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CRITICAL MAPPINGS OF RIEMANNIAN MANIFOLDS AND THE SLICE THEOREM FOR AN ACTION OF THE DIFFEOMORPHISM GROUP

BY

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Introduction. D. D. B l e e c k e r [2] considers maps from one C^∞ connected and compact Riemannian manifold M to another C^∞ Riemannian manifold N which are critical for all invariantly defined functionals on the space of maps. To define this notion, the author considers $\mathcal{M}$ (resp. $\mathcal{N}$) to be the space of all C^∞ Riemannian metrics on M (resp. N) studied by D. E b i n in [3], the space $\mathcal{D}(M)$ (resp. $\mathcal{D}(N)$) of diffeomorphisms of M (resp. N) and the left action ρ of $\mathcal{D}(M)\times\mathcal{D}(N)$ on $\mathcal{M}\times C^\infty(M, N)\times N$ given by:

$$\rho((\sigma, \eta), (g, f, g')) = (\sigma^{-1*} g, \eta \circ f \circ \sigma^{-1}, \eta^{-1*} g').$$

If (N, g') is fixed, $\mathcal{D}(M)\times I_{g'}$ acts on the submanifold $\mathcal{M}\times C^\infty(M, N)\times\{g'\}$ in the same way as before. Let $\bar{\rho}$ this action. The isotropy subgroup I_f of $\mathcal{D}(M)\times\mathcal{D}(N)$ at $(g, f, g')\in\mathcal{M}\times C^\infty(M, N)\times\mathcal{N}$ consists of all pairs of isometries $(\sigma, \eta)\in I_g\times I_{g'}$ such that $\eta\circ f\circ\sigma^{-1}=f$.

The following is the Bleecker's definition:

Definition. *Let (N, g') be a Riemannian manifold. We say that: $f:(M, g)\to(N, g')$ is critical if for all pairs $(s, \Phi)\in S^2(M)\times C^\infty_f(M, TN)$ which are normal to the orbit of (g, f, g') under the action $\bar{\rho}$ at (g, f, g') and are left fixed by the isotropy subgroups I_f, we have $\Phi=0$.*

Here $S^2(M)\times C^\infty_f(M, TN)$ is cannonically identified with the tangent space at the point (g, f, g') to the manifold $\mathcal{M}\times C^\infty(M, N)\times\{g'\}$ (see [7], [8] f.e.).

In this interesting work, Bleecker shows that, what is really needed to „justify" the previous definition is a slice theorem for the action of $\mathcal{D}(M)\times I_{g'}$ on $\mathcal{M}\times C^\infty(M, N)\times\{g'\}$ and indicates one crucial property needed to produce a slice.

In this note, in analogy with the slice theorem for the action of $\mathcal{D}(M)$ on $\mathcal{M}$[3], we shall prove that the orbit $\bar{\rho}(\mathcal{D}(M)\times I_{g'}, (g, f, g'))$ is a smooth submanifold of $\mathcal{M}\times C^\infty(M, N)\times\{g'\}$, as well as the existence of a slice for the action $\bar{\rho}$. These facts justify the Bleecker's definition.

In order to sharpen these notions, we work in the corresponding Sobolev spaces. A superscript denotes the Sobolev class $H^s=W^{s,2}=L^2_s$ (see [3]). We always assume that $s>(q/2)+1$, $q=\max(\dim M, \dim N)$. Specifically, we need the fact that $\mathcal{D}^s(M)$ is a manifold and a topological group in which right multiplication is C^∞ and that the action of $\mathcal{D}^{s+1}(M)$ on $\mathcal{M}^s$ is continuous. Thus, $\mathcal{D}^{s+1}(M)\times I_{g'}$ is also a topological group which acts at the left on the manifold $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$.

§ 1. Statement of the results

Theorem 1. *Let $(g, f, g')\in\mathcal{M}^{s+r}\times H^s(M, N)\times\mathcal{N}^{s+r}$ and $\mathcal{O}^s_{(g,f,g')}$ be its orbit under $\mathcal{D}^{s+r+1}(M)\times I_{g'}$. Then $\mathcal{O}^s_{(g,f,g')}\subset\mathcal{M}^s\times H^s(M, N)\times\{g'\}$ is a C^r closed submanifold with tangent space at (g, f, g') given by:*

$$T_{(g,f,g')}\,\mathcal{O}^s_{(g,f,g')}=\{(-L_X g,\ Y\circ f-Tf\circ X,\ 0)\quad \text{for all}\quad X\in\mathcal{X}^{s+r}(M)$$

and for all Killing fields Y on $(N, g')\}$.

Theorem 2. *If $(g, f, g')\in\mathcal{M}^{s+r}\times H^s(M, N)\times\mathcal{N}^{s+r}$, $r\geq 1$, then the action of $\mathcal{D}^{s+r+1}(M)\times I_{g'}$ on $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$ admits a slice $\tilde{S}$ at (g, f, g'); i.e. $\tilde{S}$ is a submanifold containing (g, f, g') such that:*

1) if $(\sigma, \eta)\in I_f$, $\tilde{\rho}((\sigma, \eta), \tilde{S})=\tilde{S}$;

2) if $(\sigma, \eta)\in\mathcal{D}^{s+r+1}(M)\times I_{g'}$ and $\tilde{\rho}((\sigma, \eta), \tilde{S})\cap\tilde{S}\neq\emptyset$ then $(\sigma, \eta)\in I_f$.

3) there is a local cross-section $\chi:\mathcal{D}^{s+r+1}(M)\times I_{g'}/I_f\to\mathcal{D}^{s+r+1}(M)\times I_{g'}$, defined on a neighborhood U of the identity coset such that $([\sigma, \eta], (g_1, f_1, g'))\to\tilde{\rho}(\chi([\sigma, \eta], (g_1, f_1, g'))$ is a homeomorphism of $U\times\tilde{S}$ with a neighborhood of (g, f, g')

§ 2. Proof of the theorem 1. Let $\Psi:\mathcal{D}^{s+r+1}(M)\times I_{g'}\to\mathcal{M}^s\times H^s(M, N)\times\{g'\}$ be the orbit map through (g, f, g'), $\Psi(\sigma, \eta)=\tilde{\rho}((\sigma, \eta), (g, f, g'))=(\sigma^{-1*}g,\ \eta\circ f\circ\sigma^{-1},\ g')=(\Psi_1(\sigma), \Psi_2(\sigma,\eta), \Psi_3(\eta))$. For fixed $(g, f, g')\in\mathcal{M}^{s+r}\times H^s(M, N)\times\mathcal{N}^{s+r}$, the map Ψ has its values in $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$, and is of the class C^r, $r\geqslant 0$. Indeed, the map Ψ is C^r iff the maps Ψ_1, Ψ_2, Ψ_3 are C^r. The map Ψ_1 (resp. Ψ_3) is the composition of the C^r mapping $\mathcal{D}^{s+r+1}(M)\to\mathcal{D}^{s+1}(M)$ (resp. $\mathcal{D}^{s+r+1}(N)\to\mathcal{D}^{s+1}(N)$), $\sigma\to\sigma^{-1}$ (resp. $\eta\to\eta^{-1}$) with the C^r-mapping $A_M:\mathcal{D}^{s+1}(M)\times\mathcal{M}^{s+r}\to\mathcal{M}^s$ (resp. $A_N:\mathcal{D}^{s+1}(N)\times\mathcal{N}^{s+r}\to\mathcal{N}^s$). Related to the mapping Ψ_2, it is the composition of C^r-map $\mathcal{D}^{s+r+1}(M)\to\mathcal{D}^{s+1}(M)$ $(\sigma\to\sigma^{-1})$ with C^∞-map $\mathcal{R}_{\sigma^{-1}}:H^s(M, N)\to H^s(M, N)$ (α-lemma) and with C^{r+1}-map $\omega_\eta:H^s(M, N)\to H^s(M, N)$ (ω-lemma). However, it is unknown whether, in general, the composition of two H^s-maps is again H^s. But in all known proofs one needs that one of the maps be a diffeomorphism or be C^∞. Hence the compositions in the definitions of Ψ_2 present no problem. Moreover, the mapping Ψ_2 is C^r, $r\geqslant 0$. For $r\geqslant 1$, Ψ has tangent (derivative) in identity given by:

$$T_{(\mathrm{id},\,\mathrm{id})}\Psi:\mathcal{X}^{s+r+1}(M)\times\mathcal{X}_0^{s+r+1}(N)\to S_2^s(M)\times H_f^s(M, TN),$$

$$(X, Y)\to(-L_X g,\ Y\circ f-Tf\circ X).$$

Here, $\mathcal{X}_0^{s+r+1}(N)$ stands for Killing fields of the class W^{s+r+1} related to g' on N, $S_2^s(M)$ denotes the symmetric two tensor of class W^s and $H_f^s(M, TN)$ the vector fields along f of the class W^s because $T_f H^s(M, N)\simeq H_f^s(M, TN)$. L denotes Lie differentiation.

At the point $(\sigma, \eta)\in\mathcal{D}^{s+r+1}(M)\times I_{g'}$ we have:

$$(T_{(\sigma,\eta)}\Psi)(X\circ\sigma,\ Y\circ\eta)=(\sigma^*(-L_X g),\ T\eta\circ Y\circ f-T\eta\circ Tf\circ X),$$

which is just $T_{(\mathrm{id},\,\mathrm{id})}\Psi$ extended to be right invariant. Thus $\ker T_{(\sigma,\eta)}\Psi$

is $\ker T_{(\mathrm{id},\,\mathrm{id})}\Psi$ extended to be right invariant over $\mathcal{D}^{s+r+1}(M)\times I_{g'}$. But $\ker T_{(\mathrm{id},\,\mathrm{id})}\Psi$ is a finite dimensional C^r subbundle of $T(\mathcal{D}^{s+r+1}(M)\times I_{g'})$ because the principal part of the first order operator $T_{(\mathrm{id},\,\mathrm{id})}\Psi$ is $-L_X g$ which is known to have an injective symbol ([1]). By splitting lemma ([1]), the range of $T_{(\mathrm{id},\,\mathrm{id})}\Psi$ is closed with a closed complement. Then, (see [7]), it results that $\Psi(\mathcal{D}^{s+r+1}(M)\times I_{g'})=\mathcal{O}^s_{(g,f,g')}$ is an immersed submanifold of $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$. To finish the proof we need to verify that Ψ is open onto its image and has closed range. The sufficient condition for Ψ to be open with closed range was pointed out by Fischer and Marsden in [5]:

a) The range of Ψ near $(\sigma, \eta)\in\mathcal{D}^{s+r+1}(M)\times I_{g'}$ depends only on $\Psi(\sigma, \eta)$, i.e. if $\Psi(\sigma_1, \eta_1)=\Psi(\sigma_2, \eta_2)$, then $\Psi(U_1)=\Psi(U_2)$ for some neighborhood U_1 of (σ_1, η_1) and U_2 of (σ_2, η_2);

b) if $\Psi(\sigma_n, \eta_n)\to(g_1, f_1, g')$ then (σ_n, η_n) has a convergent subsequence in $\mathcal{D}^{s+r+1}(M)\times I_{g'}$.

By the right invariance of Ψ the condition a) holds. The condition b) results from Prop. 6.13 [3] applied for Ψ_1 and Ψ_3.

§3. Proof of theorem 2. Together with the result from Theorem 1, in the proof of Theorem 2, the first step will be to construct the normal bundle of $\mathcal{O}^s_{(g,f,g')}$ in $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$. For this aim it is needed to define a weak and strong Riemannian structure on $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$ and we show that the normal bundle induced by the weak Riemannian structure is smooth. Therefore, for each $(g, f, g')\in\mathcal{M}^s\times H^s(M, N)\times\{g'\}$, we must define an inner product $<,>_{(g,f,g')}$ on the tangent space $T_{(g,f,g')}\mathcal{M}^s\times H^s(M, N)\times\{g'\}\simeq S_2^s(M)\times H_f^s(M, TN)$ which induces the H^0-topology on $S_2^s(M)\times H_f^s(M, TN)$, and an inner product $<,>^s_{(g,f,g')}$ which induces the H^s-topology on the same space. Indeed, we have: $<(s_1, \Phi_1), (s_2, \Phi_2)>_{(g,f,g')}=<s_1, s_2>_g+<\Phi_1, \Phi_2>$, $<(s_1, \Phi_1), (s_2, \Phi_2)>^s_{(g,f,g')}=<s_1, s_2>^s_g+<\Phi_1, \Phi_2>^s$ where $<s_1, s_2>_g$, $<s_1, s_2>^s_g$ are the inner products defined by Ebin in [3] on $\mathcal{M}^s$, $<\Phi_1, \Phi_2>=\int_M g'(\Phi_1(x), \Phi_2(x))\,\mu_g$, $<\Phi_1, \Phi_2>^s=\int_M<\Phi_1(x), \Phi_2(x)>^s_{g'}\,\mu_g$.

The smooth weak Riemannian structure on $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$ such constructed permits to construct an affine connection and the exponential map. Since the proof proceeds in the same way as in [3], [4] and [8] (similarly with the existence of a spray for the metric on $\mathcal{D}^s(M)$), we shall omit the details.

Now, we use suggestions from [2] and the proof of Theorem 7.1 from [3], to show that the normal bundle of $\mathcal{O}^s_{(g,f,g')}$ in $\mathcal{M}^s\times H^s(M, N)\times\{g'\}$, with respect to the previous weak Riemannian metric, is a smooth subbundle of $T(\mathcal{M}^s\times H^s(M, N)\times\{g'\}\mid\mathcal{O}^s_{(g,f,g')}$. We shall find a smooth surjective vector bundle map $P:T(\mathcal{M}^s\times H^s(M, N)\times\{g'\}\mid\mathcal{O}^s_{(g,f,g')}\to T(\mathcal{O}^s_{(g,f,g')})$ whose kernel is this normal bundle. Let P on the fiber at (g, f, g') be $P(s, \Phi)=2\operatorname{Div}(s)^\#-f^t_*\Phi$. In [2], Bleecker proves that $\ker P$ is the chosen normal bundle. Proceeding along the standard lines of the Ebin Slice Theorem, it is straightforward to show that the action of $\mathcal{D}^{s+r+1}(M)\times I_{g'}$ on $M^s\times H^s(M, N)\times\{g'\}$ has a slice. As in [3] one can also allow $s=\infty$. This completes the proof.

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SUBMANIFOLDS IN LAGRANGE GEOMETRY

BY

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In this paper we study the differential geometry of the submanifolds in Lagrange spaces. Recall that a Lagrange space is a smooth manifold carrying a regular Lagrangian i.e. a smooth function defined on the tangent bundle, sucht that the Hessian with respect to the tangential coordinates is nondegenerate. Using the Euler-Lagrange equations associated with a regular Lagrangian, we can obtain a nonlinear connection. Next we get a Riemannian metric on the tangent bundle, similar to the Sasaki metric. For a submanifold of the given Lagrange space, its tangent bundle is thought of as a submanifold in the tangent bundle of the Lagrange space. A nonlinear connection is induced on the submanifold from the given nonlinear connection but the induced metric on the tangent bundle to the submanifold is no longer the Sasaki metric obtained from the induced nonlinear connection and the induced Lagrangian. However some interesting results arise from this situation. Writing down the Gauss formula, we get the linear connection on the tangent bundle of the submanifold induced from the Levi Civita connection of the Sasaki metric on the tangent bundle to the given Lagrange space, as well as the second fundamental form. Next, from the Weingarten formulas we get the linear connection induced on the normal bundle to the tangent bundle of the submanifold. Finally we get the relations between the curvature tensor field of the Levi Civita connection and the curvature tensor fields of the induced connections.

Functions, vector fields and geometric objects we consider are assumed to be smooth (i.e. C^∞). Throughout this paper we use the well known summation convention. The ranges for the indices used in this paper are: $\{1, ..., n\}$ for $h, i, j, k, l, p, ...$; $\{\hat{1}, ..., \hat{n}\} = \{n+1, ..., 2n\}$ for $\hat{h}, \hat{i}, \hat{j}, \hat{k}, \hat{l}, \hat{p}, ...$; $\{1, ..., m\}$ for $\alpha, \beta, \gamma, \sigma, \tau, ...$; $m < n$, $\{\hat{1}, ..., \hat{m}\} = \{n+1, ..., n+m\}$ for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\sigma}, \hat{\tau}, ...$; $\{m+1, ..., n\}$ for $a, b, c, d, ...$; $\{\widehat{m+1}, ..., \hat{n}\} = \{n+m+1, ..., 2n\}$ for $\hat{a}, \hat{b}, \hat{c}, \hat{d}, ...$.

1. **Lagrange spaces.** Let $\tilde{M}$ be a smoth n-dimensional manifold and denote by $T\tilde{M}$ its tangent bundle. The natural differentiable structure is considered on $T\tilde{M}$ and let $(x^1, ..., x^n, y^1, ..., y^n)$ be the local coordinates on $T\tilde{M}$ induced from the local coordinates $(x^1, ..., x^n)$ on $\tilde{M}$. Recall that the local coordinate transformations on $T\tilde{M}$ have the following special form: