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ABOUT THE GLOBAL RIGIDITY OF SUBMANIFOLDS OF CERTAIN HOMOGENEOUS SPACES

BY

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To professor Adolf Haimovici for his 75th birth day

Introduction. In the research of submanifolds of a homogeneous space the main problem is to determine a *fundamental system of invariants*, namely a minimal system of geometrical objects (functions, tensors, connections, a.s.o.) which determines the submanifold modulo a transformation of the fundamental group of the space. Such a problem was solved first by K. F. Gauss (1827) for the surfaces of the Euclidean space. The Erlangen-Program (1872) clarified and put the problem in its largest generality, as we know it today. The moving frame method of Elie Cartan (1937) theoretically solved this problem in the most general case. However, from Gauss till now, without interruption, many studies were dedicated to this problem. It is almost impossible to mention all them but we remark here some of the most recent, which we know, only to prove that this problem is yet a living one ([1], [2]). In this paper, we present a solution of this classical problem, solution which, in a local form, was successfully applied in several cases ([4], [5], [6], [7]), cases which proved to be hard to be attacked in other ways. This solution seems to be remarkable by its generality, by its „almost-global" character and by its „almost-algorithmical" form.

Some specifications: by an „almost global property" we understand a property concerning a dense open subset of a topological space (dense = a subset whose closure is the respective space). The closure of a subset D of a topological space will be denoted by $\bar{D}$.

§1. **Preliminaries: homogeneous spaces and their submanifolds.** In this section we present (in order to precise them), some (in general well-known) fundamental facts concerning homogeneous spaces and their submanifolds.

a) *Homogeneous spaces.* Let G be an r-dimensional Lie group and H a closed subgroup of G (dim $H=h$). H does not contain any invariant non-trivial subgroup of G.

The *homogeneous space*, defined by G and H, is the pair (G, B) where: $B=G/_H=\{aH \mid a\in G\}$ is the set of left cosets of H in G endowed with the unique structure of real C^ω-manifold for which G is an effective transitive group of C^ω left-transformations of B ([3] theorem 4.2, p. 132).

Usually G is called the *fundamental group*, B the *fundamental space* and H the *isotropy-group* of (G, B).

The transformation of B determined by an arbitrary element $a\in G$ will be denoted by τ_a. We shall have:

$$\tau_a : bH\in B\to \tau_a(bH)=abH\in B. \tag{1}$$

The dimension of B is $n=r-h$. The connected component of B to which the element eH belongs (e is the identity of G) will be denoted by B_0. The elements of B will be denoted also by x, x_0, ... and then $\tau_a(x)=ax$.

For every homogeneous space (G, B) the following three structures are given:

1. The principal bundle $\xi=(G, B, \pi, H)$ where the projection π is the C^ω-map $\pi : a\in G\to aH\in B$.

2. The base $\{\tilde\omega^p \mid p=\overline{1, r}\}$ of the linear space of left-invariant differential exterior 1-forms of G, so that the cosets $\{aH \mid a\in G\}$ should be the leaves of the involutive differential system defined by the Pfaff-equations $\tilde\omega^q=0$, $q=\overline{1, n}$. ($\tilde\omega^q=0$ $q=\overline{1, n}$, determines the vertical foliation of ξ).

Consequently, for the structure-constants $\{C^p_{p_1p_2} \mid p, p_1, p_2=\overline{1, r}\}$ from the Maurer-Cartan equations, we shall have $C^q_{s_1s_2}=0$; $q=\overline{1, n}$; $s_1, s_2=\overline{n+1, r}$.

3. A preferential atlas $\mathcal{A}_0$ of B is determined as follows: Let $h_0(U_0, \chi_0)$ be a fixed allowable local chart of B. For every $a\in G$ we shall determine another local chart of B, denoted by $h=\bar\tau_a(h_0)$ where if $h(U, \chi)$ then $U=\tau_a(U_0)$, $\chi=\chi_0\circ\tau_{a^{-1}}$. The set $\{\bar\tau_a(h_0) \mid a\in G\}$ is the atlas $\mathcal{A}_0$. Obviously, $\mathcal{A}_0$ is a C^ω-atlas and G is a simple transitive group of left-transformations of $\mathcal{A}_0$.

b) *Submanifolds of* (G, B). *A submanifold of* $(G.B)$ is a pair (M, j) where:

M is a real m-dimensional paracompact and connected C^∞-manifold and

$j : M\to B$ is a C^∞-immersion.

If $j(M)=\overline{j(M)}$ then we say that (M, j) is a *complete* submanifold of (G, B).

Two submanifolds, (M, j) and (M', j'), of (G, B) are *congruent* („equal") if there is a C^∞-diffeomorphism $f : M\to M'$ and an element $a\in G$ so that $\tau_a\circ j=j'\circ f$.

Two submanifolds (M, j) and (M', j') of (G, B) are *almostcongruent* if there are two dense open subsets $D\subset M$, $D'\subset M'$ so that the submanifolds $(D, j \mid D)$ and $(D', j' \mid D')$ to be congruent.

Remark. If the submanifolds (M, j) and (M', j') are almost-congruent, then $\overline{j(M)}=\overline{j(D)}\supseteq j(M)\supseteq j(D)$, $\overline{j'(M')}=\overline{j'(D')}\supseteq j'(M')\supseteq j'(D')$ and for the element $a\in G$, implicated in the definition of the relation of almost-congruence, we shall have $\tau_a(\overline{j(M)})=\overline{j'(M')}$.

c) *Normalized submanifolds*. A submanifold (M, j) of (G, B) is *normalized* if there is a C^∞-immersion $\nu : M\to G$, called *normalization of* (M, j), so that $\pi\circ\nu=j$. (M, j) is *almost-normalized* if there is a dense open subset D of M, so that the submanifold $(D, j \mid D)$ is normalized.

Two normalization ν, ν' of two submanifolds (M, j), (M', j') of (G, B) are *congruent* if there is a C^∞-diffeomorphism $f : M\to M'$ and an element $a\in G$ so that $\nu'\circ f=L_a\circ\nu$. (L_a is the left-translation of G determined by $a\in G$).

Obviously, if two submanifolds of (G, B) are normalized or almost normalized and the implicated normalizations are congruent then these two submanifolds are congruent or almost-congruent.

We shall use the following result which is a direct particularization of the theorem 2.3, p. 238, [11].

If (M, j) and (M', j') are two normalized submanifolds of (G, B), the normalizations being ν and ν', then the necessary and sufficient conditions in order that the normalizations ν, ν' to be congruent, are: the existence of a C^∞-diffeomorphism $f : M\to M'$ so that $\nu^*(\tilde\omega^p)=(\nu'\circ f)^*(\tilde\omega^p)$, $p=\overline{1, r}$, where $\{\tilde\omega^p \mid p=\overline{1, r}\}$ is an arbitrary fixed base of the linear space of left-invariant differential exterior 1-forms of G.

§ 2. Regular homogeneous spaces and regular submanifolds of them. In this section we shall define the two notions mentioned in the title and we shall establish that every regular submanifold of a regular homogeneous space is almost-normalizable.

a) *Regular homogeneous spaces*. Let (G, B) be a homogeneous space for which a preferential atlas $\mathcal{A}_0$ is fixed (see §1, a). For an arbitrary pair of natural numbers (m, k), $1\leqslant m\leqslant n-1$, $1\leqslant k$, we shall associate to (G, B) a real, finite-dimensional, C^∞-manifold, denoted by $\mathcal{D}^k_m(B; \mathcal{A}_0)$, defined as follows: Let $(x, h, (z^i_{a_1}, z^i_{a_1a_2}, \ldots, z^i_{a_1,\ldots,a_k}))$ be a system of arbitrary elements where: $h(U, \chi)\in\mathcal{A}_0$; $x\in U$, $i=\overline{1, n}$; $a_1, \ldots, a_k=\overline{1, m}$ and $(z^i_{a_1\ldots a_k})\in\mathbb{R}^n$. Two such systems will be equivalent, i.e.

$$(x, h, (z^i_{a_1}, \ldots, z^i_{a_1\ldots a_k}))\sim(x', h', (z'^i_{a_1}, \ldots, z'^i_{a_1\ldots a_k})) \tag{2}$$

if

$$x=x', \quad z'^i_{a_1}=\frac{\partial\varphi^i}{\partial x^{i_1}}z^{i_1}_{a_1}, \quad z'^i_{a_1a_2}=\frac{\partial^2\varphi^i}{\partial x^{i_1}\partial x^{i_2}}z^{i_1}_{a_1}z^{i_2}_{a_2}+\frac{\partial\varphi^i}{\partial x^{i_1}}z^{i_1}_{a_1a_2}, \ldots \tag{3}$$

where $\dfrac{\partial\varphi^i}{\partial x^j}$ is the partial derivative of the function $x'^i=\varphi^i(x^1, \ldots, x^h)$, calculated in $x\in B$. The functions $x'^i=\varphi^i(x^1, \ldots, x^n)$ $i=\overline{1, n}$ represent the transformation of coordinates from h to h'. We remark that there functions are also the local representation in h or in h' of the transformation $\tau_{a^{-1}} : B\to B$, where $\bar\tau_a(h)=h'$.

The relation defined by (2) and (3) is an equivalence-relation. The generic point of $\mathcal{D}^k_m(B; \mathcal{A}_0)$, denoted by $[(x, h, (z^i_{a_1}, \ldots, z^i_{a_1\ldots a_k}))]$, is an equivalence class determined by the former relation.

The differentiable structure of $\mathcal{D}^k_m(B; \mathcal{A}_0)$ is determined by a C^ω atlas $\mathcal{A}^k_m$ associated to the preferential atlas $\mathcal{A}_0$ of B. The atlas $\mathcal{A}^k_m$ is determined by $\mathcal{A}_0$ in this way: to $h(U, \chi)$ we shall associate the local chart $h^k_m(U^k_m, \chi^k_m)$ of $\mathcal{A}^k_m$, where $U^k_m=\{[(x, h, (z^i_{a_1}, \ldots, z^i_{a_1\ldots a_k}))] \mid x\in U\}$ and $\chi^k_m([(x, h, (z^i_{a_1}, \ldots, z^i_a))])=(x^1, \ldots, x^n, z^1_1, \ldots, z^n_1, z^1_2, \ldots, z^n_2, \ldots, z^n_{mm\ldots m})\in\mathbb{R}^{n+mn+n[m(m+1)]}+\ldots$ Otherwise, the atlas $\mathcal{A}^k_m$ is an adapted atlas of the fibre bundle $(\mathcal{D}^k_m, \pi^k_m, B)$.

We shall use frequently the surjective open C^ω-map

$$\pi^k_m : \mathcal{D}^k_m(B; \mathcal{A}_0)\to B \quad \text{where} \quad \pi^k_m([(x, h, (z^i_{a_1}, \ldots, z^i_{a_1\ldots a_n}))])=x.$$

Definition 1. *An homogeneous space (G, B) is regular if for every natural member m, $(1 \leq m \leq n-1)$, it can be determined a minimal natural number k_m, $k_m \geq 1$, and an open subset $\mathcal{U}_m \subset \mathcal{D}^k_m(B ; \mathcal{A}_0)$ so that :*

1. *$\pi^{k_m}_m(\mathcal{U}_m)$ is a dense subset of B.*
2. *It is given a C^ω-map $\nu_m : \mathcal{U}_m \to G$ so that $\pi \circ \nu_m = \pi^{k_m}_m \mid \mathcal{U}_m$.*

b) *Auxiliary remarks concerning the differentiable manifolds*, [10]. A simplicial complex K is a set of simplices of the Euclidean space E_{n+1} so that :

1. Every face of a simplex of K is an element of K.
2. The intersection of two elements of K is an element of K.
3. Each point of $[K]$, ($[K]$ is the polyhedron defined by K, i.e. the set of points which belong to simplices of K) has an open neighbourhood in E_{n+1} intersecting only a finite number of simplices of K.

A *weak-triangulation* of a real m-dimensional manifold M is a homeomorphism $F : M \to F(M) \subseteq E_{n+1}$, $m \leq n+1$, where $F(M)$ is a polyhedron $[K]$ (determined by a simplicial complex K), endowed with the induced topology.

In [10] is presented a detailed proof of the following lemma :

Let M be a real m-dimensional paracompact connected with (or without) boundary manifold. If a weak-triangulation of M is given then a dense open subset of M contractible to a given point $x \in M$ is determined by means of this triangulation.

This lemma is fundamental in our considerations.

In the present Note, by a simplex of F, we understand the inverse-image in F of a simplex which belongs to $[K]$, ($[K] = F(M)$).

From the proof of the former lemma, it follows that if x_0 belongs to the interior of the m-simplex $\Delta_0 \in F$, then for every other m-simplex $\Delta \in F$ there is a *unique* sequence $\Delta_0, \Delta_1, \ldots, \Delta_h = \Delta$, so that $\Delta_j \in F$, $\sigma_j = \Delta_j \cap \Delta_{j+1} \neq \emptyset$, $(j = \overline{0, h-1})$, (where σ_j is an $(m-1)$-simplex of F) and the open subset of M, $\operatorname{int} \Delta_0 \cup \operatorname{int} \sigma_0 \cup \operatorname{int} \Delta_1 \cup \operatorname{int} \sigma_1 \cup \ldots \cup \operatorname{int} \Delta_h$ is included in D (D is the dense open subset of M, contractible to x_0, determined in the proof of the lemma, by means of F).

We remark that by barycentric divisions of triangulations and by refinements of atlases of M we can obtain a weak-triangulation F of M and an atlas $\mathcal{A}_F$ of M so that there is a bijective map from F to $\mathcal{A}_F$. For every m-simplex Δ of F the corresponding local chart $h_\Delta(U_\Delta, \chi_\Delta) \in \mathcal{A}_F$ will be characterized by the inclusion $\Delta \subset U$. Moreover we suppose that every U_Δ intersects only the m-simplices of F which have common faces with Δ.

In the following we shall suppose that the above circumstances are satisfied by F. We shall say that the dense contractible open subset $D \subset M$ determined by means of F and the atlas $\mathcal{A}_F$ are *associated to the weak-triangulation F.*

c) *Regular submanifolds.*

Theorem 1. *Let (M, j) be an m-diemensional submanifold of the homogeneous space (G, B). For every natural number $k > 0$ there is a dense open subset $D \subset M$ so that a C^∞-immersion $\mu : D \to \mathcal{D}^k_m(B ; \mathcal{A}_0)$, satisfying the condition $\pi^k_m \circ \mu = j \mid D$, can be determined.*

Proof. By virtue of [10] we can determine for M a contractible dense open subset $D \subset M$, consequently parallelizable. (If M is parallelizable then we shall take $D = M$.) For the set D we shall fix a parallelization $\{e_a \mid a = \overline{1, m}\}$ and with this the linear connection Γ, uniquely determined by $\nabla_\Gamma e_a = 0$, $(a = \overline{1, m})$.

The C^∞-map $\mu : D \to \mathcal{D}^k_m(B ; \mathcal{A}_0)$ will be determined in this way : for an arbitrary point $\mu_0 \in D$ we consider two local charts, $h_M(U_M, \chi_M)$ of $M(\mu_0 \in U_M)$ and $h(U, \chi) \in \mathcal{A}_0$, $j(\mu_0) \in U$). The local representation in h_M, h of the immersion j will be $x^i = \varphi^i(u^1, \ldots, u^m)$, $i = \overline{1, m}$. By means of the fields $\{e_a \mid a = \overline{1, m}\}$, of the linear connection Γ and of the functions $\{\varphi^i(u^1, \ldots, u^m) \mid i = \overline{1, m}\}$ we shall determine the real numbers

$$y^i_\alpha = \left(\frac{\partial \varphi^i}{\partial u^\alpha}\right)_{u_0}, \quad y^i_{\alpha_1 \alpha_2} = y^i_{\alpha_1 \mid \alpha_2}, \ldots, \quad y^i_{\alpha_1 \ldots \alpha_k} = y^i_{\alpha_1 \ldots \alpha_{k-1} \mid \alpha_k} \tag{4}$$

and

$$z^i_{a_1} = y^i_{\alpha_1} e^{\alpha_1}_{a_1}, \ldots, z^i_{a_1 \ldots a_k} = y^i_{\alpha_1 \ldots \alpha_k} e^{\alpha_1}_{a_1} \ldots e^{\alpha_k}_{a_k}, \tag{5}$$
$$(a_1, \ldots, a_k = \overline{1, m} ; \alpha_1, \ldots, \alpha_k = \overline{1, m}, i = \overline{1, n}).$$

Here $y_{\alpha_1 \mid \alpha_2}$ is the covariant derivative of y_{α_1} by means of Γ and $(e^\alpha_a(u_0))$ $(\alpha = \overline{1, m})$ are the local coordinate of e_a in h_M. We remark that the real numbers $z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}$ are determined by the point $u_0 \in D$, the vectors $e_a(u_0)$, $(a = \overline{1, m})$, and the local charts h_M, h. These numbers are invariant to the change of h_M. If we change the local chart $h \in \mathcal{A}_0$ by another one, $h' \in \mathcal{A}_0$, $(x_0 \in U')$, then the numbers $(z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k})$ will be changed by the relations (3). Consequently, we associate to $u_0 \in D$ the point $[(j(u_0), h, (z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}))] \in \mathcal{D}^k_m(B : \mathcal{A}_0)$, point uniquely determined by the fixation of the parallelization $\{e_a \mid a = \overline{1, m}\}$ of D. In this way the C^∞-map $\mu : D \to \mathcal{D}^k_m(B ; \mathcal{A}_0)$ is determined. The condition $\pi^k_m \circ \mu = j \mid D$ is obviously satisfied and it assures that μ is an immersion. Q.E.D.

Analysing the proof of this theorem, we again remark that the obtained C^∞-immersion $\mu : D \to \mathcal{D}^k_m(B ; \mathcal{A}_0)$ is determined by D, $\{e_a\}$ and j. We shall say that μ is *associated to* $[D, \{e_a\}, j]$.

Definition 2. *Let (M, j) be a submanifold of a regular homogeneous space (G, B). This submanifold is regular if a parallelizable dense open subset $D \subset M$ and a parallelization $\{e_a \mid a = \overline{1, m}\}$ of D can be determined so that the C^∞-immersion $\mu : D \to \mathcal{D}^k_m(B ; \mathcal{A}_0)$ associated to $[D, \{e_a\}, j]$ satisfy the condition : $j(D) \subset \mathcal{U}_m$.*

Corollary. *If (M, j) is a regular submanifold of a regular homogeneous space (G, B), the elements D, $\{e_a\}$ being fixed, then (M, j) has an almost-normalization given by $\nu_m \circ \mu : D \to G$.*

For a regular submanifold of a regular homogeneous space, the condition of the existence of structure D, $\{e_a\}$, μ is not restrictive (every smooth manifold has weak-triangulations). The condition $\mu(D) \subset \mathcal{U}_m$ is restrictive, implying a certain „regularity" of M.

§3. The almost-global existence and rigidity of regular submanifolds of regular homogeneous space. a) *The almost-global existence.* This problem can be formulated as follows : Let be given two real C^∞-manifolds, M and B, dim $M = m$, dim $B = n$, $m \leq n$. What properties must M have and what structures must be given on M so that the existence and the determination of a C^∞-immersion $j: M \to B$ should be assured ?

We solve here this problem but the immersion j will be defined only on a dense open subset D of M. In this way we obtain a theorem of almost-global existence.

Theorem 2 (of almost-global existence). *Let B and M be two real paracompact C^∞-manifolds, M is connected,* dim $B=n$, dim $M=m$, $m \leqslant n$. *If we suppose the following geometrical objects given:*

1) *a weak-triangulation F of M,*

2) *a contractible dense open subset $D \subset M$ and an atlas $\mathcal{A}_F$ of M, both associated to F,*

3) *a parallelization $\{e_a \mid a=\overline{1, m}\}$ of D and implicitely the system of m exterior 1-forms $\{\omega^a \mid a=\overline{1, m}\}$ determined by the conditions $\omega^a(e_b)=\delta^a_b$, $a, b=\overline{1, m}$.*

4) *a system of C^∞-functions $\{a^i_a: D \to \mathbf{R} \mid i=\overline{1, n},\ a=\overline{1, m}\}$ and for the last two system the following conditions are satisfied:*

$$\forall u \in D \mid \text{rang } (a^i_a(u))=m \tag{6}$$

and

$$d(a^i_a \omega^a)=0, \tag{7}$$

then, for every local chart $h(U, \chi)$ of B, there is a C^∞-immersion $j: D \to U \subset B$ so that the local representation in h_Δ, h of j, where h_Δ is arbitrary from $\mathcal{A}_F$, satisfies the relation $(\partial x^i / \partial u^\alpha)\, du^\alpha = a^i_a \omega^a$.

Proof. We consider an arbitrary local chart $h_\Delta(U_\Delta, \chi_\Delta) \in \mathcal{A}_F$ and the product-manifold $U \times U_\Delta$ having as global chart $h \times h_\Delta$. On the manifold $U \times U_\Delta$ we consider the involutive m-dimensional differential system (Chevalley-distribution) given in $h \times h_\Delta$ by

$$V_\alpha = \frac{\partial}{\partial u^\alpha} + a^i_a \omega^a_\alpha \frac{\partial}{\partial x^i}; \quad a,\ \alpha=\overline{1, m}; \quad i=\overline{1, n}; \tag{8}$$

where in $h_\Delta \omega^a = \omega^a_\alpha du^\alpha$. For every $\Delta \in F$ we shall consider the maximal integral manifold of (8), uniquely determined in this manner: for $\Delta_0 \in F$ we consider an arbitrary fixed point $(x_0, u_0) \in U \times \text{int } \Delta_0 \subset U \times U_{\Delta_0}$. The maximal integral manifold of (8), defined on $U \times U_{\Delta_0}$, will be uniquely determined by the condition that it contains the point (x_0, u_0). For an arbitrary m-simplex $\Delta \in F$ we have an unique sequence $\Delta_0, \Delta_1, \ldots, \Delta_h = \Delta$ which joins Δ to Δ_h (§2, *b*)). The maximal integral manifold of the system (8) associated to $U \times U_{\Delta_h}$ will be uniquely determined on by the condition that it is contiguous with the maximal integral manifold of (8) associated to $U \times U_{\Delta_{h-1}}$ a.s.o. (We remark that $U \times U_{\Delta_h} \cap U \times U_{\Delta_{h-1}} \neq \varnothing$ in virtue of relations $\sigma_{h-1} = \Delta_{h-1} \cap \Delta_h$, $U \times \sigma_{h-1} \subset U \times U_{\Delta_{h-1}} \cap U \times U_{\Delta_h}$. In order to be contiguous, it is sufficient that the considered integral manifolds have a common point $(x_1, u_1) \in U \times \text{int } \sigma_{h-1}$).

The maximal integral manifold of (8) defined on $U \times U_\Delta$, can be locally represented in $h \times h_\Delta$ by the relations $u^\alpha = u^\alpha$, $x^i = \varphi^i(u^1, \ldots, u^m)$ and so it defines a C^∞-map $j_\Delta: U_\Delta \cap D \cap \Delta \to U$. The functions $x^i = \varphi^i(u^1, \ldots, u^m)$,

$i=\overline{1, n}$, obviously satisfy the relations $\partial \varphi^i / \partial u^\alpha = a^i_a \omega^a_\alpha$ and by virtue of (6), rang $(\partial \varphi^i / \partial u^\alpha)=m$, so that the C^∞-map j_Δ is an immersion. For all the pairs $\Delta, \Delta' \in F$, having as intersection an $(m$-1)-simplex whose interior is included in D, the immersions j_Δ, $j_{\Delta'}$, are contiguous. It follows that we can define the C^∞-immersion $j: D \to U$ by the relation $j \mid U_\Delta \cap D \cap \Delta = j_\Delta$. Q.E.D.

b) *The almost global rigidity.* This problem can be formulated as follows: Let (M, j_1), (M, j_2) be two submanifolds of homogeneous space (G, B). What are the conditions which being satisfied by these two submanifolds, they should be congruent ? An answer can be obtained immediately putting together the theorem 1 and the proposition presented in §1, c) as a particularization of the theorem 2.3, p. 238 from [11]. Obviously, in this way, we shall obtain only the almost-congruence of the considered submanifolds. We shall have:

Theorem 3 (*of almost-global rigidity*). *Let (G, B) be a regular homogeneous space. We shall suppose that a base $\{\tilde{\omega}^p \mid p=\overline{1, r}\}$ of the linear space of left-invariant exterior 1-forms of G is fixed.*

Let (M, j_1), (M, j_2) be two submanifolds of (G, B). Let D be a parallelizable dense open subset of M with a parallelization $\{e_a \mid a=\overline{1, m}\}$ fixed. We shall denote by $\mu_1, \mu_2: D \to \mathcal{D}^{k_m}_m(B, \mathcal{A}_0)$ the C^∞-immersions associated to $[D, \{e_a\}, j_1]$, respectively to $[D, \{e_a\}, j_2]$ (theorem 1).

The sufficient conditions in order that the submanifolds (M, j_1), (M, j_2) be almost-congruent are

$$\mu_1(D) \subset \mathcal{N}_m,\ \mu_2(D) \subset \mathcal{N}_m, \tag{9}$$

$$\mu_1^*(\nu_m^*(\tilde{\omega}^p)) = \mu_2^*(\nu_m^*(\tilde{\omega}^p));\ p=\overline{1, r}. \tag{10}$$

Proof. By (9) the submanifolds (M, j_1) and (M, j_2) are regular submanifolds of (G, B) and they are almost-normalized by the C^∞-immersions $\nu_m \circ \mu_1$ and $\nu_m \circ \mu_2$ (Corollary of theorem 1 and Definition 2). By virtue of (10) and above mentioned proposition from §1, c), it follows that the submanifolds $(D, j_1 \mid D)$ and $(D, j_2 \mid D)$ are congruent so that the submanifolds (M, j_1), (M, j_2) are almost-congruent. Q.E.D.

Remark. If in the former theorem the submanifolds are complete (equivalently, $j_1(M)=\overline{j_1(M)}$, $j_2(M)=\overline{j_2(M)}$), then they are congruent. In this case the theorem 3 is a theorem of global rigidity.

c) *A fundamental system of invariants.* We shall consider a special class of regular homogeneous spaces.

Definition 3. *A special regular homogeneous space is a regular homogeneous space (G, B) for which are fixed: a base $\{\tilde{\omega}^p \mid p=\overline{1, r}\}$ of the linear space of left-invariant 1-forms of G, a preferential atlas $\mathcal{A}_0$ of B, for every natural numbers m $(1 \leqslant m \leqslant n-1)$ it is determined a natural number $k_m > 0$, the open subset $\mathcal{N}_m \subset \mathcal{D}^{k_m}_m(B; \mathcal{A}_0)$ and the C^∞-map $\nu_m: \mathcal{N}_m \to G$. For these elements, in the atlas $\mathcal{A}^{k_m}_m$ of $\mathcal{D}^{k_m}_m$ associated to $\mathcal{A}_0$, the following two conditions are satisfied:*

1) *The relations $[(x, h, (z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}))] \in \mathcal{N}_m$ are written only by means of the coordinates $z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}$. We shall denote these relations by $\mathcal{R}(z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k})$.*

2) *The coefficients of the exterior 1-forms* $\nu_m^*(\tilde{\omega}^p)$, $p=\overline{1, r}$, *are written only by means of the coordinates* $z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}$.

Remark. It (G, B) is a special regular homogeneous space then in a local chart of atlas $\mathcal{A}^k_{m^m}$ of $\mathcal{D}^k_{m^m}(B;\mathcal{A}_0)$ we shall have $\gamma^*_m(\tilde{\omega}^p)=G^p_i dx^i + \Sigma G^p_{ia_1,\ldots,a_h} dz^i_{a_1 \ldots a_h}$ where, by condition 2) from the Definition 3, G^p_i and $G^p_{a_1,\ldots,a_h}$ are functions only of $z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_{k_h}}$. By means of these functions, for every regular m-manifold M for which the C^∞-immersion $\mu: D \to \mathcal{H}_m$ is given, we can calculate the functions $g^p_a: D \to \mathbf{R}$, defined by the relations $\mu^*(\nu^*_m(\tilde{\omega}^p)) = g^p_a \omega^a$.

All these preparations being done we are able to state *the fundamental theorem*:

Theorem 4. *Let (G, B) be a special regular homogeneous space with all elements from the Definition 3 fixed and with the relations $\mathcal{R}(z^i_a, \ldots, z^i_{a_1 \ldots a_k})$ and the coefficients G^p_i, $G^p_{ia_1 \ldots a_h}$ of the 1-forms $\nu^*_m(\tilde{\omega}^p)$ determined.*

Let M be a real m-dimensional paracompact, connected C^∞-manifold for which the following four structures are fixed:

1) a weak-triangulation F of M,

2) a contractible dense open subset $D \subset M$ and an atlas $\mathcal{A}_F$ of M, both associated to F,

3) a C-parallelization $\{e_a \mid a=\overline{1, m}\}$ of D and implicitly the system of 1-forms $\{\omega^a \mid a=\overline{1, m}\}$, defined on D, determined by the relations $\omega^a(e_b)=\delta^a_b$.

4) a system of C^∞-functions $\{a^i_a: D \to \mathbf{R} \mid i=\overline{1, n}, a=\overline{1, m}\}$. By means of systems $\{e_a\}$, $\{\omega^a\}$ and $\{a^i_a\}$ the functions

$$z^i_a = a^i_a, \quad z^i_{a_1 a_2} = (z^i_a \omega^a_{\alpha_1})\, a_2\, e^{\alpha_1}_{a_1} e^{\alpha_2}_{a_2}, \quad z^i_{a_1 a_2 a_3} = (z^i_a \omega^a_{\alpha_1})_{|\alpha_2 \alpha_3} e^{\alpha_1}_{a_1} e^{\alpha_2}_{a_2} e^{\alpha_3}_{a_3}, \ldots \tag{11}$$

are calculated. By means of these functions the coefficients

$$\{g^p_a: D \to \mathbf{R} \mid p=\overline{1, r};\ a=\overline{1, m}\} \tag{12}$$

defined by $\mu^*(\nu^*_m(\tilde{\omega}^p)) = g^p_a \omega^a$, are calculated.

If for the systems 3), 4) and the functions (11), (12) *the following four conditions*

$$C_1 .\ \mathrm{rang}\,(a^i_a(u)) = m,\ (\forall u \in D) \qquad C_2 .\ d(a^i_a \omega^a) = 0, \qquad C_3 .\ \mathcal{R}(z^i_a, \ldots, z^i_{a_1 \ldots a_k} \ldots a_k),$$

$$C_4 .\ dg^p_a \wedge \omega^a + g^p_a d\omega^a = C^p_{p_1 p_2} g^{p_1}_{a_1} \cdot g^{p_2}_{a_2} \omega^{a_1} \wedge \omega^{a_2}$$

(where $C^p_{p_1 p_2}$ are the constants from Maurer-Cartan equations written for $\{\omega^p\}$) are satisfied,

then, a local chart $h(U, \chi) \in \mathcal{A}_0$, $U \subset \pi^k_{m^m}(\mathcal{H}_m)$, being fixed, there is a C^∞-immersion $j: D \to U \subset B$ so that the submanifold (D, j) is normalized by the C^∞-immersion $\nu = \nu_m \circ \mu$ where μ is the C^∞-immersion given by

$$\mu: u \in D \to [(j(a), h, (z^i_{a_1}, \ldots, z^i_{a_1 \ldots a_k}))] \in \mathcal{H}_m.$$

All the submanifolds (D, j) obtained in this way are congruent. If the map $j: D \to B$ can be prolonged in a map $\bar{j}: M \to B$, then the submanifold $(M, \bar{j})$

is almost-normalized and two such submanifolds are almost-congruent. If the submanifold $(M, \bar{j})$ is complete then they are congruent.

Proof. The structures 1)—4) (fixed for M) and the hypothesis (13), C_1, C_2 assure, in virtue of theorem 2, the existence of a C^∞-immersion $j: D \to U \subset B$. By means of (11) we can determine for (D, j) the C^∞-immersion $\mu: u \in D \to [(j(u), h, (z^i_a, \ldots, z^i_{a_1 \ldots ak_m}))] \in \mathcal{D}^k_{m^m}(B, \mathcal{A}_0)$. In virtue of (13), C_3, $\mu(D) \subset \mathcal{H}_m$, and so (D, j) is a regular submanifold. This submanifold is normalized and a normalization will be the C^∞-immersion $\nu = \nu_m \circ \mu: D \to G$. For this normalization we shall have $\nu^*(\tilde{\omega}^p) = g^p_m \omega^a$. The conditions (13), C_4 are implicated by the Maurer-Cartan equations satisfied by $\{\tilde{\omega}^p \mid p=\overline{1, r}\}$. Consequently, for two submanifolds $(D; j_1)$ and (D, j_2), determined by the given structure of M, the conditions of the theorem 3 are satisfied and these submanifolds are congruent. Q.E.D.

In order to apply the fundamental theorem for the submanifold of a given special regular homogeneous space we must cover the following step (we have to realize something like an algorithm):

Step 1. The fixation of G, H, B, $\{\tilde{\omega}^p\}$, $\mathcal{A}_0$, k_m, $\mathcal{D}^k_{m^m}$, $\mathcal{H}_m$, $\mathcal{R}(z^i_a, \ldots z^i_{a_1 \ldots ak_m})$, ν_m, $\mathcal{A}^k_{m^m}$, $\nu^{*}_m(\tilde{\omega}^p)$.

Step 2. The fixation of M, F, D, $\mathcal{A}_F$, $\{e_a\}$, $\{a^i_a\}$ and the verification of conditions (13) C_1, C_2 by these elements. We remark that the step 2 is independent of the homogeneous space considered in the step 1.

Step 3. By means of $\{e_a\}$, $\{a^i_a\}$ the calculation of the functions $(z^i_a, \ldots, z^i_{a_1 \ldots a_{km}})$ from (11) and by means of these functions the calculations of the functions $\{g^p_a\}$ determined by the relations $\mu^*(\nu^*_m(\tilde{\omega}^p)) = g^p_a \omega^a$. The verifications of the conditions (13), C_3, C_4 by the former elements. All these steps covered, in virtue of theorem 4, we can conclude that the structure $(F, D, \mathcal{A}_F, \{e_a\}, \{a^i_a\})$ (from the step 2) given on the m-dimensional manifold M, constitute a complete (non-minimal) system of invariants of M respective to the homogeneous space (G, B). This structure assure the existence of a C^∞-immersion $j: D \to U \subset B$ and all submanifolds (D, j) of (G, B) obtained in this way are congruent.

§4. **Examples.** We shall apply the fundamental theorem (i.e. the forms „algorithm") to the study of submanifolds of affine and projective space, the both being special regular homogeneous spaces.

a) *Affine space. Step* 1: The fundamental group is $GA(n, \mathbf{R}) = \{(a^i_j, a^i) \mid i, j=\overline{1, n}, a^i_j, a^i \in \mathbf{R}, \det(a^i_j) \neq 0\}$. If $a=(a^i_j, a^i)$, $b=(b^i_j, b^i)$ then $a \cdot b = (a^i_h b^h_j, a^i_h b^h + a^i)$. The group $GA(n, \mathbf{R})$ has a global chart $h(GA(n, \mathbf{R}), \chi)$ where $\chi((a^i_j, a^i)) = (X^i_j, X^i) \in \mathbf{R}^{n^2+n}$ and $a^i_j = X^i_j$, $a^i = X^i$.

The isotropy group is $H = \{(a^i_j, a^i) \in GA(n, \mathbf{R}) \mid a^i = 0, i=\overline{1, n}\}$.

The fundamental space $A_n = GA(n, \mathbf{R})/H = \mathbf{R}^n$ because $(a^i_j, a^i) \sim (b^i_j, b^i)$ if and only if $a^i = b^i$ $(i=\overline{1, n})$.

In the global chart h of $GA(n, \mathbf{R})$, the fixed base of the linear space of left-invariant 1-forms of $GA(n, \mathbf{R})$ will be $\tilde{\omega}^i_j = \tilde{X}^i_h dX^h_j$, $\tilde{\omega}^i = \tilde{X}^i_h dX^h$ where $(\tilde{X}^i_h)$ is the inverse matrix of the matrix (X^i_j). The Maurer-Cartan equations are $d\tilde{\omega}^i_j = -\tilde{\omega}^i_h \wedge \tilde{\omega}^h_j$, $d\tilde{\omega}^i = -\tilde{\omega}^i_h \wedge \tilde{\omega}^h$ and the structure-constants $C^{iqs}_{jpr} = -\delta^i_p \delta^q_r \delta^s_j$, $C^{iq}_{jpr} = 0$, $C^{iqs}_{pr} = 0$, $C^{iq}_{pr} = -\delta^i_q \delta^q_r$.

The preferential atlas of $A_n = \mathbf{R}^n$ will be determined by the canonical chart $h_0(\mathbf{R}^n, 1_{\mathbf{R}^n})$.

The natural number k_m $(1 \leqslant m \leqslant n-1)$ will be determined by the inequalities $C^1_m + C^2_{m+1} + \dots + C^{k_m-1}_{m+k_m-2} = Q < n \leqslant Q + C^{k_m}_{m+k_m-1}$.

$\mathcal{D}^{k_m}_m$ is the C^∞-manifold defined in §2, a).

$\mathcal{H}_m \subset \mathcal{D}^{k_m}_m$ is the open subset constituted by the points $[(x, h_0, (z^i_a, \dots z^i_{a_1 \dots a_{k_m}}))] \in \mathcal{D}^{k_m}_m$ for which, in the canonical chart h_0

$$\det\ (z^i_1, \dots, z^i_m, z^i_{11}, \dots, z^i_{mm}, \dots, z^i_{1\dots 1}, \dots, z^i_{b_1 \dots b_{k_m}}) \neq 0. \tag{14}$$

(the indices of the $n-Q$ final columns are took in the lexicographical order). We remark that $\pi^{k_m}_m(\mathcal{H}_m) = A_n$ and the relation (14) represents the relation $\mathcal{R}(z^i_a, \dots, z^i_{a_1 \dots a_{k_m}})$.

The map $\nu_m : \mathcal{H}_m \to \mathrm{GA}(n, \mathbf{R})$ is defined by the relation

$$\nu_m([(x, h_0, (z^i_a, \dots, z^i_{a_1 \dots a_{k_m}}))]) := (z^i_1, \dots, z^i_{b_1 \dots b_{k_m}}, x^i) \in \mathrm{GA}(n, \mathbf{R}). \tag{15}$$

We shall have $\pi \circ \nu_m = \pi^{k_m}_m \mid \mathcal{H}_m$.

The natural number k_m, the open subset $\mathcal{H}_m \subset \mathcal{D}^{k_m}_m(A_n;\ \mathcal{A}_0)$ and the map $\nu_m : \mathcal{H}_m \to \mathrm{GA}(n, \mathbf{R})$ being fixed, the relations $\pi^{k_m}_m(\mathcal{H}_m) = A_n$ and $\pi \circ \nu_m = \pi^{k_m}_m \mid \mathcal{H}_m$ being satisfied it follows that the homogeneous space $(\mathrm{GA}(n, \mathbf{R}), A_n)$ is regular (Def. 2).

The atlas $\mathcal{A}^{k_m}_m$ of $\mathcal{D}^{k_m}_m$, associated to $\mathcal{A}_0$, is defined in §2, a).

The C^∞-map ν_m will be represented in the local chart $h_0 \in \mathcal{A}^{k_m}_m$ (associated to $h_0 \in \mathcal{A}_0$) and in h of GA $(m, \mathbf{R})$, by the relations

$$X^i_1 = z^i_1,\ X^i_2 = z^i_2, \dots, X^i_m = z^i_m,\ X^i_{m+1} = z^i_{11}, \dots, X^i_n = z^i_{b_1 \dots b_{km}},\ X^i = x^i,\ i = \overline{1, n}. \tag{16}$$

By means of these relations we can determine the coefficients of the 1-forms $\nu^*_m(\tilde{\omega}^i_j)$, $\nu^*_m(\tilde{\omega}^i)$ in the local chart h_0. We shall have

$$\nu^*_m(\tilde{\omega}^i_j) = X^i_h dX^h_j,\ \nu^*_m(\tilde{\omega}^i) = \tilde{X}^i_h dX^h, \tag{17}$$

where X^i_j, X^i, $\tilde{X}_j{}^i$ are determined by (16).

The relations (14) and (17) written in the local chart $\bar{h}_0$ of $\mathcal{D}^{k_m}_m$ show that $(\mathrm{GA}(n, \mathbf{R}), A_n)$ is a special regular homogeneous space.

Step 2. For a given real m-dimensional paracompact connected C^∞-manifold M the following structures are fixed: A weak-triangulation F, i.e. a set of indexed m-simplices $\{\Delta_{om_1 \cdots m_p} \mid 1 \leqslant m_1 \leqslant m+1,\ 1 \leqslant m_2, \dots, m_p \leqslant m\}$ so that they covered M, $\Delta_{om_1 \dots m_{p-1}} \cap \Delta_{om_1 \dots m_{p-1} m_p}$ is an $(m-1)$-simplex of F and the union of the open subsets $\operatorname{int} \Delta_0 \cup \operatorname{int} (\Delta_0 \cap \Delta_{om_1}) \cup \operatorname{int} \Delta_{om_1} \cup \dots \cup \operatorname{int} \Delta_{om_1 \dots m_p}$ is a dense contractible open subset, denoted by D, of M [10];

An associated to F atlas $\mathcal{A}_F = \{h_{om_1 \dots m_p}(U_{om_1 \dots m_p}, \chi_{om_1 \dots m_p}) \mid \Delta_{om_1 \dots m_p} \in F\}$;

a parallelization $\{e_a \mid a = \overline{1, m}\}$ of D and the associate system of 1-forms $\{\omega^a \mid a = \overline{1, m}\}$, defined on D, determined by the relations $\omega^a(e_{a_1}) = \delta^a_{a_1}$;

a system of C^∞-functions $\{a^i_a : D \to \mathbf{R} \mid i = \overline{1, n};\ a = \overline{1, m}\}$.

For these elements we shall verify the conditions (13) C_1, C_2.

Step 3. By means of $\{e_a\}$, $\{a^i_a\}$ and (11) we shall calculate the functions $(z^i_a, \dots, z^i_{a_1 \dots a_{k_m}})$.

In order to calculate the C^∞-functions g^i_{ja}, $g^i_a : D \to \mathbf{R}$ defined by the relations $\mu^*(\nu^*_m(\tilde{\omega}^i_j)) = g^i_{ja}\omega^a$, $\mu^*(\nu^*_m(\tilde{\omega}^i)) = g^i_a \omega^a$, we shall use (16) and (17). The wanted expressions will be:

$$g^i_{ja} = X^i_h dX^h_j(e_a),\ g^i_a = \tilde{X}_h{}^i dX^h(e_a), \tag{18}$$

where $dX^i = z^i_a \omega^a$ and we must replace (X^i_j) by $(z^i_1, z^i_2, \dots, z^i_{b_1 \dots b_{km}})$.

We shall verify the conditions (13), C_3, C_4 by the obtained functions $z^i_a, \dots, z^i_{a_1 \dots a_m}$, g^i_{ja}, g^i_a. The conditions (13), C_4 are

$$dg^i_{ja} \wedge \omega^a + g^i_{ja} d\omega^a = -g^i_{ha_1} g^h_{ja_2} \omega^{a_1} \wedge \omega^{a_2},\ dg^i_a \wedge \omega^a + g^i_a d\omega^a = -g^i_{ha_1} g^h_{a_2} \omega^{a_1} \wedge \omega^{a_2}, \tag{19}$$

Conclusion. The system $(F, D, \mathcal{A}_F, \{e_a\}, \{a^i_a\})$ defined for M, satisfying (13) C_1, C_2, (14) and (19) is a complete system of invariants for M in the affine space (the homogeneous space $(\mathrm{GA}(n, \mathbf{R}), A_n)$). The local chart $h(U, \chi) \in A_*$ utilized in the theorem 4 is here the canonical chart $h_0(\mathbf{R}^n, 1_{\mathbf{R}^n})$.

b) *Projective space. Step* 1. The fundamental group is $\mathrm{GP}(n, \mathbf{R}) = \mathrm{GL}(n+1, \mathbf{R})/D(n+1, \mathbf{R})$ where $\mathrm{GL}(n+1, \mathbf{R}) = \{(a^B_A) \mid A, B = \overline{1, n+1}, \det (a^B_A) \neq 0\}$ and $D(n+1, \mathbf{R}) = \{(\rho \delta^B_A) \mid \rho \in \mathbf{R}, \rho \neq 0\}$. An arbitrary element of $\mathrm{GP}(n, \mathbf{R})$ will be denoted by $a = (\rho a^B_A)$. For $\mathrm{GP}(n, \mathbf{R})$ we shall consider the atlas constituted by $(n+1)^2$ local charts $h^B_A(U^A_B, \chi^B_A)$ where $U^B_A = \{a \in \mathrm{GP}(n, \mathbf{R}) \mid a^B_A \neq 0\}$, $\chi^B_A : a \in U^B_A \to (X^D_C) \in \mathbf{R}^{n^2+2n}$, $(C \neq A,\ D \neq B,\ X^D_C = a^D_C / a^B_A)$. We shall use the local chart h^{n+1}_{n+1} and for this chart $X^i_{n+1} = X^i$, $X^{n+1}_j = X_j$.

The isotropy group is $H = \{(\rho a^B_A) \in \mathrm{GP}(n, \mathbf{R}) \mid a^1_{n+1} = \dots = a^n_{n+1} = 0\}$. In h^{n+1}_{n+1} the subgroup H is characterized by $X^i = 0$, $(i = \overline{1, n})$.

The fundamental space P_n is $\mathrm{GP}(n, \mathbf{R})/H$. Because $(\rho a^B_A) \sim (\rho b^B_A)$ if and only if $a^B_{n+1} = \rho b^B_{n+1} (A = \overline{1, n+1})$ it follows that $P_n = \{(\rho \mathscr{X}^A) \mid \mathscr{X}^A \in \mathbf{R},\ A = \overline{1, n+1},\ \rho \in \mathbf{R},\ \rho \neq 0,\ \Sigma(\mathscr{X}^A)^2 \neq 0\}$.

In the local chart h^{n+1}_{n+1} of $\mathrm{GP}(n, \mathbf{R})$ the fixed base of the linear space of left-invariant 1-forms will be

$$\tilde{\omega}^i_j = \tilde{A}^i_h dX^h_j - \tilde{A}^i_h X^h dX_j + \delta^i_j X_h \tilde{A}^h_k X^l,\ \tilde{\omega}^i = \tilde{A}^i_h dX^h,$$

$$\tilde{\omega}_j = -X_h \tilde{A}^h_k dX^k_j + (1 + X_h \tilde{A}^h_k X^k)\, dX_j,\ (h, i, j, k = \overline{1, n}),$$

where $A^i_j = X^i_j - X_j X^i$ and $(\tilde{A}^i_j)$ is the inverse matrix of (A^i_j). The Maurer-Cartan equations are $d\tilde{\omega}^i_j = -\tilde{\omega}^i_h \wedge \tilde{\omega}^h_j - \tilde{\omega}^i \wedge \tilde{\omega}_j + \delta^i_j \tilde{\omega}^h \tilde{\omega}_h$, $d\tilde{\omega}^i = -\tilde{\omega}^i_h \wedge \tilde{\omega}^h$, $d\tilde{\omega}_j = -\tilde{\omega}^h_j \wedge \tilde{\omega}_h$.

The preferential atlas $\mathcal{A}_0$ of P_n will be determined by the local chart $h_{n+1}(U_{n+1}, \chi_{n+1})$ where $U_{n+1} = \{(\rho \mathscr{X}^A) \in P_n \mid \mathscr{X}^{n+1} \neq 0\}$, $\chi_{n+1} : (\rho \mathscr{X}^A) \in U_{n+1} \to (x^i) \in \mathbf{R}^n$, $x^i = \mathscr{X}^i / \mathscr{X}^{n+1}$, $i = \overline{1, n}$.

The natural number k_m will be determined by the same inequalities as in affine space. $\mathcal{D}^{k_m}_m$ is the C^∞-manifold defined in §2, a). $\mathcal{H}_m \subset \mathcal{D}^{k_m}_m$ is the open subset constituted by all the points $[(x, h_{n+1}, (z^i_a, \dots, z^i_{a_1 \dots a_{km}}))] \in \mathcal{D}^{k_m}_m$ for which, in the local chart h_{n+1} of P_n the relation (14) is satisfied. We shall have $\pi^{k_m}_m(\mathcal{H}_m) = U_{n+1}$ and the relations $\mathcal{R}(z^i_a, \dots, z^i_{a_1 \dots a_{km}})$ are represented also by (14).

The C^{ω}-map $\nu_m : \mathcal{N}_m \to GP(n, \mathbf{R})$ is defined by

$$\nu_m([(x, h_{n+1}, (z_a^i, \ldots, z^i_{a_1 \ldots a_{km}}))]) =$$

$$= \rho \begin{pmatrix} z_1^i \; z_2^i \ldots z_m^i \; z_{11}^i \ldots z^i_{b_1 \ldots b_{km}} \mathcal{X}^i \\ 0 \;\; 0 \quad 0 \;\; 0 \ldots 0 \qquad 1 \end{pmatrix} \in GP(n, \mathbf{R}). \tag{20}$$

The atlas $\mathcal{A}^{k_m}_m$ of $\mathcal{D}^{k_m}_m$ associated to $\mathcal{A}_0$ is defined in §2, a).

The map ν_m will be represented in the local chart $\overline{h}_{n+1} \in \mathcal{A}^{k_m}_m$(associated to $h_{n+1} \in \mathcal{A}_0$) and in h^{n+1}_{n+1} of $GP(n, \mathbf{R})$ by the relations

$$X_1^i = z_1^i, \; X_2^i = z_2^i \ldots X_m^i = z_m^i, \; X^i_{m+1} = z^i_{11}, \ldots, X^i_n = z^i_{b_1 \ldots b_{km}},$$
$$X^i = x^i, \; X_j = 0, \; i, j = \overline{1, n}. \tag{21}$$

By means of these relations we can determine the coefficients of the 1-forms $\nu_m^*(\tilde{\omega}_j^i)$, $\nu_m^*(\tilde{\omega}^i)$, $\nu_m^*(\tilde{\omega}_j)$ in the local chart h_{n+1}. We shall have

$$\nu_m^*(\tilde{\omega}_j^i) = \tilde{X}_h^i dX_j^h, \quad \nu_m^*(\tilde{\omega}^i) = \tilde{X}_h^i dX^h, \quad \nu_m^*(\tilde{\omega}_j) = 0. \tag{22}$$

It is obviously that $(GP(n\ \mathbf{R}), P_n)$ is a special regular homogeneous space.

Step 2 is the same as in the affine case (we remarked that in general this step is independent of the considered homogeneous space). We shall suppose that for the considered m-dimensional paracompact connected C^{∞}-manifold M, the structure of the step 2 $(F, D, \mathcal{A}_F, \{e_a\}, \{a_a^i\})$, satisfying the conditions (13), C_1, C_2, is given.

Step 3. By means of $\{e_a\}$, $\{a_a^i\}$ and (11) we shall calculate the functions $(z_a^i, \ldots, z^i_{a_1 \ldots a_{km}})$. In the same way as in the affine space, utilizing (21), (22), we shall calculate the functions g^i_{ja}, g^i_a, g_{ja}. We shall have

$$g^i_{ja} = \tilde{X}_h{}^i \, dX_j^h(e_a), \quad g_a^i = \tilde{X}_h{}^i \, dX^h(e_a), \quad g_{ja} = 0 \tag{23}$$

where $dX^i = z_a^i \omega^a$ and the elements (X_j^i) are replaced by $(z_a^i, \ldots, z^i_{b_1 \ldots b_{km}})$ from (21). We shall verify the conditions (13), C_4 (implicated by the Maurer-Cartan equations satisfied by $\tilde{\omega}_j^i$, $\tilde{\omega}^i$, $\tilde{\omega}_j$) which here coincide with (19).

Conclusion. The system $(F, D, \mathcal{A}_F, \{e_a\}, \{a_h^i\})$ defined on M, satisfying the conditions (13) C_1, C_2 (14), (19) is a complete system of invariants for M in the homogeneous space $(GP(n, \mathbf{R}), P_n)$. The local chart utilized in the theorem 4 is here h_{n+1}. We remark that this system and the conditions satisfied by it are exact the same as for the affine space $(GA(n, \mathbf{R}), A_n)$. The reasons are: the affine space is subordinated to the projective space in sense of Erlangen-Program (i.e. $GA(n, \mathbf{R}) \subset GP(n, \mathbf{R})$ and $A_n \subset P_n$ and for the projective space the relations $\nu_m(\mathcal{N}_m) \subset GA(n, \mathbf{R})$ and $\pi^{k_m}_m(N_m) \subset U_{n+1} = A_n = \mathbf{R}^n$ are satisfied. Two submanifolds (D, j_1), (D, j_2) will be congruent by a projective transformation which preserve the set $U_{n+1}(j_1(D) \subset U_{n+1}, j_2(D) \subset U_{n+1})$, i.e. by an affine transformation.

REFERENCES

1. Bernard D. — *Espaces homogènes et repère mobile*, Colloquio International de Geometria Differential, Santiago de Compostela, Septembre 1978.
2. Green L. M. — *The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces*, Duke Math. J., 45 (1978), 735—779.
3. Helgason S. — *Differential geometry and symmetric spaces*. Academic Press, New York and London 1962 (Russian edition : izd. „Mir" Moskva, 1964).
4. Papuc I. D. — *Sur la théorie des variétés des espaces Kleinéens à groupe linéaire complètement reductible*, C.R. Acad. Sci. Paris, 257, (1963), 589—591.
5. Papuc I. D. — *Sur les variétés différentiables plongées dans un espace Klein à group linéaire complètement reductible*, An. şt. Univ. Iaşi, S. 1, IX, (2) (1963), 423—436.
6. Papuc I. D. — *Une nouvelle méthode dans la théorie des variétés d'un espace homogène à groupe linéaire*, An. şt. Univ. Iaşi, S.1. Mat. X, (1) (1964), 159—182.
7. Papuc I. D. — *Sur la géométrie différentielle des variétés d'un espace axial*, An. şt. Univ. Iaşi, S. 1, XI, (1) (1965), 299—312.
8. Papuc I. D. — *Sur la normalisation locale des variétés d'un espace homogène*, An. şt. Univ. Iaşi, S. 1. Mat. XII, (1) (1966), 139—148.
9. Papuc I. D. — *Asupra normalizării globale a subvarietăţilor unui spaţiu omogen*, An. Univ. Timişoara, seria mat.-fiz. vol. V (1967), 177—191.
10. Papuc I., D. — *A Lemma of differential topology*, Sem. de geometrie şi topologie, Universitatea din Timişoara, nr. 87, 1986, p. 1—9.
11. Sternberg S. — *Lectures on Differential Geometry* : Prentice Hall. Inc. Englewood Cliffs, N. J. 1964 (Russian edition : izd. „Mir", Moskva, 1970).

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