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SUBMANIFOLDS IN LAGRANGE GEOMETRY

BY

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In this paper we study the differential geometry of the submanifolds in Lagrange spaces. Recall that a Lagrange space is a smooth manifold carrying a regular Lagrangian i.e. a smooth function defined on the tangent bundle, sucht that the Hessian with respect to the tangential coordinates is nondegenerate. Using the Euler-Lagrange equations associated with a regular Lagrangian, we can obtain a nonlinear connection. Next we get a Riemannian metric on the tangent bundle, similar to the Sasaki metric. For a submanifold of the given Lagrange space, its tangent bundle is thought of as a submanifold in the tangent bundle of the Lagrange space. A nonlinear connection is induced on the submanifold from the given nonlinear connection but the induced metric on the tangent bundle to the submanifold is no longer the Sasaki metric obtained from the induced nonlinear connection and the induced Lagrangian. However some interesting results arise from this situation. Writing down the Gauss formula, we get the linear connection on the tangent bundle of the submanifold induced from the Levi Civita connection of the Sasaki metric on the tangent bundle to the given Lagrange space, as well as the second fundamental form. Next, from the Weingarten formulas we get the linear connection induced on the normal bundle to the tangent bundle of the submanifold. Finally we get the relations between the curvature tensor field of the Levi Civita connection and the curvature tensor fields of the induced connections.

Functions, vector fields and geometric objects we consider are assumed to be smooth (i.e. C^∞). Throughout this paper we use the well known summation convention. The ranges for the indices used in this paper are : $\{1, \ldots, n\}$ for $h, i, j, k, l, p, \ldots$; $\{\hat{1}, \ldots, \hat{n}\} = \{n+1, \ldots, 2n\}$ for $\hat{h}, \hat{i}, \hat{j}, \hat{k}, \hat{l}, \hat{p}, \ldots$; $\{1, \ldots, m\}$ for $\alpha, \beta, \gamma, \sigma, \tau, \ldots$; $m < n$, $\{\hat{1}, \ldots, \hat{m}\} = \{n+1, \ldots, n+m\}$ for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\sigma}, \hat{\tau}, \ldots$; $\{m+1, \ldots, n\}$ for $a, b, c, d, \ldots$; $\{\widehat{m+1}, \ldots, \hat{n}\} = \{n+m+1, \ldots, 2n\}$ for $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \ldots$.

1. **Lagrange spaces.** Let $\tilde{M}$ be a smoth n-dimensional manifold and denote by $T\tilde{M}$ its tangent bundle. The natural differentiable structure is considered on $T\tilde{M}$ and let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be the local coordinates on $T\tilde{M}$ induced from the local coordinates $(x^1, \ldots, x^n)$ on $\tilde{M}$. Recall that the local coordinate transformations on $T\tilde{M}$ have the following special form :

$$(1)\qquad \tilde{x}^i=\tilde{x}^i(x^1,\dots,x^n),\qquad \tilde{y}^i=\frac{\partial\tilde{x}^i}{\partial x^k}y^k,\qquad i=1,\dots,n.$$

A *regular Lagrangian* on $\tilde{M}$ is a smooth real valued function L defined on $T\tilde{M}$ such that the matrix with the entries

$$(2)\qquad \tilde{g}_{ik}=\frac{\partial^2 L}{\partial y^i\partial y^k}$$

is everywhere nondegenerate. Throughout this paper we shall make the supplimentary assumption that the matrix $\tilde{g}_{ik}$ is positive. A *Lagrange space* is a smooth manifold $\tilde{M}$ carrying a regular Lagrangian. Recall that from the Euler-Lagrange equations associated with L, we can obtain a nonlinear connection on $\tilde{M}$ defined by

$$(3)\qquad \tilde{N}^i_k=\frac{1}{2}\frac{\partial\tilde{G}^i}{\partial y^k},$$

where

$$\tilde{G}^i=\tilde{g}^{ih}\left(\frac{\partial^2 L}{\partial x^j\partial y^h}y^j-\frac{\partial L}{\partial x^h}\right).$$

A nonlinear connection determines a direct sum decomposition of the tangent bundle to $T\tilde{M}$

$$(4)\qquad TT\tilde{M}=VT\tilde{M}\oplus HT\tilde{M}.$$

A local frame in the vertical distribution $VT\tilde{M}$ is $(\partial/\partial y^1,\dots,\partial/\partial y^n)$ and a local frame in the horizontal distribution $HT\tilde{M}$ is $(\delta/\delta x^1,\dots,\delta/\delta x^n)$ where

$$(5)\qquad \frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\tilde{N}^k_i\frac{\partial}{\partial y^k}.$$

Remark that the transformation rule for the local frame $(\partial/\partial y^i,\ \delta/\delta x^i)$ adapted to the direct sum decomposition (4) is

$$(6)\qquad \frac{\partial}{\partial y^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\partial}{\partial\tilde{y}^k},\qquad \frac{\delta}{\delta x^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\delta}{\delta\tilde{x}^k}.$$

Scalar (inner) products, denoted by the same letter g, are defined on $VT\tilde{M}$ and $HT\tilde{M}$ by

$$g\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k}\right)=\tilde{g}_{ik},\qquad g\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\right)=\tilde{g}_{ik}.$$

The complete integrability of the differential system defined by $HT\tilde{M}$ over $T\tilde{M}$ is equivalent to the vanishing of the following tensor field

$$(7)\qquad \tilde{R}^h_{ik}=\frac{\delta\tilde{N}^h_k}{\delta x^i}-\frac{\delta\tilde{N}^h_i}{\delta x^k},$$

obtained from the computation of the brackets $[\delta/\delta x^i,\ \delta/\delta x^k]$. The local frame of the cotangent bundle to $T\tilde{M}$, dual to the local frame $(\partial/\partial y^i,\ \delta/\delta x^i)$, is $(\delta y^i,\ dx^i)$, where

$$(5')\qquad \delta y^i=dy^i+\tilde{N}^i_k dx^k.$$

Considering the 1-form

$$(8)\qquad \theta=\frac{\partial L}{\partial y^i}dx^i,$$

globally defined on $T\tilde{M}$ we obtain for its exterior differential

$$(9)\qquad d\theta=\tilde{g}_{ik}\delta y^i\wedge dx^k.$$

The following identities obtained from $d^2\theta=0$ are valid on a Lagrange manifold

$$(10)\qquad \tilde{g}_{ih}\tilde{R}^h_{jk}+\tilde{g}_{jh}\tilde{R}^h_{ki}+\tilde{g}_{kh}\tilde{R}^h_{ij}=0,$$

$$(11)\qquad \frac{\delta\tilde{g}_{ij}}{\delta x^k}-\frac{\delta\tilde{g}_{kj}}{\delta x^i}=\tilde{g}_{ih}\frac{\partial\tilde{N}^h_k}{\partial y^j}-\tilde{g}_{kh}\frac{\partial\tilde{N}^h_i}{\partial y^j}.$$

Next on $T\tilde{M}$ we may define a Riemannian metric $\tilde{G}$ of Sasaki type

$$(12)\qquad \tilde{G}=\tilde{g}_{ik}dx^i dx^k+\tilde{g}_{ik}\delta y^i\delta y^k.$$

It follows that the distributions $VT\tilde{M}$, $HT\tilde{M}$ are orthogonal to each othe with respect to $\tilde{G}$ and the scalar products on $VT\tilde{M}$, $HT\tilde{M}$ induced from $\tilde{G}$ by restriction do coincide with the scalar products g defined above.

In [6] the first author has obtained the local coordinate expressions of the Levi Civita connection $\tilde{D}$ of the metric $\tilde{G}$

$$(13)\qquad \tilde{D}'_i\frac{\partial}{\partial y^j}=\tilde{C}^k_{ij}\frac{\partial}{\partial y^k}+\tilde{b}^k_{ij}\frac{\delta}{\delta x^k},\qquad \tilde{D}_i\frac{\partial}{\partial y^j}=\tilde{\Phi}^k_{ij}\frac{\partial}{\partial y^k}+\tilde{D}^k_{ji}\frac{\delta}{\delta x^k},$$

$$(14)\qquad \tilde{D}'_i\frac{\delta}{\delta x^j}=\tilde{D}^k_{ij}\frac{\delta}{\delta x^k}-\tilde{P}^k_{ij}\frac{\partial}{\partial y^k},\qquad \tilde{D}_i\frac{\delta}{\delta x^j}=\tilde{F}^k_{ij}\frac{\delta}{\delta x^k}+\tilde{a}^k_{ij}\frac{\partial}{\partial y^k},$$

where $\tilde{D}'_i=\tilde{D}_{\partial/\partial y^i}$, $\tilde{D}_i=\tilde{D}_{\delta/\delta x^i}$ and

$$\tilde{C}^k_{ij}=\frac{1}{2}\tilde{g}^{kh}\frac{\partial\tilde{g}_{hj}}{\partial y^i},\qquad \tilde{\Phi}^k_{ij}=\tilde{F}^k_{ij}=\frac{1}{2}\tilde{g}^{kh}\left(\frac{\delta\tilde{g}_{hj}}{\delta x^i}+\frac{\delta\tilde{g}_{hi}}{\delta x^j}-\frac{\delta\tilde{g}_{ij}}{\delta x^h}\right),$$

$$(15)\qquad \tilde{b}^k_{ij}=\tilde{P}^k_{ij}=\frac{\partial\tilde{N}^k_j}{\partial y^i}-\tilde{\Phi}^k_{ij},\qquad \tilde{D}^k_{ij}=\tilde{C}^k_{ij}+\frac{1}{2}\tilde{g}^{kh}\tilde{R}_{ijh},$$

$$\tilde{R}_{ijh}=\tilde{g}_{ik}\tilde{R}^k_{jh},\qquad \tilde{a}^k_{ij}=-\left(\tilde{C}^k_{ij}+\frac{1}{2}\tilde{R}^k_{ij}\right).$$

Denote by $\tilde{\nabla}$ the linear connection induced from $\tilde{D}$ on the distributions $VT\tilde{M}$ and $HT\tilde{M}$ over $T\tilde{M}$

$$\tilde{\nabla}_i \frac{\partial}{\partial y^j} = \tilde{C}^k_{ij} \frac{\partial}{\partial y^k}, \quad \tilde{\nabla}_i \frac{\partial}{\partial y^j} = \tilde{F}^k_{ij} \frac{\partial}{\partial y^k}, \tag{16}$$

$$\tilde{\nabla}_i \frac{\delta}{\delta x^j} = \tilde{D}^k_{ij} \frac{\delta}{\delta x^k}, \quad \tilde{\nabla}_i \frac{\delta}{\delta x^j} = \tilde{F}^k_{ij} \frac{\delta}{\delta x^k}. \tag{17}$$

Next the local coordinate expressions of the curvature tensor fields of $\tilde{D}$ and $\tilde{\nabla}$ can be obtained. Further, the first author has obtained in [6] the relations between these tensor fields.

2. **Submanifolds in Lagrange spaces.** Since we study the local properties of the submanifolds in Lagrange spaces, we may assume that such submanifolds are given as geometric images of appropriate injective immersions. Let $\tilde{M}$ be a Lagrange manifold with the regular Lagrangian L and suppose that $f: M \to \tilde{M}$ is an injective immersion with local coordinate expression

$$x^i = x^i(u^\alpha), \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \tag{18}$$

The differential of f is a smooth mapping $Df: TM \to T\tilde{M}$, given in local coordinates by

$$x^i = x^i(u^\alpha), \qquad y^i = \frac{\partial x^i}{\partial u^\alpha} v^\alpha, \tag{19}$$

where (u^α, v^α) are local coordinates on TM induced from the local coordinates (u^α) on M. The images of the natural local frames of TM by Df are given by

$$D^2f\left(\frac{\partial}{\partial u^\alpha}\right) = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial u^\sigma \partial u^\alpha} v^\sigma \frac{\partial}{\partial y^i}, \quad D^2f\left(\frac{\partial}{\partial v^\alpha}\right) = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial y^i}. \tag{20}$$

Since D^2f is injective, we shall identify from now on, $D^2f(\partial/\partial u^\alpha)$, $D^2f(\partial/\partial v^\alpha)$ with $\partial/\partial u^\alpha$, $\partial/\partial v^\alpha$ respectively. Further we shall use the notations from the theory of submanifolds in Finsler spaces:

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{0\alpha} = \frac{\partial^2 x^i}{\partial u^\sigma \partial u^\alpha} v^\sigma, \ldots$$

Then

$$\frac{\partial}{\partial u^i} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{0\alpha} \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i}. \tag{20'}$$

Consider the restriction of L to (the image by f) of M. Since we suppose that the matrix $[g_{ik}]$ is positive, the matrix with the entries

$$g_{\alpha\beta} = \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta} = \tilde{g}_{ik} B^i_\alpha B^k_\beta \tag{21}$$

is positive too, hence nondegenerate. Thus the restriction of L to TM defines on M a Lagrange space structure. From the formulae (20) we see that the image of VTM by D^2f is a vector subbundle of the restriction of VTM to

$Df(TM)$. Denote by $VTM^\perp$ the orthogonal complement of VTM in $Df^*(VTM)$ with respect to the inner product g and let

$$X_a = B^i_a \frac{\partial}{\partial y^i}, \quad a = m+1, \ldots, n \tag{22}$$

be a local orthonormal frame in $VTM^\perp$. Then

$$\tilde{g}_{ik} B^i_a B^k_\alpha = 0, \quad \tilde{g}_{ik} B^i_a B^k_b = \delta_{ab}. \tag{23}$$

The change matrix from the local frame $(\partial/\partial y^i)$ of $Df^*(VTM)$ to the local frame $(\partial/\partial v^\alpha, X_a)$ is determined by laying by the blocks $[B^i_\alpha]$, $[B^i_a]$. Denote by A^α_i, A^a_i the entries of the inverse of this change matrix. Then

$$A^\alpha_k B^k_\beta = \delta^\alpha_\beta, \quad A^\alpha_k B^k_a = 0, \quad A^a_k B^k_\alpha = 0, \quad A^a_k B^k_b = \delta^a_b, \quad B^i_\alpha A^\alpha_k + B^i_a A^a_k = \delta^i_k. \tag{24}$$

Denoting by $\tilde{g}^{ik}$ the entries of the inverse of the matrix $[\tilde{g}_{ik}]$ and by $g^{\alpha\beta}$ the entries of the inverse of the matrix $[g_{\alpha\beta}]$ we get

$$g^{\alpha\beta} = \tilde{g}^{ik} A^\alpha_i A^\beta_k, \quad \delta^{ab} = \tilde{g}^{ik} A^a_i A^b_k, \quad 0 = \tilde{g}^{ik} A^\alpha_i A^a_k, \tag{25}$$

$$\tilde{g}_{ik} = g_{\alpha\beta} A^\alpha_i A^\beta_k + \delta_{ab} A^a_i A^b_k, \quad \tilde{g}^{ik} = g^{\alpha\beta} B^i_\alpha B^k_\beta + \delta^{ab} B^i_a B^k_b, \tag{26}$$

$$g_{\alpha\beta} A^\beta_k = \tilde{g}_{ki} B^i_\alpha, \quad \tilde{g}^{ki} A^\alpha_k = g^{\alpha\beta} B^i_\beta, \quad A^a_k = \tilde{g}_{ki} B^i_a, \quad B^i_a = \tilde{g}^{ik} A^a_k. \tag{27}$$

We shall now obtain the induced nonlinear connection on the submanifold M. Since the first formula (20'), we get by using the local frame $(\partial/\partial y^i, \delta/\delta x^i)$ on $T\tilde{M}$:

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i} + \left(B^i_{0\alpha} + B^k_\alpha \tilde{N}^i_k\right) \frac{\partial}{\partial y^i}.$$

On the other hand, in $Df^*(VTM)$ we may use the local frame $(\partial/\partial v^\alpha, X_a)$ and denote by N^β_α, N^a_α the corresponding coefficients

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i} + N^\beta_\alpha \frac{\partial}{\partial v^\beta} + N^a_\alpha X_a.$$

Using (20'), (22) we get

$$B^i_{0\alpha} + \tilde{N}^i_k B^k_\alpha = B^i_\sigma N^\sigma_\alpha + B^i_a N^a_\alpha. \tag{28}$$

Then

$$N^\beta_\alpha = A^\beta_i (B^i_{0\alpha} + \tilde{N}^i_k B^k_\alpha), \quad N^a_\alpha = A^a_i (B^i_{0\alpha} + \tilde{N}^i_k B^k_\alpha). \tag{28'}$$

It follows easily that N^β_α define a nonlinear connection on M. This is the nonlinear connection on M induced from the nonlinear connection defined by $\tilde{N}^i_k$ on M. Then the horizontal distribution HTM is defined by this induced nonlinear connection and in HTM we have local frame

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta} = B^i_\alpha \frac{\delta}{\delta x^i} + N^a_\alpha X_a. \tag{29}$$

Thus the image $D^2f(HTM)$ is no longer a subbundle in $HT\tilde{M}$.

Finally, denote by NTM the vector subbundle of $Df^*(T\widetilde{TM})$ defined as the orthogonal complement of $VTM\oplus HTM\oplus VTM^{\perp}$ with respect to the metric $\widetilde{G}$. Then it follows that in NTM we have the local frame (Y_a) where

$$Y_a = B_a^i \frac{\delta}{\delta x^i}. \tag{30}$$

Thus we get the following direct sum decomposition:

$$Df^*(T\widetilde{TM}) = VTM\oplus HTM\oplus VTM^{\perp}\oplus NTM$$

and $(\partial/\partial v^{\alpha},\ \delta/\delta u^{\alpha},\ X_a,\ Y_a)$ is a local frame adapted to this direct sum decomposition. Though VTM and NTM are orthogonal each other and both orthogonal to $HTM\oplus VTM^{\perp}$, the terms HTM, $VTM^{\perp}$ are not orthogonal each other with respect to $\widetilde{G}$. In fact we have

$$\widetilde{G}\left(\frac{\delta}{\delta u^{\alpha}},\ X_a\right) = N_{\alpha}^a. \tag{31}$$

From the formulae (20′), (29) we get

$$\frac{\partial}{\partial y^i} = A_i^{\alpha}\frac{\partial}{\partial v^{\alpha}} + A_i^a X_a,\qquad \frac{\delta}{\delta x^i} = A_i^{\alpha}\left(\frac{\delta}{\delta u^{\alpha}} - N_{\alpha}^a X_a\right) + A_i^a Y_a. \tag{32}$$

We shall introduce, for later use, the following notation

$$C_{\alpha}^i = B_a^i N_{\alpha}^a. \tag{33}$$

Then the formula (28) may be written in the form

$$B_{0\alpha}^i + \widetilde{N}_k^i B_{\alpha}^k = B_{\beta}^i N_{\alpha}^{\beta} + C_{\alpha}^i \tag{28″}$$

and we get easily the following relations

$$N_{\alpha}^a = A_i^a C_{\alpha}^i,\qquad A_i^{\beta} C_{\alpha}^i = 0. \tag{34}$$

The relation between the tensor field $R_{\alpha\beta}^{\gamma}$ and the tensor field $\widetilde{R}_{ij}^k$ defined by the complete integrability problem for HTM and $H\widetilde{TM}$ respectively is obtained as follows

Proposition 1. *We have*

$$\widetilde{R}_{ij}^k B_{\alpha}^i B_{\beta}^j - (B_{\alpha}^i C_{\beta}^j - C_{\alpha}^i B_{\beta}^j)\frac{\partial \widetilde{N}_i^k}{\partial y^j} = B_{\gamma}^k R_{\alpha\beta}^{\gamma} + \frac{\delta C_{\beta}^k}{\delta u^{\alpha}} - \frac{\delta C_{\alpha}^k}{\delta u^{\beta}}. \tag{35}$$

Proof. Use the definition of $R_{\alpha\beta}^{\gamma}$ and the formula (29)

$$-R_{\alpha\beta}^{\gamma}\frac{\partial}{\partial v^{\gamma}} = \left[\frac{\delta}{\delta u^{\alpha}},\ \frac{\delta}{\delta u^{\beta}}\right] = \left(\frac{\delta B_{\beta}^k}{\delta u^{\alpha}} - \frac{\delta B_{\alpha}^k}{\delta u^{\beta}}\right)\frac{\delta}{\delta x^k} + \left(\frac{\delta C_{\beta}^k}{\delta u^{\alpha}} - \frac{\delta C_{\alpha}^k}{\delta u^{\beta}}\right)\frac{\partial}{\partial y^k} +$$
$$+ B_{\alpha}^i B_{\beta}^j\left[\frac{\delta}{\delta x^i},\ \frac{\delta}{\delta x^j}\right] + (B_{\alpha}^i C_{\beta}^j - B_{\beta}^i C_{\alpha}^j)\left[\frac{\delta}{\delta x^i},\ \frac{\partial}{\partial y^j}\right].$$

Since B_{α}^k does not depend on v^{α}, the coefficient of $\delta/\delta x^k$ vanishes. Then use (20′) and

$$\left[\frac{\delta}{\delta x^i},\ \frac{\delta}{\delta x^j}\right] = -\widetilde{R}_{ij}^k\frac{\partial}{\partial y^k},\qquad \left[\frac{\delta}{\delta x^i},\ \frac{\partial}{\partial y^j}\right] = \frac{\partial \widetilde{N}_i^k}{\partial y^j}\frac{\partial}{\partial y^k},$$

to get (35).

Let us study the relation existing between the induced nonlinear connection and the intrinsic nonlinear connection on M defined by the restriction of L to M. Denote $\mathring{N}_{\beta}^{\alpha} = (1/2)\,\partial G^{\alpha}/\partial v^{\beta}$ where G^{α} is obtained from L by using the Euler Lagrange equations on TM.

Proposition 2. *Denote*

$$G^c = A_i^c(\widetilde{G}^i + B_{00}^i),\qquad \mathring{N}_{\alpha}^a = \frac{1}{2}\frac{\partial G^a}{\partial v^{\alpha}}. \tag{36}$$

Then

$$N_{\alpha}^{\beta} = \mathring{N}_{\alpha}^{\beta} + \frac{1}{2}A_k^{\beta}\frac{\partial B_c^k}{\partial v^{\alpha}}G^c,\qquad N_{\alpha}^a = \mathring{N}_{\alpha}^a + \frac{1}{2}A_k^a\frac{\partial B_c^k}{\partial v^{\alpha}}G^c. \tag{37}$$

Proof. After a straightforward but quite long computation we get

$$G^{\alpha} = A_i^{\alpha}(\widetilde{G}^i + B_{00}^i). \tag{38}$$

Thus

$$\widetilde{G}^i + B_{00}^i = B_{\beta}^i G^{\beta} + B_c^i G^c. \tag{39}$$

Next differentiate with respect to v^{α}

$$\widetilde{N}_k^i B_{\alpha}^k + B_{0\alpha}^i = B_{\beta}^i \mathring{N}_{\alpha}^{\beta} + B_c^i \mathring{N}_{\alpha}^c + \frac{1}{2}\frac{\partial B_c^i}{\partial v^{\alpha}}G^c.$$

Using (23), (24) and (28) we get (37).

Finally, we obtain some other relations satisfied by the induced connection by using the restriction of the 1-form θ to TM. Firstly remark the following formulae:

$$dx^i = B_{\alpha}^i du^{\alpha},\qquad \delta y^i = (B_{0\alpha}^i + \widetilde{N}_k^i B_{\alpha}^k)du^{\alpha} + B_{\alpha}^i dv^{\alpha} = B_{\alpha}^i \delta v^{\alpha} + C_{\alpha}^i du^{\alpha}.$$

Then differentiate $\theta = Df^*(\theta)$:

$$d\theta = \widetilde{g}_{ik}B_{\alpha}^i B_{\beta}^k \delta v^{\alpha}\wedge du^{\beta} + \widetilde{g}_{ik}B_a^i N_{\alpha}^a B_{\beta}^k du^{\alpha}\wedge du^{\beta} = g_{\alpha\beta}\delta v^{\alpha}\wedge du^{\beta}.$$

Using this expression of $d\theta$ we get the following identities similar to (10) and (11)

$$g_{\alpha\delta}R_{\beta\gamma}^{\delta} + g_{\beta\delta}R_{\gamma\alpha}^{\delta} + g_{\gamma\delta}R_{\alpha\beta}^{\delta} = 0, \tag{40}$$

$$\frac{\delta g_{\alpha\beta}}{\delta u^{\gamma}} - \frac{\delta g_{\alpha\gamma}}{\delta u^{\beta}} + g_{\gamma\delta}\frac{\partial N_{\beta}^{\delta}}{\partial v^{\alpha}} - g_{\beta\delta}\frac{\partial N_{\gamma}^{\delta}}{\partial v^{\alpha}} = 0. \tag{41}$$

Finally, differentiate in (28) with respect to v^{β} to get

$$\frac{\partial N_{\beta}^{\gamma}}{\partial v^{\alpha}} - \frac{\partial N_{\alpha}^{\gamma}}{\partial v^{\beta}} = A_i^{\gamma}\left(\frac{\partial B_c^i}{\partial v^{\beta}}N_{\alpha}^c - \frac{\partial B_c^i}{\partial v^{\alpha}}N_{\beta}^c\right) = \frac{\partial A_i^{\gamma}}{\partial v^{\alpha}}C_{\beta}^i - \frac{\partial A_i^{\gamma}}{\partial v^{\beta}}C_{\alpha}^i, \tag{42}$$

$$\frac{\partial N^a_\beta}{\partial v^\alpha}-\frac{\partial N^a_\alpha}{\partial v^\beta}=A^a_i\left(\frac{\partial B^i_c}{\partial v^\beta}N^c_\alpha-\frac{\partial B^i_c}{\partial v^\alpha}N^c_\beta\right)=\frac{\partial A^a_i}{\partial v^\alpha}C^i_\beta-\frac{\partial A^a_i}{\partial v^\beta}C^i_\alpha. \tag{43}$$

3. **The Gauss-Weingarten equations.** The tangent bundle TM of M is thought of as a submanifold in the tangent bundle $T\tilde{M}$ of $\tilde{M}$. However let us remark that the normal bundle $VTM^\perp\oplus NTM$ to TTM in $Df^*(TT\tilde{M})$ is not orthogonal to TTM with respect to $\tilde{G}$. In fact, according to (31), $VTM^\perp$ is not orthogonal to HTM. The Riemannian metric $G=(Df)^*\tilde{G}$ induced from $\tilde{G}$ by Df has the following local expression

$$G=g^*_{\alpha\beta}du^\alpha du^\beta+g_{\alpha\beta}\delta v^\alpha\delta v^\beta, \tag{44}$$

where

$$g^*_{\alpha\beta}=g_{\alpha\beta}+\delta_{ab}N^a_\alpha N^b_\beta\,, \tag{45}$$

Remark that G is no longer the Sasaki metric defined by the induced nonlinear connection and the scalar product on VTM corresponding to the Lagrange structure induced on M. However it is quite interesting to study the results arising from this situation. Obviously, these results are similar to the corresponding classical results from the theory of submanifolds in Riemannian space but we should point out some differences arising from the fact that the „normal" bundle $VTM^\perp\oplus NTM$ is not orthogonal to TTM.

Let X, Y be two vector fields tangent to TM i.e. $X, Y\in\Gamma(TTM)$. The vector field $\tilde{D}_XY$ has a component D_XY tangent to TM and a „normal" component $b(X, Y)$. Thus the Gauss equation is written as follows

$$\begin{aligned}&\tilde{D}_XY=D_XY+b(X,Y),\ X,\ Y\in\Gamma(TTM),\ D_XY\in\Gamma(TTM),\\&\qquad b(X,Y)\in\Gamma(VTM^\perp\oplus NTM).\end{aligned} \tag{46}$$

Just like in the classical case we may prove

Proposition 3. *D is a linear connection on TM and b is bilinear.*

Now we should express the local coordinate expressions of D and b. To do this we write

$$\begin{aligned}
\tilde{D}_{\hat\alpha}\frac{\partial}{\partial v^\beta}&=C^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma}+b^\gamma_{\alpha\beta}\frac{\delta}{\delta u^\gamma}+H^a_{\hat\alpha\hat\beta}X_a+K^a_{\hat\alpha\hat\beta}Y_a,\\
\tilde{D}_\alpha\frac{\partial}{\partial v^\beta}&=\Phi^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma}+\bar{D}^\gamma_{\alpha\beta}\frac{\delta}{\delta u^\gamma}+H^a_{\alpha\hat\beta}X_a+K^a_{\alpha\hat\beta}Y_a,\\
\tilde{D}_{\hat\alpha}\frac{\delta}{\delta u^\beta}&=D^\gamma_{\alpha\beta}\frac{\delta}{\delta u^\gamma}-P^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma}+H^a_{\hat\alpha\beta}X_a+K^a_{\hat\alpha\beta}Y_a,\\
\tilde{D}_\alpha\frac{\delta}{\delta u^\beta}&=F^\gamma_{\alpha\beta}\frac{\delta}{\delta u^\gamma}+a^\gamma_{\alpha\beta}\frac{\partial}{\partial v^\gamma}+H^a_{\alpha\beta}X_a+K^a_{\alpha\beta}Y_a.
\end{aligned} \tag{47}$$

The condition that D is torsion free implies

$$\begin{aligned}
&C^\gamma_{\alpha\beta}=C^\gamma_{\beta\alpha},\quad b^\gamma_{\alpha\beta}=b^\gamma_{\beta\alpha},\quad H^a_{\hat\alpha\hat\beta}=H^a_{\hat\beta\hat\alpha},\quad K^a_{\hat\alpha\hat\beta}=K^a_{\hat\beta\hat\alpha},\\
&F^\gamma_{\alpha\beta}=F^\gamma_{\beta\alpha},\quad a^\gamma_{\alpha\beta}=a^\gamma_{\beta\alpha}-R^\gamma_{\alpha\beta},\quad H^a_{\alpha\beta}=H^a_{\beta\alpha},\quad K^a_{\alpha\beta}=K^a_{\beta\alpha}\,,\\
&P^\gamma_{\alpha\beta}=\frac{\partial N^\gamma_\beta}{\partial v^\alpha}-\Phi^\gamma_{\beta\alpha},\quad \bar{D}^\gamma_{\alpha\beta}=D^\gamma_{\beta\alpha},\quad H^a_{\hat\alpha\beta}=H^a_{\beta\hat\alpha}\,,\quad K^a_{\hat\alpha\beta}=K^a_{\beta\hat\alpha}\,.
\end{aligned} \tag{48}$$

The condition that $\tilde{G}$ is preserved by $\tilde{D}$ gives us

$$\begin{aligned}
&C^\sigma_{\alpha\beta}g_{\sigma\gamma}+C^\sigma_{\alpha\gamma}g_{\beta\sigma}=\frac{\partial g_{\beta\gamma}}{\partial v^\alpha}\,,\quad \Phi^\sigma_{\alpha\beta}g_{\sigma\gamma}+\Phi^\sigma_{\alpha\gamma}g_{\beta\sigma}=\frac{\delta g_{\beta\gamma}}{\delta u^\alpha}\,,\\
&g^*_{\sigma\gamma}D^\sigma_{\alpha\beta}+g^*_{\beta\sigma}D^\sigma_{\alpha\gamma}+\delta_{ab}(H^a_{\hat\alpha\beta}N^b_\gamma+H^a_{\hat\alpha\gamma}N^b_\beta)=\frac{\partial g^*_{\beta\gamma}}{\partial v^\alpha}\,;\\
&g^*_{\sigma\gamma}F^\sigma_{\alpha\beta}+g^*_{\beta\sigma}F^\sigma_{\alpha\gamma}+\delta_{ab}(H^a_{\alpha\beta}N^b_\gamma+H^a_{\alpha\gamma}N^b_\beta)=\frac{\delta g^*_{\beta\gamma}}{\delta u^\alpha}\,,\\
&P^\sigma_{\alpha\gamma}g_{\sigma\beta}=g^*_{\sigma\gamma}b^\sigma_{\alpha\beta}+\delta_{ab}H^a_{\hat\alpha\hat\beta}N^b_\gamma,\quad -a^\sigma_{\alpha\gamma}g_{\beta\sigma}=g^*_{\sigma\gamma}D^\sigma_{\beta\alpha}+\delta_{ab}H^a_{\alpha\hat\beta}N^b_\gamma.
\end{aligned} \tag{49}$$

The explicite expresions of D and b are given as follows

$$\begin{aligned}
&C^\gamma_{\alpha\beta}=A^\gamma_k\tilde{C}^k_{ij}B^i_\alpha B^j_\beta,\quad b^\gamma_{\alpha\beta}=A^\gamma_k\tilde{b}^k_{ij}B^i_\alpha B^j_\beta,\quad H^a_{\hat\alpha\hat\beta}=A^a_k\tilde{C}^k_{ij}B^i_\alpha B^j_\beta-N^a_\sigma b^\sigma_{\alpha\beta},\\
&K^a_{\hat\alpha\hat\beta}=A^a_k\tilde{b}^k_{ij}B^i_\alpha B^j_\beta\,,\quad \Phi^\gamma_{\alpha\beta}=A^\gamma_k(B^k_{\alpha\beta}+\tilde{F}^k_{ij}B^i_\alpha B^j_\beta+\tilde{C}^k_{ij}C^i_\alpha B^j_\beta),\\
&D^\gamma_{\alpha\beta}=A^\gamma_k(\tilde{D}^k_{ij}B^i_\beta B^j_\beta+\tilde{b}^k_{ij}B^i_\alpha C^j_\beta),\quad H^a_{\alpha\hat\beta}=A^a_k(B^k_{\alpha\beta}+\tilde{F}^k_{ij}B^i_\alpha B^j_\beta+\\
&\qquad+\tilde{C}^k_{ij}C^i_\alpha B^j_\beta)-N^a_\sigma D^\sigma_{\beta\alpha}\,,\quad K^a_{\alpha\hat\beta}=A^a_k(\tilde{D}^k_{ij}B^i_\alpha B^j_\beta+\tilde{b}^k_{ij}C^i_\alpha B^j_\beta),\\
&P^\gamma_{\alpha\beta}=A^\gamma_k\left(\tilde{P}^k_{ij}B^i_\alpha B^j_\beta-\frac{\partial C^k_\beta}{\partial v^\alpha}-\tilde{C}^k_{ij}B^i_\alpha C^j_\beta\right),\quad F^\gamma_{\alpha\beta}=A^\gamma_k(B^k_{\alpha\beta}+\tilde{F}^k_{ij}B^i_\alpha B^j_\beta+\\
&\qquad+\tilde{D}^k_{ij}C^i_\alpha B^j_\beta+\tilde{D}^k_{ij}C^i_\beta B^j_\alpha+\tilde{b}^k_{ij}C^i_\alpha C^j_\beta),\\
&a^\gamma_{\alpha\beta}=A^\gamma_k\left(\frac{\delta C^k_\beta}{\delta u^\alpha}+\tilde{a}^k_{ij}B^i_\alpha B^j_\beta-\tilde{P}^k_{ij}C^i_\alpha B^j_\beta+\tilde{F}^k_{ij}B^i_\alpha C^j_\beta+\tilde{C}^k_{ij}C^i_\alpha C^j_\beta\right),\\
&H^a_{\alpha\beta}=A^a_k\left(\frac{\delta C^k_\beta}{\delta u^\alpha}+\tilde{a}^k_{ij}B^i_\alpha B^j_\beta-\tilde{P}^k_{ij}C^i_\alpha B^j_\beta+\tilde{F}^k_{ij}B^i_\alpha C^j_\beta+\tilde{C}^k_{ij}C^i_\alpha C^j_\beta\right)-N^a_\sigma F^\sigma_{\alpha\beta},\\
&K^a_{\alpha\beta}=A^a_k(B^k_{\alpha\beta}+\tilde{F}^k_{ij}B^i_\alpha B^j_\beta+\tilde{D}^k_{ij}C^i_\alpha B^j_\beta+\tilde{D}^k_{ij}C^i_\beta B^j_\alpha+\tilde{b}^k_{ij}C^i_\alpha C^j_\beta).
\end{aligned} \tag{50}$$

The expressions of $C^\gamma_{\alpha\beta}$, $\Phi^\gamma_{\alpha\beta}$ as determined by the induced metric on TM can be obtained if we use the following result

Proposition 4. *The following formula is valid on TTM*

$$T^\gamma_{\alpha\beta}=\Phi^\gamma_{\alpha\beta}-\Phi^\gamma_{\beta\alpha}=-\frac{1}{2}\left(\frac{\partial N^\gamma_\beta}{\partial v^\alpha}-\frac{\partial N^\gamma_\alpha}{\partial v^\beta}\right). \tag{51}$$

Proof. Differentiate with respect to v^β in (28'') to obtain

$$B^i_\sigma\left(\frac{\partial N^\sigma_\beta}{\partial v^\alpha}-\frac{\partial N^\sigma_\alpha}{\partial v^\beta}\right)+\frac{\partial C^i_\beta}{\partial v^\alpha}-\frac{\partial C^i_\alpha}{\partial v^\beta}=0.$$

Next consider

$$T^{\gamma}_{\alpha\beta}=A^{\gamma}_{k}\tilde{C}^{k}_{ij}(C^{i}_{\alpha}B^{j}_{\beta}-B^{i}_{\alpha}C^{j}_{\beta})=\frac{1}{2}A^{\gamma}_{k}\tilde{g}^{kh}\frac{\partial g_{jh}}{\partial y^{i}}(C^{i}_{\alpha}B^{j}_{\beta}-B^{i}_{\alpha}C^{j}_{\beta}).$$

and use the formulae (33), (23)–(27) to get

$$T^{\gamma}_{\alpha\beta}=\frac{1}{2}\left(-\frac{\partial A^{\gamma}_{k}}{\partial v^{\alpha}}C^{k}_{\beta}+\frac{\partial A^{\gamma}_{k}}{\partial v^{\beta}}C^{k}_{\alpha}\right)=-\frac{1}{2}\left(\frac{\partial N^{\gamma}_{\beta}}{\partial v^{\alpha}}-\frac{\partial N^{\gamma}_{\alpha}}{\partial v^{\beta}}\right).$$

Thus the torsion of $\Phi^{\gamma}_{\alpha\beta}$ is completely determined by the induced non-linear connection. Next we obtain just like in the classical case;

$$C^{\gamma}_{\alpha\beta}=\frac{1}{2}g^{\gamma\sigma}\frac{\partial g_{\sigma\beta}}{\partial v^{\alpha}};\quad \Phi^{\gamma}_{\alpha\beta}=\frac{1}{2}g^{\gamma\sigma}\left(\frac{\delta g_{\sigma\beta}}{\delta u^{\alpha}}+\frac{\delta g_{\alpha\sigma}}{\delta u^{\beta}}-\frac{\delta g_{\alpha\beta}}{\delta u^{\sigma}}\right)+ \tag{52}$$
$$+\frac{1}{2}(T^{\gamma}_{\alpha\beta}-g^{\gamma\tau}g_{\beta\sigma}T^{\sigma}_{\alpha\tau}+g^{\gamma\tau}g_{\alpha\sigma}T^{\sigma}_{\tau\beta}).$$

The obtaining of $D^{\gamma}_{\alpha\beta}$, $F^{\gamma}_{\alpha\beta}$ is much more difficult.

Now let $X\in\Gamma(TTM)$ be a vector field tangent to TM and let $Y\in\Gamma(VTM^{\perp}\oplus NTM)$ be a „normal" section to TM. The Weingarten formula is obtained by decomposing $\tilde{D}_{X}Y$ in the tangent part and the „normal" part:

$$\tilde{D}_{X}Y=D^{\perp}_{X}Y-A_{Y}X;\quad D^{\perp}_{X}Y\in\Gamma(VTM^{\perp}\oplus NTM),\quad A_{Y}X\in\Gamma(TTM). \tag{53}$$

Proposition 5. *$D^{\perp}$ is a linear connection and A is bilinear.* The local coordinate expressions are as follows:

$$\begin{aligned}
&\tilde{D}_{\hat{\alpha}}X_{a}=\Gamma^{\hat{b}}_{\hat{\alpha}\hat{a}}X_{b}+\Gamma^{b}_{\hat{\alpha}\hat{a}}Y_{b}-A^{\hat{\beta}}_{\hat{a}\hat{\alpha}}\frac{\partial}{\partial v^{\beta}}-A^{\beta}_{\hat{a}\hat{\alpha}}\frac{\delta}{\delta u^{\beta}},\\
&\tilde{D}_{\alpha}X_{a}=\Gamma^{\hat{b}}_{\alpha\hat{a}}X_{b}+\Gamma^{b}_{\alpha\hat{a}}Y_{b}-A^{\hat{\beta}}_{\hat{a}\alpha}\frac{\partial}{\partial v^{\beta}}-A^{\beta}_{\hat{a}\alpha}\frac{\delta}{\delta u^{\beta}},\\
&\tilde{D}_{\hat{\alpha}}Y_{a}=\Gamma^{\hat{b}}_{\hat{\alpha}a}X_{b}+\Gamma^{b}_{\hat{\alpha}a}Y_{b}-A^{\hat{\beta}}_{a\hat{\alpha}}\frac{\partial}{\partial v^{\beta}}-A^{\beta}_{a\hat{\alpha}}\frac{\delta}{\delta u^{\beta}},\\
&D_{\alpha}Y_{a}=\Gamma^{\hat{b}}_{\alpha a}X_{b}+\Gamma^{b}_{\alpha a}Y_{b}-A^{\hat{\beta}}_{a\alpha}\frac{\partial}{\partial v^{\beta}}-A^{\beta}_{a\alpha}\frac{\delta}{\delta u^{\beta}},
\end{aligned} \tag{54}$$

where:

$$A^{\hat{\beta}}_{\hat{a}\hat{\alpha}}=-A^{\beta}_{k}\left(\frac{\partial B^{k}_{a}}{\partial v^{\alpha}}+\tilde{C}^{k}_{ij}B^{i}_{a}B^{j}_{\alpha}\right),\quad A^{\beta}_{\hat{a}\hat{\alpha}}=-A^{\beta}_{k}\tilde{b}^{k}_{ij}B^{i}_{a}B^{j}_{\alpha},$$

$$\Gamma^{\hat{b}}_{\hat{\alpha}\hat{a}}=A^{b}_{k}\left(\frac{\partial B^{k}_{a}}{\partial v^{\alpha}}+\tilde{C}^{k}_{ij}B^{i}_{a}B^{j}_{\alpha}\right)+N^{b}_{\delta}A^{\delta}_{\hat{a}\hat{\alpha}},\quad \Gamma^{b}_{\hat{\alpha}\hat{a}}=A\tilde{b}^{k}_{ij}B^{i}_{a}B^{j}_{\alpha},$$

$$A^{\hat{\beta}}_{\hat{a}\alpha}=-A^{\beta}_{k}\left(\frac{\delta B^{k}_{a}}{\delta u^{\alpha}}+\tilde{F}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}+\tilde{C}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}\right),\quad A^{\beta}_{\hat{a}\alpha}=-A^{\beta}_{k}(\tilde{D}^{k}_{ji}B^{i}_{\alpha}B^{j}_{a}+$$
$$+\tilde{b}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}),\quad \Gamma^{\hat{b}}_{\alpha\hat{a}}=A^{b}_{k}\left(\frac{\delta B^{k}_{a}}{\delta u^{\alpha}}+\tilde{F}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}+\tilde{C}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}\right)+N^{b}_{\sigma}A^{\sigma}_{\hat{a}\alpha},$$

$$\begin{aligned}
&\Gamma^{b}_{\alpha\hat{a}}=A^{b}_{k}(\tilde{D}^{k}_{ji}B^{i}_{\alpha}B^{j}_{a}+\tilde{b}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}),\qquad A^{\hat{\beta}}_{a\hat{\alpha}}=A^{\beta}_{k}\tilde{P}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a},\\
&A^{\beta}_{a\hat{\alpha}}=-A^{\beta}_{k}\left(\frac{\partial B^{k}_{a}}{\partial v^{\alpha}}+\tilde{D}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}\right),\quad \Gamma^{\hat{b}}_{\hat{\alpha}a}=-A^{b}_{k}\tilde{P}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}+N^{b}_{\sigma}A^{\sigma}_{a\hat{\alpha}},\\
&\Gamma^{b}_{\hat{\alpha}a}=A^{b}_{k}\left(\frac{\partial B^{k}_{a}}{\partial v^{\alpha}}+\tilde{D}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}\right),\quad A^{\hat{\beta}}_{a\alpha}=-A^{\beta}_{k}(\tilde{a}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}-\tilde{P}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}),\\
&A^{\beta}_{a\alpha}=-A^{\beta}_{k}\left(\frac{\delta B^{k}_{a}}{\delta u^{\alpha}}+\tilde{F}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}+\tilde{D}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}\right),\quad \Gamma^{\hat{b}}_{\alpha a}=A^{b}_{k}(\tilde{a}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}-\\
&-\tilde{P}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a})+N^{b}_{\sigma}A^{\sigma}_{a\alpha},\quad \Gamma^{b}_{\alpha a}=A^{b}_{k}\left(\frac{\delta B^{k}_{a}}{\delta u^{\alpha}}+\tilde{F}^{k}_{ij}B^{i}_{\alpha}B^{j}_{a}+\tilde{D}^{k}_{ij}C^{i}_{\alpha}B^{j}_{a}\right).
\end{aligned} \tag{55}$$

From the orthogonality conditions satisfied by the local frame $\partial/\partial v^{\alpha}$, $\delta/\delta u^{\alpha}$, X_{a}, Y_{a}) we get some relations satisfied by the coefficients. Since the condition $\tilde{G}(\partial/\partial v^{\beta}, X_{a})=0$ one obtains:

$$H^{a}_{\hat{\alpha}\hat{\beta}}=g_{\beta\gamma}A^{\hat{\gamma}}_{\hat{a}\alpha}-N^{a}_{\sigma}b^{\sigma}_{\alpha\beta},\qquad H^{a}_{\alpha\hat{\beta}}=g_{\beta\sigma}A^{\hat{\sigma}}_{\hat{a}\alpha}-N^{a}_{\sigma}D^{\sigma}_{\beta\alpha}. \tag{56}$$

Next, since the condition $\tilde{G}\left(\frac{\partial}{\partial v^{\beta}}, Y_{a}\right)=0$, we get

$$K^{a}_{\hat{\alpha}\hat{\beta}}=g_{\beta\sigma}A^{\hat{\sigma}}_{a\hat{\alpha}},\quad K^{a}_{\alpha\hat{\beta}}=g_{\beta\sigma}A^{\hat{\sigma}}_{a\alpha}. \tag{57}$$

The following relations involve the derivatives of N^{a}_{β}, since $\tilde{G}\left(\frac{\delta}{\delta u^{\beta}}, X_{a}\right)=N^{a}_{\beta}$.

$$H^{a}_{\hat{\alpha}\beta}=\frac{\partial N^{a}_{\beta}}{\partial v^{\alpha}}+g^{*}_{\beta\sigma}A^{\sigma}_{\hat{a}\hat{\alpha}}-N^{a}_{\sigma}D^{\sigma}_{\alpha\beta}-\sum_{b=m+1}^{n}N^{b}_{\beta}\Gamma^{\hat{b}}_{\hat{\alpha}\hat{a}},\quad H^{a}_{\alpha\beta}=\frac{\delta N^{a}_{\beta}}{\delta u^{\alpha}}+ \tag{58}$$
$$+g^{*}_{\beta\sigma}A^{\sigma}_{\hat{a}\alpha}-N^{a}_{\sigma}F^{\sigma}_{\alpha\beta}-\sum_{b=m+1}^{n}N^{b}_{\beta}\Gamma^{\hat{b}}_{\alpha\hat{a}}.$$

Next, since the condition $\tilde{G}\left(\frac{\delta}{\delta u^{\beta}}, Y_{a}\right)=0$, we get

$$K^{a}_{\hat{\alpha}\beta}=g^{*}_{\beta\sigma}A^{\sigma}_{a\hat{\alpha}}-\sum_{b=m+1}^{n}N^{b}_{\beta}\Gamma^{\hat{b}}_{\hat{\alpha}a},\quad K^{a}_{\alpha\beta}=g^{*}_{\beta\sigma}A^{\sigma}_{a\alpha}-\sum_{b=m+1}^{n}N^{b}_{\beta}\Gamma^{\hat{b}}_{\alpha a}. \tag{59}$$

The last conditions $\tilde{G}(X_{a}, Y_{b})=0$, $\tilde{G}(X_{a}, X_{b})=\delta_{ab}$, $\tilde{G}(Y_{a}, Y_{b})=\delta_{ab}$ imply:

$$\Gamma^{b}_{\hat{\alpha}\hat{a}}+\Gamma^{\hat{a}}_{\hat{\alpha}b}=N^{a}_{\sigma}A^{\sigma}_{b\hat{\alpha}},\qquad \Gamma^{b}_{\alpha\hat{a}}+\Gamma^{\hat{a}}_{\alpha b}=N^{a}_{\sigma}A^{\sigma}_{b\alpha}, \tag{60}$$

$$\Gamma^{\hat{b}}_{\hat{\alpha}\hat{a}}+\Gamma^{\hat{a}}_{\hat{\alpha}\hat{b}}=N^{b}_{\sigma}A^{\sigma}_{\hat{a}\hat{\alpha}}+N^{a}_{\sigma}A^{\sigma}_{\hat{b}\hat{\alpha}},\qquad \Gamma^{\hat{b}}_{\alpha\hat{a}}+\Gamma^{\hat{a}}_{\alpha\hat{b}}=N^{b}_{\sigma}A^{\sigma}_{\hat{a}\alpha}+N^{a}_{\sigma}A^{\sigma}_{\hat{b}\alpha}, \tag{61}$$

$$\Gamma^{b}_{\hat{\alpha}a}+\Gamma^{a}_{\hat{\alpha}b}=0,\qquad \Gamma^{b}_{\alpha a}+\Gamma^{a}_{\alpha b}=0. \tag{62}$$

4. The Gauss-Codazzi structure equations. Denote by $\tilde{K}$ the curvature tensor field of the connection $\tilde{D}$ and by K the curvature tensor field of

D. Denote also: $(\bar{\nabla}_X b)(Y, Z) = D^{\perp}_X(b(Y, Z)) - b(D_X Y, Z) - b(Y, D_X Z)$; $X, Y, Z \in \Gamma(TTM)$, the mixed covariant derivative of the $(VTM^{\perp} \oplus NTM)$-valued bilinear mapping b. If we use the definition of $\tilde{K}$:

$$\tilde{K}(X, Y)Z = \tilde{D}_X \tilde{D}_Y Z - \tilde{D}_Y \tilde{D}_X Z - \tilde{D}_{[X, Y]} Z; \quad X, Y, Z \in \Gamma(TTM),$$

and the Gauss-Weingarten equations, we get the Gauss-Codazzi structure equations:

$$\tilde{K}(X, Y)Z = [K(X, Y)Z + A_{b(X,Z)}Y - A_{b(Y,Z)}X] + [(\bar{\nabla}_X b)(Y, Z) - (\bar{\nabla}_Y b)(X, Z)] \tag{63}$$

for $X, Y, Z \in \Gamma(TTM)$. We should remark that the first bracket in (63) is the tangent part of $\tilde{K}(X, Y)Z$ while the second bracket is the „normal" part. Thus, by denoting $\tilde{K}(X, Y)Z = \tilde{K}^t(X, Y)Z + \tilde{K}^n(X, Y)Z$, we get

$$(63') \quad \begin{aligned} &K(X, Y)Z = \tilde{K}^t(X, Y)Z - A_{b(X,Z)}Y + A_{b(Y,Z)}X; \\ &(\bar{\nabla}_X b)(Y, Z) - (\bar{\nabla}_Y b)(X, Z) = \tilde{K}^n(X, Y)Z; \quad X, Y, Z \in \Gamma(TTM). \end{aligned}$$

For $W \in \Gamma(TTM)$ we may use the metric $\tilde{G}$ to get a relation similar to the classical Gauss structure equation:

$$(64) \quad \begin{aligned} \tilde{G}(K^t(X, Y)Z, W) = {} & \tilde{G}(K(X, Y)Z, W) + \tilde{G}(b(X, Z), b(Y, W)) - \\ & - \tilde{G}(b(Y, Z), b(X, W)) - (\bar{\nabla}_Y \tilde{G})(W, b(X, Z)) + (\bar{\nabla}_X \tilde{G})(W, b(Y, Z)), \end{aligned}$$

where $(\bar{\nabla}_X G)(W, b(Y, Z)) = X(\tilde{G}(W, b(Y, Z)) - \tilde{G}(D_X W, b(Y, Z)) - \tilde{G}(W, D^{\perp}_X b(Y, Z))$.

Now we can get easily the local coordinate expressions corresponding to the Gauss-Codazzi structure equations. Denote

$$K\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right)\frac{\partial}{\partial v^\gamma} = K^{\hat{\tau}}_{\gamma\alpha\beta}\frac{\partial}{\partial v^\tau} + K^{\tau}_{\gamma\alpha\beta}\frac{\delta}{\delta u^\tau},$$

and so on and use similar notations for $\tilde{K}$. Then the first few local coordinate expressions of the Gauss-Codazzi equations are given by:

$$K^{\hat{\tau}}_{\gamma\alpha\beta} = A^{\tau}_h \tilde{K}^{\hat{h}}_{kij} B^i_\alpha B^j_\beta B^k_\gamma + A^{\hat{\tau}}_{a\alpha} H^a_{\beta\gamma} - A^{\hat{\tau}}_{a\beta} H^a_{\alpha\gamma} + A^{\hat{\tau}}_{a\alpha} K^a_{\beta\gamma} - A^{\hat{\tau}}_{a\beta} K^a_{\alpha\gamma},$$

$$K^{\tau}_{\gamma\alpha\beta} = A^{\tau}_h \tilde{K}^h_{kij} B^i_\alpha B^j_\beta B^k_\gamma + A^{\tau}_{a\alpha} H^a_{\beta\gamma} - A^{\tau}_{a\beta} H^a_{\alpha\gamma} + A^{\tau}_{a\alpha} K^a_{\beta\gamma} - A^{\tau}_{a\beta} K^a_{\alpha\gamma},$$

$$K^{\hat{\tau}}_{\gamma\alpha\beta} = A^{\tau}_h \tilde{K}^h_{kij} B^i_\alpha B^j_\beta B^k_\gamma + A^{\tau}_h \tilde{K}^h_{kij} B^i_\alpha B^j_\beta C^k_\gamma + A^{\hat{\tau}}_{a\alpha} H^a_{\beta\gamma} - A^{\hat{\tau}}_{a\beta} H^a_{\alpha\gamma} + A^{\hat{\tau}}_{a\alpha} K^a_{\beta\gamma} - A^{\hat{\tau}}_{a\beta} K^a_{\alpha\gamma},$$

$$K^{\tau}_{\gamma\alpha\beta} = A^{\tau}_h (\tilde{K}^h_{kij} B^k_\gamma + \tilde{K}^h_{kij} C^k_\gamma) B^i_\alpha B^j_\beta + A^{\tau}_{a\alpha} H^a_{\beta\gamma} - A^{\tau}_{a\beta} H^a_{\alpha\gamma} + A^{\tau}_{a\alpha} K^a_{\beta\gamma} - A^{\tau}_{a\beta} K^a_{\alpha\gamma}.$$

Next, from the second relation (63'), we get:

$$K^{\hat{a}}_{\gamma\alpha\beta} = \frac{\partial H^a_{\beta\gamma}}{\partial v^\alpha} - \frac{\partial H^a_{\alpha\gamma}}{\partial v^\beta} + \Gamma^a_{\alpha b} H^b_{\beta\gamma} - \Gamma^a_{\beta b} H^b_{\alpha\gamma} + \Gamma^{\hat{a}}_{\alpha b} K^b_{\beta\gamma} - \Gamma^a_{\beta b} K^b_{\alpha\gamma} + H^a_{\tau\alpha} C^{\tau}_{\beta\gamma} - $$
$$- H^a_{\tau\beta} C^{\tau}_{\alpha\gamma} + H^a_{\tau\alpha} b^{\tau}_{\beta\gamma} - H^a_{\tau\beta} b^{\tau}_{\alpha\gamma}.$$

$$K^a_{\gamma\alpha\beta} = \frac{\partial K^a_{\beta\gamma}}{\partial v^\alpha} - \frac{\partial K^a_{\alpha\gamma}}{\partial v^\beta} + \Gamma^a_{\alpha b} H^b_{\beta\gamma} - \Gamma^a_{\beta b} H^b_{\alpha\gamma} + \Gamma^a_{\alpha b} K^b_{\beta\gamma} - \Gamma^a_{\beta b} K^b_{\alpha\gamma} + K^a_{\tau\alpha} C^{\tau}_{\beta\gamma} - $$
$$- K^a_{\tau\beta} C^{\tau}_{\alpha\gamma} + K^a_{\tau\alpha} b^{\tau}_{\beta\gamma} - K^a_{\tau\beta} b^{\tau}_{\alpha\gamma}.$$

The other local coordinate expressions are obtained similarly.

Next, we get the Ricci structure equation by expressing the curvature tensor field induced in $VTM^{\perp} \oplus NTM$ from the curvature tensor field of $\tilde{D}$.

$$(65) \quad \begin{aligned} \tilde{K}(X, Y)Z = {} & [D_Y A_Z X - D_X A_Z Y + A_{D^{\perp}_X} Y - A_{D^{\perp}_Y} X + A_Z([X, Y]) + \\ & + [D^{\perp}_X D^{\perp}_Y Z - D^{\perp}_Y D^{\perp}_X Z - D^{\perp}_{[X,Y]} Z + b(Y, A_Z X) - b(X, A_Z Y)]; \end{aligned}$$

$X, Y \in \Gamma(TTM)$, $Z \in \Gamma(VTM^{\perp} \oplus NTM)$, where the first bracket is the tangent part of $K(X, Y)Z$, while the second bracket is the „normal" part. Denote by $K^{\perp}$ the curvature tensor field of the „normal" linear connection $D^{\perp}$. Then (65) implies:

$$(66) \quad \begin{aligned} &K^{\perp}(X, Y)Z = \tilde{K}^n(X, Y)Z + b(X, A_Z Y) - b(Y, A_Z X) \quad X, Y \in \Gamma(TTM), \\ &Z \in \Gamma(VTM^{\perp} \oplus NTM). \end{aligned}$$

For $W \in \Gamma(VTM^{\perp} \oplus NTM)$ we may use the metric $\tilde{G}$ to get a relation similar to the classical Ricci structure equation:

$$(67) \quad \begin{aligned} &\tilde{G}(\tilde{K}^n(X, Y)Z, W) = \tilde{G}(K^{\perp}(X, Y)Z, W) + \tilde{G}(A_Z X, A_W Y) - \\ &- \tilde{G}(A_Z Y, A_W X) + (\bar{\nabla}_Y \tilde{G})(A_Z X, W) - (\bar{\nabla}_X \tilde{G})(A_Z Y, W); \\ &X, Y \in \Gamma(TTM), \ Z \in \Gamma(VTM^{\perp} \oplus NTM). \end{aligned}$$

Finally, from the relation (66), we can get easily the local coordinate expressions of the „normal" curvature tensor field $K^{\perp}$. Denote

$$K^{\perp}\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right) X_a = K^{\perp\hat{b}}_{a\alpha\beta} X_b + K^{\perp b}_{a\alpha\beta} Y_b, \quad \text{and so on.}$$

Then, by using (66), we get

$$K^{\perp\hat{b}}_{a\alpha\beta} = A^b_h \tilde{K}^{\hat{h}}_{kij} B^i_\alpha B^j_\beta B^k_a - N^b_\sigma A^\sigma_h \tilde{K}^h_{kij} B^i_\alpha B^j_\beta B^k_a + A^{\hat{\gamma}}_{a\beta} H^b_{\alpha\gamma} + A^{\gamma}_{\alpha\beta} H^b_{\alpha\gamma} - A^{\hat{\gamma}}_{a\alpha} H^b_{\beta\gamma} - A^{\gamma}_{a\alpha} H^b_{\beta\gamma},$$

$$K^{\perp b}_{a\alpha\gamma} = A^b_h \tilde{K}^h_{kij} B^i_\alpha B^j_\beta B^k_a + A^{\hat{\gamma}}_{a\beta} K^b_{\alpha\gamma} + A^{\gamma}_{a\beta} K^b_{\alpha\gamma} - A^{\hat{\gamma}}_{a\alpha} K^b_{\beta\gamma} - A^{\gamma}_{a\alpha} K^b_{\beta\gamma}.$$

The other local coordinate expressions of $K^{\perp}$ are obtained similarly.

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ALMOST SEMIQUATERNION STRUCTURES EXISTENCE AND CONNECTIONS

BY

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An almost semiquaternion structure is a degenerate, hipercomplex structure defined by the semiquaternion algebra [3]. The study of such a structure is similar to that of the quaternion and antiquaternion structure [1, 16, 17, 18, ...].

In the present paper we deal with two problems: examples of the semiquaternion structures and linear connections.

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§1. **Definitions.** *Let M be a differentiable manifold,* dim $M=4n$ *(the necessity of this assumation will be seen leter in the paper) and $\tau^p_q(M)$ the module of tensor fields of type (p, q) on M.*

Definition 1.1. *The structure $SQ=(F_1, F_2, F_3)$, $F_i\in\tau^1_1(M)$, $i=\overline{1, 3}$, satisfying:*

$$(1.1)\qquad F_1^2=-I \quad , \quad F_2^2=F_3^2=0 \quad , \quad \text{rank } F_2=2n$$

$$F_1F_2=-F_2F_1=F_3\,;\quad F_1F_3=-F_3F_1=-F_2\,;\quad F_2F_3=F_3F_2=0$$

is called an almost semiquaternion structure, shorthly (SQ)-structure on M.

Let us consider structures defined by two linear complex tensor fields $J_1, J_2\in\tau^1_1(M)$, satisfying:

$$(1.2)\qquad J_1\,.\,J_2+J_2\,.\,J_1=\alpha I \qquad (\alpha\in R,\ I\in\tau^1_1(M)\ \text{identity}).$$

Such structures appeared firstly in H. W a k a k u w a [15] and a study of them was made by F. T r i c e r r i in [13], [14].

If $\alpha^2<4$ we have almost quaternion structures, if $\alpha^2>4$ we have almost antiquaternion structures. Finally, the case $\alpha^2=4$ is strongly related to the almost semiquatern'on structures. However, the later case is not studied by Tricerri. Hence, once again, we see the importance of the present study.

An (SQ)-structure is a hipercomplex structure defined by the semiquaternion algebra ([3], p. 325). In fact, using left regular representation of semiquaternion algebra in a $4n$-dimensional space we obtain [10]:

$$S_0=I_{4n},\ S_1=\begin{pmatrix}0 & -I & 0 & 0\\ I & 0 & 0 & 0\\ 0 & 0 & 0 & -I\\ 0 & 0 & I & 0\end{pmatrix},\ S_2=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ I & 0 & 0 & 0\\ 0 & -I & 0 & 0\end{pmatrix},\ S_3=S_1\,.\,S_2.$$