SUMAR

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ON CERTAIN NYKODYM TYPE THEOREMS FOR MULTIMEASURES

BY

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In the present paper we first establish a theorem of Brooks-Jewett type for multimeasures defined on a quasi-σ-ring with values in a locally convex space and then, using some ideas of L. Drewnowski [2], we show the equivalence between the Brooks-Jewett, Vitali-Hahn-Saks and Nykodym theorems for exhausting multimeasures defined on a quasi-σ-ring with values in a locally convex space.

§ 1. Basic notions, terminology and notations. Let Ω be a set and let $\mathcal{P}(\Omega)$ be the family of all subsets of Ω organized as a Boolean ring for the symmetric difference and intersection of sets, where the sum $\oplus$ in $\mathcal{P}(\Omega)$ is given by the symmetric difference and the product $\odot$ by the intersection of the sets. In this ring the empty set, Φ, is the null element and the set Ω is the unity.

We consider $\mathcal{P}(\Omega)$ endowed with the order topology $\mathcal{O}$. The topology $\mathcal{O}$ is given in the following way: For every $A\in\mathcal{P}(\Omega)$, $A' = \Omega\setminus A$, $[0, A] = \{B\in\mathcal{P}(\Omega);\ B\subset A\}$; if A, $B\in\mathcal{P}(\Omega)$, $[A, B] = \{E\in\mathcal{P}(\Omega);\ A\subset E\subset B\}$, $A\oplus[0, B] = \{A\oplus C;\ C\subset B\} = [A\setminus B,\ A\cup B]$.

If $\mathcal{F}$ is the family of finite sets in Ω, the family $\mathcal{F}' = \{A\subset\Omega;\ A'\in\mathcal{F}\}$ is directed by inclusion and if Ω is infinite, then $\mathcal{F}'$ ia a filter on Ω.

For every $A\in\mathcal{P}(\Omega)$, the family

$$\mathcal{B}(A) = \{A\oplus[0, F] : F\in\mathcal{F}'\},$$

is a base of neighborhoods of A for the topology $\mathcal{O}$.

A subring $\mathcal{R}$ of $\mathcal{P}(\Omega)$ is said to be a *quasi-σ-ring* of sets of Ω (see Haydon [8]) iff for every disjoint sequence $(A_n)_{n\in N}$ in $\mathcal{R}$ there exists an infinite subset M of N with $\bigcup_{n\in N} A_n\in\mathcal{R}$ ($N = \{0,1,\ldots\}$ and $N^* = N\setminus\{0\}$).

Let X be a Hausdorff locally convex topological vector space, τ its topology and $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin in X. Using notations as in [3], $\mathcal{A}(X)$ is the family of all nonempty subsets of X and $\mathcal{C}(X)$ the family of closed sets of X, respectively.

$\bar{A}$ denotes the closure of A in X. If we define the equivalence relation "ρ" on $\mathcal{A}(X)$ by

$$A\ \rho\ B \leftrightarrow \bar{A} = \bar{B}, \tag{1}$$

then the factorization of the semigroup $(\mathcal{A}(X), +)$ by ρ, where $+$ is the usual addition in X, may be identified with $(\mathcal{C}(X), \dot{+})$, where the addition $\dot{+}$ is defined by

(2) $$A \dot{+} B = \overline{A + B}.$$

If $\mathcal{U}$ is the invariant uniformity on X compatible with τ, then uniformity $\tilde{\mathcal{U}}$ on $\mathcal{A}(X)$ defined by the following base of vicinities

$$\mathcal{W}(U) = \{(A, B) \in \mathcal{A}(X) \times \mathcal{A}(X);\ A \subset B \dot{+} U,\ B \subset A \dot{+} U\},\ U \in \mathcal{V},$$

is said to be the *exponential extension* of $\mathcal{U}$ to $\mathcal{A}(X)$.

The topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$ will be denoted by $\tilde{\tau}$.

Using the equivalence relation (1) we may identify $(\mathcal{A}(X), \tilde{\tau})/\rho$ with $(\mathcal{C}(X), \tilde{\tau})$ and $(\mathcal{C}(X), \tilde{\mathcal{U}})$ with the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})$.

If X is metrizable, then $(\mathcal{A}(X), \tilde{\mathcal{U}})$ becomes a semi-metrizable space for the Hausdorff distance between sets. If, in addition, X is complete then so is $\mathcal{C}(X)$. If $(X, \mathcal{U})$ is a complete uniform space then so is $(\mathcal{C}(X), \tilde{\mathcal{U}})$.

If $(E_n)_{n \in N}$ is a sequence in $\mathcal{A}(X)$ we say that the series $\sum_{n=1}^{\infty} E_n$ is $\tilde{\tau}$-*convergent* if the sequence $S_n = \sum_{k=1}^{n} E_k$, $\tilde{\tau}$-converges in $\mathcal{A}(X)$ and we write $(\tilde{\tau}) \sum_{n=1}^{\infty} E_n$. We also denote by $^*\Sigma$ a finite sum or a series with respect to $\dot{+}$.

It is easy to see that $(\mathcal{C}(X), \dot{+}; \tilde{\mathcal{U}})$ is a Hausdorff uniform commutative semigroup with the unit $\{0\}$. Furthermore the semigroup $\mathcal{C}(X)$ has the important property that the law of cancellation.

(3) $$A \dot{+} C = B \dot{+} C \to A = B, \text{ for every } A, B, C \in \mathcal{C}(X),$$

holds in it.

According to a result of W e b e r [15] there exists a family $\mathcal{P} = \{p\}$ of semi-invariant pseudo-metrics, taking values in $[0, 1]$ which generates the uniformity $\tilde{\mathcal{U}}$. We recall that the pseudo-metric p is semi-invariant if:

(4) $$p(A \dot{+} C,\ B \dot{+} C) \le p(A, B),$$

for every $A, B, C \in \mathcal{C}(X)$. If $p \in \mathcal{P}$ is an arbitrary pseudo-metric we denote by

(5) $$|A|_p = p(A, O),$$

where O represents the set $\{0\}$ from $\mathcal{C}(X)$.

Since every p is semi-invariant, it is easy to verify the following properties:

(6) $$|A \dot{+} B|_p \le |A|_p + |B|_p, \text{ for every } A, B \in \mathcal{C}(X);$$

(7) $$|A \dot{+} B|_p \ge |A|_p - |B|_p, \text{ for every } A, B \in \mathcal{C}(X);$$

(8) $$p(A \dot{+} B \dot{+} C,\ A' \dot{+} B' \dot{+} C') \ge |A|_p - |A'|_p - p(A, A') - 2|B'|_p - |C|_p - |C'|_p, \text{ for every } A, B, C, A', B', C' \in \mathcal{C}(X).$$

In the following we shall consider the correspondences from a ring $\mathcal{R}$ of subsets of Ω to the locally convex space X, that is set functions from $\mathcal{R}$ to $\mathcal{A}(X)$ or $\mathcal{C}(X)$.

Definition 1.1. A correspondence μ from $\mathcal{R}$ to X is said to be:

I) *additive* (respectively *subadditive*) iff

(9) $$\mu(A \cup B) = (\text{respectively } \subset)\ \mu(A) \dot{+} \mu(B),$$

for every $A, B \in \mathcal{R}$ such that $A \cap B = \emptyset$.

II) σ-*additive* (respectively σ-*subadditive*) iff

(10) $$\mu\Big(\bigcup_{n=0}^{\infty} A_n\Big) = (\text{resp. } \subset)\ (\tilde{\tau})^*\sum_{n=0}^{\infty} \mu(A_n),$$

for every disjoint sequence $(A_n)_{n \in N}$ in $\mathcal{R}$ with $\bigcup_{n \in N} A_n \in \mathcal{R}$.

Definition 1.2. I) A family $(\mu_i)_{i \in I}$ of correspondences from $\mathcal{R}$ to X is said to be *uniformly exhausting* iff for every disjoint sequence $(A_n)_{n \in N}$ in $\mathcal{R}$, $\mu_i(A_n) \to 0$ as $n \to \infty$, uniformly with respect to $i \in I$.

II) In particular, if the family $\{\mu_i;\ i \in I\} = \{\mu\}$ (it consists of a single correspondence μ) we say that μ is *exhausting*.

Definition 1.3. A correspondence μ from $\mathcal{R}$ to X is said to be a *multimeasure* iff:

I) $\mu(\Phi) = 0$, where 0 denotes the set $\{0\}$ in $\mathcal{C}(X)$, and

II) μ is additive.

Let μ be a multimeasure on $\mathcal{R}$. Put

(11) $$\tilde{\mu}(A) = \sup\{|\mu(A \cap B)|_p;\ B \in \mathcal{R}\},$$

for every $A \in \mathcal{R}$. This function $\tilde{\mu}: \mathcal{R} \to \bar{R}_+$ is called the *p-semi-variation* of μ. It is easy to see that $\tilde{\mu}$ is a bounded exhausting submeasure on $\mathcal{R}$ in the *Drewnowski sense* (see [1]).

§ 2. Theorem of Brooks-Jewett. In this section we establish for multimeasures a version of *Drewnowski-type* (see [2]) for *Brooks-Jewett theorem*.

Theorem of Brooks-Jewett. *Let $\mathcal{R}$ be a quasi-σ-ring of subsets of Ω and let $(\mathcal{C}(X), \dot{+}; \tilde{\mathcal{U}})$ with the above signification. If $(\mu_n)_{n \in N}$ is a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim_{n \to \infty} \mu_n(A) = \mu_0(A)$ for every $A \in \mathcal{R}$, then the set $\{\mu_n;\ n \in N\}$ is uniformly exhausting.*

Proof. We reason by contradiction. Assume therefore there exists a sequence $(\mu_n)_{n \in N^*}$ of exhausting multimeasures such that for every $A \in \mathcal{R}$ there exists the limit

$$\lim_{n \to \infty} \mu_n(A) = \mu_0(A),$$

but the set $(\mu_n)_{n\in N^*}$ is not uniformly exhausting. Then there exist a pseudo-metric $p\in\mathcal{P}$ ($\mathcal{P}$ is the family of pseudo-metrics which generates the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{C}(X)$), an $\varepsilon>0$ and a disjoint sequence $(A_n)_{n\in N}$ in $\mathcal{R}$ such that

$$|\mu_n(A_n)|_p\geq\varepsilon, \text{ for every } n\in N^*.$$

Since $\mathcal{R}$ is a quasi-σ-ring of sets of Ω, $\mathcal{R}$ satisfies the Subsequential Interpolation Property (see F. J. F r e n i c h i [6]) and then taking into account that $\tilde{\mu}_0$ is a submeasure in Drewnowski sense there exists $B_0\in\mathcal{R}$ and an infinite subset N_0 of N such that

$$\tilde{\mu}_0(B_0)<\frac{\varepsilon}{12},$$

$A_n\subset B_0$, for every $n\in N_0$ and $A_n\cap B_0=\varnothing$ for $n\in N\setminus N_0$. Indeed, to see that it is sufficient to apply the arguments used in the proof of lemma 3 of [6], considering a partition $(E_i)_{i\in N}$ of N with infinite subsets of N. Inductively, we can construct a decreasing sequence $(B_j)_{j\in N}$ in $\mathcal{R}$, a sequence $(N_j)_{j\in N}$ of infinite subsets of N and a sequence $(n_j)_{j\in N}$ of natural numbers such that $n_0=0$, $n_j\in N_{j-1}$, n_j is strictly smaller than the first element of N_j and $N_j\subset N_{j-1}$ ($N_{-1}=N$), $B_j\cap A_n=\varnothing$ if $n\in N\setminus N_j$ and $A_n\subset B_j$ if $n\in N_j$, $\mu_{n_j}(B_j)<\dfrac{\varepsilon}{12}$ and

$$\sum_{i<j} p(\mu_{nj}(A_{ni}),\ \mu_0(A_{ni}))<\frac{\varepsilon}{12}$$

Let us denote by

$$\mathcal{R}[B_0]=\{A\cap B_0;\ A\in\mathcal{R}\}.$$

It is easy to see that $\mathcal{R}[B_0]$ is a Boolean algebra contained in $\mathcal{R}$. Since the sequences $(B_j)_{j\in N}$ and $(A_{nj})_{j\in N}$ belong to $\mathcal{R}[B_0]$, using Theorem 4 of [6] we can find an infinite subset M of N and an element $A\in\mathcal{R}[B_0]$ such that for every $k\in M$, $A=A_{n_k}\cup(\cup\{A_{n_i};\ i\in M,\ i<k\})\cup(A\setminus(\cup\{A_{n_i};\ i\in M,\ i\leqslant k\}))$ where $(A\setminus\cup\{A_{n_i};\ i\in M,\ i\leqslant k\})\subset B_k$. Taking into account that μ_{n_k} and μ_0 are additive set functions, from (8) we obtain

$$p(\mu_{n_k}(A),\ \mu_0(A))>\varepsilon-\frac{\varepsilon}{12}-\frac{\varepsilon}{12}-\frac{2\varepsilon}{12}-\frac{\varepsilon}{12}-\frac{\varepsilon}{12}=\frac{\varepsilon}{2},$$

for every $k\in M$, which contradicts the pointwise convergence of the sequence $(\mu_n)_{n\in N^*}$.

§ 3. The equivalence between Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems for multimeasures. In this section we shall point out the relationship between Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems for exhausting multimeasures from a quasi-σ-ring $\mathcal{R}$ of sets of Ω to a locally convex space X. In the proof we shall use some ideas from [2].

First we recall that a topology Γ on a ring $\mathcal{R}$ of sets of Ω is a *Fréchet-Nikodym topology* (abbreviated an *(F-N)-topology*) if $(\mathcal{R},\oplus)$ is a topological group under Γ and the operation of intersection $(A,\ B)\to A\cap B$ is uniformly continuous on $(\mathcal{R},\oplus;\ \Gamma)$.

Each $(F\text{-}N)$-topology is associated with a family of submeasures and conversely, given a family $(\lambda_i;\ i\in I)$ of submeasures on $\mathcal{R}$, the λ_i are quasi-norms on the group $(\mathcal{R},\oplus)$ and therefore determine a unique topology $\tilde{\Gamma}$ compatible with its group structure; $\tilde{\Gamma}$ is obviously an $(F\text{-}N)$-topology.

If Γ is an $(F\text{-}N)$-topology on $\mathcal{R}$, then we say that Γ is *order-continuous* if for every decreasing sequence $(A_n)_{n\in N}$ in $\mathcal{R}$ such that $\bigcap_{n\in N} A_n=\varnothing$, then $A_n\to\varnothing$ (with respect to Γ) (In particular, this is the case for the order topology on $\mathcal{P}(\Omega)$).

We say that Γ is *exhausting* if for every disjoint sequence $(A_n)_{n\in N}$ in $\mathcal{R}$, $A_n\to\varnothing$ (with respect to Γ).

If μ is a multimeasure, we denote by $\Gamma\mu$ the $(F\text{-}N)$-topology induced by μ, that is the $(F\text{-}N)$-topology determined by the Fréchet Nikodym ecart $(A,\ B)\to\mu(A\ \Delta\ B)$. We say that the multimeasure μ is order-continuous iff $\Gamma\mu$ is order-continuous.

It is evident the meaning of family $\{\mu_n:\ n\in N\}$ uniformly order-continuous.

Now let us show that the following statements are equivalent:

(N) (Nikodym theorem). *Let $\mathcal{R}$ be a quasi-σ-ring of subsets of Ω and $(\mathcal{C}(X),\dot{+};\ \tilde{\mathcal{U}})$ as above. If $(\mu_n)_{n\in N^*}$ is a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n\to\infty}\mu_n(A)=\mu_0(A)$ for every $A\in\mathcal{R}$ and μ_n is order continuous for every $n\in N^*$, then the set $\{\mu_n;\ n\in N\}$ is uniformly order-continuous.*

(B J) (Brooks-Jewett theorem). *Let $\mathcal{R}$ be a quasi-σ-ring of subsets of Ω and let $(\mathcal{C}(X),\dot{+};\ \tilde{\mathcal{U}})$ as above. If $(\mu_n)_{n\in N^*}$ is a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n\to\infty}\mu_n(A)=\mu_0(A)$, for every $A\in\mathcal{R}$, then the set $\{\mu_n;\ n\in N\}$ is uniformly exhausting.*

(V H S) (Vitali-Hahn-Saks theorem). *Let $\mathcal{R}$ be a quasi-σ-ring of subsets of Ω, $(\mathcal{C}(X),\dot{+};\ \tilde{\mathcal{U}})$ as above and let Γ be an $(F\text{-}N)$-topology on $\mathcal{R}$. If $(\mu_n)_{n\in N^*}$ is a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n\to\infty}\mu_n(A)=\mu_0(A)$ for every $A\in\mathcal{R}$, and μ_n is Γ-continuous for every $n\in N^*$, then the set $\{\mu_n;\ n\in N\}$ is equi-Γ-continuous.*

Proof. **(N)$\Rightarrow$(BJ).** Let $(\mu_n)_{n\in N^*}$ be a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n\to\infty}\mu_n(A)=\mu_0(A)$, for every $A\in\mathcal{R}$, and $(A_k)_{k\in N}$ a disjoint sequence in $\mathcal{R}$. Let $p\in\mathcal{P}$. We define the following equivalence relation on $\mathcal{C}(X)$.

$$E\sim F\leftrightarrow p(E,\ F)=0. \tag{9}$$

If we denote by $\mathcal{C}'=\mathcal{C}(X)/\sim$ the space of all equivalence classes, it is easy to see that $\mathcal{C}'$ is a semigroup, the function $p':\ \mathcal{C}'\to R_+$ defined by

$$p'([E],\ [F])=p(E,\ F),\ \text{for every } [E],\ [F]\in\mathcal{C}',$$

is a semi-invariant metric on $\mathcal{C}'$, and the cononical projection $\pi:\ (\mathcal{C}(X),\ p)\to(\mathcal{C}',\ p')$ is continuous.

Let $(\hat{\mathcal{C}}, \hat{p})$ be the completion of the metric space $(\mathcal{C}', p')$. Extending the addition to $(\hat{\mathcal{C}}, \hat{p})$ in a natural way, the space $(\hat{\mathcal{C}}, \hat{p})$ becomes a Hausdorff complete uniform semigroup.

Let $\lambda_n = \pi \circ \mu_n$ for every $n \in N^*$. We observe that λ_n is an exhausting additive set function from $\mathcal{R}$ to $\hat{\mathcal{C}}$ and $\lim\limits_{n\to\infty} \lambda_n(A) = \lambda_0(A)$ for every $A \in \mathcal{A}$.

Using Lema 2.2 of [10] there exists a subsequence $(A_{k_m})_{m\in N}$ of $(A_k)_{k\in N}$, a quasi-σ-ring $\mathcal{S}$ containing $(A_{k_m})_{m\in N}$ and contained in $\mathcal{A}$, the σ-algebra generated by $(A_{k_m})_{m\in N}$, and a sequence $(\nu_n)_{n\in N^*}$ of σ-additive $\hat{\mathcal{C}}$-valued functions on $\mathcal{A}$ with the following properties:

a) for every $n \in N^*$, $\nu_n(A) = \lambda_n(A)$, $(\forall) A \in \mathcal{R} \cap \mathcal{A}$;

b) if $(\lambda_n)_{n\in N^*}$ is pointwise convergent on $\mathcal{R}$ then (ν_n) is pointwise convergent on $\mathcal{S}$;

c) the restriction of ν_n to $\mathcal{S}$ is exhausting.

Now, using Corollary 2.3 of [11] it follows that each $\nu_n|_{\mathcal{S}}$ is order continuous. Since $\mathcal{S}$ is a quasi-σ-ring, $\mathcal{S}$ has the Subsequential Interpolation Property (see Frenichi [6]) and then, by virtue of hypothesis (N), the set $\{\nu_n|_{\mathcal{S}} ; n \in N^*\}$ is uniformly order continuous. Using again that $\mathcal{S}$ is a quasi-σ-ring we obtain a subsequence $(A_{k_{m_j}})_{j\in N}$ of (A_{k_m}) such that $\bigcup\limits_{j\in N} A_{k_{m_j}} \in \mathcal{S}$.

Let $E_j = \bigcup\limits_{i\geq j} A_{k_{m_i}}$, for every $j \in N$. It is easy to see that $(E_j)_{j\in N}$ is a decreasing sequence in $\mathcal{S}$ such that $\bigcap\limits_{j=1}^{\infty} E_j = \emptyset$ and $E_j = A_{k_{m_j}} \cup E_{j+1}$. Thus we have

$$|\nu_n(A_{k_{m_j}})|_{\hat{p}} \leq |\nu_n(E_j)|_{\hat{p}} + |\nu_n(E_{j+1})|_{\hat{p}},$$

where the second side of the last inequality tends to 0 uniformly in $n \in N^*$. But then $|\lambda_n(A_{k_{m_j}})|_{\hat{p}} \to 0$ uniformly in $n \in N$ and hence $|\lambda_n(A_k)|_{\hat{p}} \to 0$ uniformly in $n \in N$. From the evident equality $|\lambda_n(A_k)|_{\hat{p}} = |\pi \circ \mu_n(A_k)|_{\hat{p}} = |[\mu_n(A_k)]_{\hat{p}}| = |\mu_n(A_k)|_{\hat{p}}$, we obtain that $|\mu_n(A_k)|_p \to 0$ uniformly in $n \in N^*$ which involves that the set $\{\mu_n ; n \in N^*\}$ is uniformly exhausting. Since μ_0 is also exhausting we see that $\{\mu_n ; n \in N\}$ is uniformly exhausting.

(BJ) $\to$ (VHS). Let $(\mu_n)_{n\in N^*}$ be a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n} \mu_n(A) = \mu_0(A)$ for all $A \in \mathcal{R}$ and μ_n is Γ-continuous for every $n \in N^*$.

According to (BJ) the set $\{\mu_n ; n \in N^*\}$ is uniformly exhausting. Using now lemma 2.4 from [10] which says: If $\mathcal{R}$ is a Boolean ring, S a uniform semigroup and M a non-empty uniformly exhausting set of additive set functions from $\mathcal{R}$ to S, then for every neighborhood V of 0 in S, there exist a finite subset K of M and a neighborhood W of 0 in S such that $\mu(A) \in V$ for all $\mu \in M$ whenever $A \in \mathcal{R}$ and $\{\mu(B \cap A) ; B \in \mathcal{R}$ and $\mu \in K\} \subset W$, we see that the set $\{\mu_n ; n \in N\}$ is uniformly Γ-continuous.

(VHS) $\Rightarrow$ (N). Let $(\mu_n)_{n\in N^*}$ be a sequence of exhausting multimeasures from $\mathcal{R}$ to X such that $\lim\limits_{n\to\infty} \mu_n(A) = \mu_0(A)$ for every $A \in \mathcal{R}$ and μ_n is order continuous for every $n \in N$.

Let us show that every μ_n is σ-additive. Let $p \in \mathcal{P}$ and $(A_k)_{k\in N}$ a disjoint sequence in $\mathcal{R}$ with $\bigcup\limits_{k\in N} A_k \in \mathcal{R}$. Let $A = \bigcup\limits_{k=1}^{\infty} A_k$. We remark that

$$A = (A \setminus \bigcup_{i=1}^{m} A_i) \cup (\bigcup_{i=1}^{m} A_i)$$

and $B_m = A \setminus \bigcup\limits_{i=1}^{m} A_i$ is a decreasing sequence in $\mathcal{R}$ with $\bigcap\limits_{m\in N} B_m = \emptyset$.

We have then

$$p(\mu_i(A), \sum_{i=0}^{m} \mu_i(A)) \leq |\mu_i(B_m)|_p, \text{ for all } i \in N^*,$$

and since $\lim\limits_{m\to\infty} |\mu_i(B_m)|_p = 0$ we obtain that μ_i is σ-additive.

We observe that if $(A_i)_{i\in N}$ is a decreasing sequence in $\mathcal{R}$ such that $\bigcap\limits_{i\in I} A_i = \emptyset$, then $\mu_n(A_i \cap B) \to 0$ uniformly in $B \in \mathcal{R}$.

Indeed, if $p \in \mathcal{P}$, since

$$|\mu_n(A_i \cap B)|_p \leq \tilde{\mu}_n(A_i \cap B) \leq \tilde{\mu}_n(A_i),$$

by virtue of Theorem 2.2 of [11] we obtain that $\tilde{\mu}_n$ is order continuous which assures the above property of (μ_n).

Let Γ_0 be the supremum of all order-continuous, exhausting $(F\text{-}N)$-topologies on $\mathcal{R}$.

It is clear that Γ_0 is also order continuous, exhausting and an $(F\text{-}N)$-topology [1]. By virtue of the above considerations, for every $n \in N$, the topology Γ_{μ_n} is order continuous. Since Γ_{μ_n} is also exhausting it follows that Γ_{μ_n} is weaker than Γ_0. Hence μ_n is Γ_0-continuous.

According to the hypothesis **(VHS)** the set $\{\mu_n ; n \in N\}$ is equi-Γ_0-continuous and therefore uniformly order continuous.

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ÜBER DIE EIGENSCHAFTEN DES RAUMES $P_D(F, fF)$

VON

KERIM KOCA und CLAUDIO WITHALM

1. Einleitung. Es seien D und D_0 einfach zusammenhängende Gebiete in C mit $D\subset\subset D_0$. (d. h. $\bar{D}\subset D_0$, D ist relativ kompakt in D_0.). Man weiß, daß in der Theorie der pseudoholomorphen Funktionen, der in $D_0\subset C$ erklärte Erzeugendenvektor $E:=(F, G)$ den Voraussetzungen

$$\text{i)} \quad \bigwedge_{z\in D_0} \operatorname{Im}(\overline{F(z)}G(z))>0\,; \quad \text{ii)} \quad \bigwedge_{z\in D_0} E\in H^1_{D_0}\times H^1_{D_0}$$

genügt, wobei $H^1_{D_0}$ den Raum der hölderstetig 1-mal differenzierbaren Funktionen bezeichnet. Die Menge der Erzeugendenvektoren wird allgemein mit E_{D_0} abgekürzt. Es sei Ω_D der Raum über R der in $D\subset\subset D_0$ reellen Vektorfunktionen $\omega:=\binom{\varphi}{\psi}\in H^1_D\times H^1_D$. Wir nennen dann $w=E.\ \omega\in E.\Omega_D$ in D eine E-pseudoholomorphe Funktion 1. Art, wenn für $z=x+iy$, $z_0=x_0+iy_0\in D$

$$(1) \qquad \bigwedge_{z_0\in D} \dot{w}:=\frac{d_E w}{dz}(z_0):=\lim_{z\to z_0}\left[E(z).\,\frac{\omega(z)-\omega(z_0)}{z-z_0}\right]$$

existiert. Wir bezeichnen mit $P_D(E)$ den Raum über R der in $D\subset\subset D_0$ erklärten E-pseudoholomorphen Funktionen 1. Art. Man sieht leicht ein, daß für den Spezialfall $A:=(1, i)\in E_{D_0}$ die Funktion $A.\ \omega$ holomorph ist. Wir werden daher zweckmäßig den Raum $P_D(A)$ als die Menge aller in D holomorphen Funktionen bezeichnen. $\dot{w}(z_0)$ in (1) heißt die komplexe Ableitung modulo E oder die komplexe E-Ableitung von w an der Stelle z_0. Wenn $w\in P_D(A)$ ist, so ist

$$d_A w/dz=\dot{w}=w_z.$$

Nach L. B e r s [2] nennt man die durch $E=(F, G)$ erklärten Funktionen

$$(2) \qquad \begin{aligned} a_E &:=-(\bar{F}G_{\bar z}-F_{\bar z}\bar{G})/\Delta, & b_E &:=(FG_{\bar z}-F_{\bar z}G)/\Delta, \\ A_E &:=-(\bar{F}G_z-F_z\bar{G})/\Delta, & B_E &:=(FG_z-F_zG)/\Delta \end{aligned}$$

die charakterischen Koeffizienten mod. E, wobei $\Delta:=F\bar{G}-\bar{F}G$ ist.

Es seien E_0, $E_1\in E_{D_0}$, dann nennt man E_1 Nachfolger von E_0 und E_0 Vorgänger von E_1, wenn die Relation $a_{E_1}=a_{E_0}$, $b_{E_1}=-B_{E_0}$ gilt.