\}_{m=1}^{\infty}$, given by

$$y_{0,n}\equiv x_0, \quad y_{m,n}=x_0+\sum_{i=0}^{n-1}\left[f(i, y_{m-1,i})-\frac{1}{t}\sum_{i=0}^{t-1}f(i, y_{m-1,i})\right]$$

for every $x_0\in[c+Mt/2,\ d-Mt/2]=D_1$, tends to a t-periodic function $y_{\infty,\cdot}: N\to R$.

Define a constant p, as follows;

$$p=\frac{1}{t}\sum_{i=0}^{t-1}f(i, y_{\infty,i}).$$

We claim that this p is p-constant for equation (E). By virtue of Lemma 3 we have $\Delta y_{\infty,n}=f(n, y_{\infty,n})-p$, i.e. $y_{\infty,\cdot}$ is a t-periodic solution of equation (E1).

We next show that p, as defined above, is unique. Suppose the contrary. Then for (E) there are two different p-constants p_1, p_2, $p_1\neq p_2$, such that the equations (E1)

$$\Delta x_n=f(n, x_n)-p_1 \quad\text{and}\quad \Delta x_n=f(n, x_n)-p_2$$

have t-periodic solutions with the same initial value x_0. Denote these solutions by u and v ($u_0=v_0=x_0$). First we notice that if p is p-constant for equation (E) and x is a t-periodic solution of (E1) then

$$p=\frac{1}{t}\sum_{i=0}^{t-1}f(i, x_i).$$

This follows from the fact that

$$x_0=x_t=x_0+\sum_{i=0}^{t-1}[f(i, x_i)-p].$$

Moreover, if $x_0\in D_1$, then $x_n\in D$ for $n\geqslant 1$. Therefore

$$|u_n-v_n| = \left|\sum_{i=0}^{n-1}\left[f(i,u_i)-\frac{1}{t}\sum_{i=0}^{t-1} f(i,u_i)\right]-\sum_{i=0}^{n-1}\left[f(i,v_i)-\frac{1}{t}\sum_{i=0}^{t-1} f(i,v_i)\right]\right| \le$$

$$\le \left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}|f(i,u_i)-f(i,v_i)|+\frac{n}{t}\sum_{i=n}^{t-1}|f(i,u_i)-f(i,v_i)|\le$$

$$\le K\left[\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1}|u_i-v_i|+\frac{n}{t}\sum_{i=n}^{t-1}|u_i-v_i|\right].$$

Putting $r_n=|u_n-v_n|$ we have

(14) $$r_n\le K\left[\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1} r_i+\frac{n}{t}\sum_{i=n}^{t-1} r_i\right].$$

Define $z_0\equiv 1$, and let

$$z_{m+1,n}=\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1} z_{m,i}+\frac{n}{t}\sum_{i=n}^{t-1} z_{m,i},\quad m\in N.$$

We obtain $z_{1,n}=2n(1-n/t)$ and by Lemma 2,

(15) $$z_{m+1,n}\le 2^{-1}\cdot 3^{-m}\, t^{m+1}.$$

We get by (14) $r_n\le K\,\|r\|\,z_{1,n}$ and consequently, by virtue of (14),

$$r_n\le K^2\|r\|\left[\left(1-\frac{n}{t}\right)\sum_{i=0}^{n-1} z_{1,i}+\frac{n}{t}\sum_{i=n}^{t-1} z_{1,i}\right]=K^2\|r\|\,z_{2,n}.$$

Hence $r_n\le K^m\|r\|\,z_{m,n}$, and then by (15)

(16) $$r_n\le K^m\|r\|\,2^{-1}\cdot 3^{-m+1}t^m=(3/2)\|r\|(Kt/3)^m.$$

The inequality (16) is satisfied for every $n\in\overline{0,t-1}$. Thus

(17) $$\|r\|\le(3/2)\|r\|(Kt/3)^m,\quad m\in N,$$

and because $Kt/3<1$ we conclude $\|r\|=0$. This means that $u_n=v_n$, for every $n\in\overline{0,t-1}$, from which it follows that $p_1=p_2$. This completes the proof.

As a consequence of Theorem 1 we have the following theorem.

Theorem 2. *Let the function f satisfy the conditions (2)–(5). The solution $x=w$ of equation (E) in the domain D with the initial condition $w_0=x_0\in D_1$ is t-periodic if and only if the p-constant for the point x_0 is zero. Furthermore the equality*

$$w_n=y_{\infty,n},\quad n\in N,$$

is satisfied, where $y_{\infty,\cdot}$ is a limit of the sequence of t-periodic functions y_m, which are given by the recurrence formula

(18) $$y_0\equiv x_0,\quad y_{m,n}=x_0+\sum_{i=0}^{n-1}\left[f(i,y_{m-1,i})-\frac{1}{t}\sum_{j=0}^{t-1} f(j,y_{m-1,j})\right],$$

for $m\ge 1$.

Proof. We first prove the necessity of the condition. Let $x=w$ be a t-periodic solution of equation (E) and $w_0=x_0$, $x_0\in D_1$. Therefore

$$w_t=x_0+\sum_{i=0}^{t-1} f(i,w_i)=w_0=x_0.$$

Hence

$$\sum_{i=0}^{t-1} f(i,w_i)=0.$$

Since for $p=0$ the solution x of equation (E1) agrees with w, then it is t-periodic. In view of Theorem 1 equation (E1) has a p-constant for the point x_0 and it is unique.

The sufficiency of the condition follows directly from the definition of a point being p-constant.

The equality $w_n=y_{\infty,n}$ follows from Theorem 1 also, and so the proof is complete.

Remark 2. For a t-periodic solution of equation (E) the following estimate holds:

$$|w_n-y_{m,n}|\le(Mt/2)(1-Kt/3)^{-1}(Kt/3)^m,\quad m>1.$$

The correctness of the above inequality is obtained from (18).

Remark 3. We observe that the search for t-periodic solutions of equation (E) is related to the calculation of solutions x_i of the equation

$$\sum_{i=0}^{t-1} f(i,x)=0,$$

or, for fixed x_0, the equation $f(0,x_0)=-\sum_{i=1}^{t-1} f(i,x)$.

On the other hand solutions of the above equations can be obtained from the calculations of the periodic functions y_m defined by (18).

Remark 4. If the assumptions (2)–(5) hold and for $x_0\in D_1$ $p(x_0)\neq 0$, then equation (E) has no periodic solutions with the initial condition x_0.

Last of all we consider the following equation

(E2) $$\Delta x_n=p_n f(x_n),$$

where the function f is continuous on R, $p: N\to R$, and p_n is t-periodic. That is, there exists a $t\in N$ such that for all $n\in N$ $p_{n+t}=p_n$. Then we can state the following theorem.

Theorem 3. *If there exists a positive constant C such that the inequality*

(19) $$\sum_{j=1}^{t}|p_j|\le C/(4\sup_{x\in[-C,C]}|f(x)|)$$

is satisfied, then the equation (E2) has a p-constant.

Proof. Let R^{t+1} be a Banach space with the norm

$$\|x\|=\max_{i\in\overline{1,t+1}}|\xi_i|$$

for $x=\{\xi_i\}_{i=1}^{t+1}\in R^{t+1}$, and let B be the subset of R^{t+1} defined by

$$x\in B \Leftrightarrow |\xi_1|\leq C/2,\ |\xi_i|\leq C, \text{ for } i=2, 3, \dots, t+1,$$

where C is the constant defined by (19). An operator A is defined by the equality

$$Ax=y=\{\eta_i\}_{i=1}^{t+1}$$

for every $x\in R^{t+1}$, where

$$\eta_1=\xi_1, \text{ and } \eta_i=\xi_1+\sum_{j=1}^{i-1} p_j f(\xi_j)-\frac{i-1}{t}\sum_{j=1}^{t} p_j f(\xi_j), \quad i=2, 3, \dots, t+1;$$

Suppose that $x=\{\xi_i\}_{i=1}^{t+1}\in B$. It is easy to see that for every $i\in\overline{2, t+1}$ we have

$$|\eta_i|\leq |\xi_1|+\sum_{j=1}^{i-1}|p_j|\,|f(\xi_j)|+\frac{i-1}{t}\sum_{j=1}^{t}|p_j|\,|f(\xi_j)|\leq$$

$$\leq\frac{1}{2}C+2\sup_{\overline{t}\in[-C, C]}|f(\overline{t})|\sum_{j=1}^{t}|p_j|\leq C.$$

Therefore, $A(B)\subseteq B$.

Let us consider a sequence $\{x^m\}_{m=1}^{\infty}$, where $x^m=\{\xi_i^m\}_{i=1}^{t+1}\in B$, such that $x^m\to x^0$, with $x^0=\{\xi_i^0\}_{i=1}^{t+1}\in B$. Then

$$\max_{i\in\overline{1, t+1}}|\xi_i^m-\xi_i^0|\to 0 \quad \text{as} \quad m\to\infty. \tag{20}$$

Let $y^m=\{\eta_i^m\}_{i=1}^{t+1}=Ax^m$, $\quad y^0=\{\eta_i^0\}_{i=1}^{t+1}=Ax^0$. Then

$$\|Ax^m-Ax^0\|=\max_{i\in\overline{1, t+1}}|\eta_i^m-\eta_i^0|=\max\{|\xi_1^m-\xi_1^0|, \max_{i\in\overline{2, t+1}}|\eta_i^m-\eta_i^0|\}. \tag{21}$$

But for $i\in\overline{2, t+1}$,

$$\begin{aligned}|\eta_i^m-\eta_i^0|&\leq|\xi_1^m-\xi_1^0|+\sum_{j=1}^{i-1}|p_j|\,|f(\xi_j^m)-f(\xi_j^0)|+\\&+\frac{i-1}{t}\sum_{j=1}^{t}|p_j|\,|f(\xi_j^m)-f(\xi_j^0)|\leq|\xi_1^m-\xi_1^0|+\\&+\sum_{j=1}^{t}|p_j|\left\{\sum_{j=1}^{i-1}|f(\xi_j^m)-f(\xi_j^0)|+\frac{i-1}{t}\sum_{j=1}^{t}|f(\xi_j^m)-f(\xi_j^0)|\right\}.\end{aligned} \tag{22}$$

Since f is a uniformly continuous function on $[-C, C]$, by virtue of (20) and (22) we obtain $|\eta_i^m-\eta_i^0|\to 0$, for $i=2, 3, \dots, t+1$. Therefore from (21) $\|Ax^m-Ax^0\|\to 0$.

Hence by the Schauder Theorem, the operator A is continuous and has a fixed point in the set B. Let $z=\{\zeta_i\}_{i=1}^{t+1}$ be this fixed point, i.e. $z=Az$. Let us define the following sequence $\{u_n\}_{n=1}^{\infty}$ by the formula

$$u_n=\zeta_i, \quad n=i+kt, \quad i=1, 2, \dots, t, \quad k=0, 1, \dots$$

Obviously the sequence $\{u_n\}$ is t-periodic. We shall prove that it satisfies equation

$$\Delta x_n=p_n f(x_n)+p,$$

that is equation (E2) has p-constant of the form

$$p=-\frac{1}{t}\sum_{j=1}^{t} p_j f(\zeta_j).$$

We can write

$$\Delta u_n=\zeta_{i+1}-\zeta_i \text{ for } n\neq kt, \qquad \zeta_1-\zeta_t \text{ for } n=kt.$$

Let $n\neq kt$, $n=i+kt$, $i<t$. We obtain

$$\Delta u_n=\zeta_{i+1}-\zeta_i=\zeta_1+\sum_{j=1}^{i} p_j f(\zeta_j)-\frac{i}{t}\sum_{j=1}^{t} p_j f(\zeta_j)-\zeta_1-\sum_{j=1}^{i-1} p_j f(\zeta_j)+\frac{i-1}{t}\sum_{j=1}^{t} p_j f(\zeta_j)=$$

$$=p_i f(\zeta_i)-\frac{1}{t}\sum_{j=1}^{t} p_j f(\zeta_j)=p_{i+kt} f(u_{i+kt})+p=p_n f(u_n)+p.$$

For $n=kt$, since $\zeta_{t+1}=\zeta_1$ we get

$$\Delta u_n=u_{n+1}-u_n=u_{kt+1}-u_{kt}=\zeta_1-\zeta_t=\zeta_{t+1}-\zeta_t=$$

$$=p_t f(\zeta_t)+p=p_{kt} f(u_n)+p=p_n f(u_n)+p,$$

and so the theorem is proved.

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