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A NOTE ON COMPLEX SUBMANIFOLDS OF CERTAIN NON-KAEHLER MANIFOLDS

BY

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§ 0. Introduction. This paper completes the work of M. Kimura [5] about complex submanifolds of certain non-Kaehler manifolds.

E. Calabi and B. Eckmann [3] proved that the product of two odd-dimensional spheres carries a non-Kaehler Hermitian structure, and A. Morimoto [7] generalized this situation to the product of two Sasakian manifolds.

In §1, the Hermitian structure of the manifolds similar to the Calabi-Eckmann manifolds is studied. In §2, complex submanifolds immersed in such manifolds are considered and we prove that they are totally geodesics and locally a product of two minimal Sasakian submanifolds.

§ 1. Preliminaries. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasakian manifold. The structure tensors verify [1]

$$\varphi^2 = -I + \eta\otimes\xi,\ \varphi\xi = 0,\ \eta\circ\varphi = 0,\ \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi),$$

$$\eta\wedge(d\eta)^n \neq 0,\ d\eta(X, Y) = g(X, \varphi Y).$$

It is easy to prove that the vector field ξ is a Killing vector field on M, i.e., $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ denotes the Lie differentiation.

Let N be an invariant submanifold of M, that is, for all $X \in \mathfrak{X}(N)$, $\varphi X \in \mathfrak{X}(N)$. It is well known the following

Lemma 1.1 ([6]). *For all $X, Y \in \mathfrak{X}(N)$, $B(\varphi X, \varphi Y) = -B(X, Y)$ where B denotes the second fundamental form of the submanifold. Furthermore, N is minimal.*

Now let $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1,2$, be two $(2n_i+1)$-dimensional Sasakian manifolds and let $\tilde{M}$ denote the Riemannian product manifold $M_1\times M_2$ with the product metric $\tilde{g}$. The (1,1)-tensor J given by

$$J(X_1, X_2) = (\varphi_1 X_1 - \eta_2(X_2)\xi_1,\ \varphi_2 X_2 + \eta_1(X_1)\xi_2), \tag{1.1}$$

for all $(X_1, X_2) \in \mathfrak{X}(\tilde{M})$, defines a complex structure on $\tilde{M}$. In fact, $(\tilde{M}, J, \tilde{g})$ is a non-Kaehler Hermitian manifold [7], and by consequence, the following relation holds [4]

(1.2) $$[J, J](\tilde X, \tilde Y) = 0, \quad \forall \tilde X, \tilde Y \in \mathcal{X}(\tilde M).$$

Clearly, the Calabi-Eckmann manifold $S^{2n+1}\times S^{2m+1}$ is an example of this situation.

Let $\tilde\xi_1=(\xi_1, 0)$, $\tilde\xi_2=(0, \xi_2)$ be two vector fields on $\tilde M$. Remark that $J\tilde\xi_1=\tilde\xi_2$. Furthermore, since $\xi_i\in\mathcal{X}(M_i)$, $i=1, 2$, are Killing vector fields, it is easy to prove the following

Lemma 1.2. *$\tilde\xi_1$ and $\tilde\xi_2$ are Killing vector fields on $\tilde M$.*

§2. Complex submanifolds of certain non-Kaehler manifolds. Let N be a $2n$-dimensional complex submanifold of $\tilde M$. Then, using (1.2), (N, J) is again an Hermitian manifold such as $\tilde M$, and M. Kimura proves [5]: i) N is a minimal submanifold of $\tilde M$; ii) the vector fields $\tilde\xi_1$ and $\tilde\xi_2$ are necessarily tangent to N.

Example. Let PC^n be the complex projective space with homogeneous coordinates $(z_0,\dots, z_n)$, and consider the natural fibration $S^{2n+1}\times S^{2n+1}\to PC^n\times PC^n$. Let M be the complex hypersurface of $PC^n\times PC^n$ defined by the equation $\sum_{i=0}^{n} z_i z_i'=0$, and N be the torus bundle over M such that the following diagram is commutative

$$\begin{tikzcd} N \arrow[r] \arrow[d] & S^{2n+1}\times S^{2n+1} \arrow[d] \\ M \arrow[r] & PC^n\times PC^n \end{tikzcd}$$

Then, N is a complex hypersurface of $S^{2n+1}\times S^{2n+1}$.

Let $\tilde\eta_1$ and $\tilde\eta_2$ denote the covariant forms of $\tilde\xi_1$ and $\tilde\xi_2$ respectively, i.e., for all $\tilde X\in\mathcal{X}(N)$, $\tilde\eta_i(\tilde X)=\tilde g(\tilde\xi_i, \tilde X)$.

Since M_1 and M_2 are Sasakian manifolds, we have

Lemma 2.1. *For all tangent vector fields $\tilde X$, $\tilde Y\in\mathcal{X}(N)$ the following relations hold*

i) $d\tilde\eta_i(J\tilde X, J\tilde Y)=d\tilde\eta_i(\tilde X, \tilde Y)$, $i=1, 2$; ii) $d\tilde\eta_i(\tilde\xi_j, \tilde X)=0$, $i, j=1, 2$.

Furthermore, since N is a Hermitian manifold, using (1.2), one can prove that $\mathcal{L}_{\tilde\xi_i}J=0$, $i=1, 2$, and we get

Lemma 2.2. *$\tilde\xi_1$ and $\tilde\xi_2$ are analytic vector fields on N.*

Now, let $\{\tilde\xi_1, \tilde\xi_2, \tilde X_3,\dots, \tilde X_{2n}\}$ be a J-basis of $\mathcal{X}(N)$. Note that $(d\tilde\eta_i)^n$, $i=1, 2$, vanishes on N by Lemma 2.1. On the other hand, the 1-forms η_i are of maximal degree on M_i and considered as 1-forms extended to the product manifold they verify $\eta_1\wedge\eta_2\wedge(d\eta_1)^{n_1}\wedge(d\eta_2)^{n_2}\neq0$. Hence, there exists $r_i\in\mathbb{Z}$, $r_i\leqslant n_i$, $i=1, 2$, such that $(d\tilde\eta_i)^{r_i}\neq0$, $(d\tilde\eta_i)^{r_i+1}=0$ and $\tilde\eta_1\wedge\tilde\eta_2\wedge(d\tilde\eta_1)^{r_1}\wedge(d\tilde\eta_2)^{r_2}\neq0$ with $r_1+r_2+1=n$.

By consequence, using Lemmas 1.1 and 2.2, we conclude that N is an Hermitian bicontact manifold [2]; and, using Theorem 4.4 in [2] and Lemma 2.1, N is locally the product of two Sasakian manifolds N_1 and N_2 of dimensions $2r_1+1$ and $2r_2+1$ respectively. Such manifolds are, in fact, maximal integral submanifolds of the distributions

$$\mathcal{D}_i=\{\tilde X\in\mathcal{X}(N)/\iota_{\tilde X}\tilde\eta_i=0,\ \iota_{\tilde X}d\tilde\eta_i=0\},\quad i=1, 2.$$

where ι denotes the inner product operator.

The normal contact metric structure (φ, ξ, η, g) on N_2 is given by

$$\varphi\tilde X=J\tilde X+\tilde\eta_2(\tilde X)\tilde\xi_1,\quad \xi=\tilde\xi_2,\quad \eta=\tilde\eta_2,\quad g=g_2,$$

and one can compute that for all $\tilde X\in\mathcal{X}(N_2)$, $\varphi\tilde X=(0, \varphi_2X_2)\in\mathcal{X}(N_2)$. Hence, this structure on N_2 coincides with that induced from M_2 as submanifold, and clearly N_2 is an invariant submanifold of the Sasakian manifold M_2; then, using Lemma 1.1, N_2 is minimal.

Analogously, N_1 is a minimal Sasakian submanifold of M_1.

Hence, we have

Theorem 2.3. *Let $\tilde M=M_1\times M_2$ be the Riemannian product of two Sasakian manifolds with the almost complex structure J defined in (1.1) and let N be a complex submanifold of $\tilde M$. Then N is locally the product $N_1\times N_2$, where N_i, $i=1$, 2, is a minimal Sasakian submanifold of M_i.*

Corollary 2.4. *Every complex submanifold of a Calabi-Eckmann manifold $S^{2n+1}\times S^{2m+1}$ is locally the product of two minimal Sasakian submanifolds, one on each sphere.*

Now, using Theorem 2.3, let $\tilde X=(X_1, X_2)$, $\tilde Y=(Y_1, Y_2)\in\mathcal{X}(N)$. Using Gauss formula, it is easy to prove that

(2.1) $$B(\tilde X, \tilde Y)=(B_1(X_1, Y_1), B_2(X_2, Y_2)),$$

where B_i denotes the second fundamental form of the submanifold N_i. Clearly we have the following

Proposition 2.5. *The complex submanifold N is totally geodesic if and only if N_1 and N_2 are totally geodesic.*

On the other hand, using (1.1), (2.1) and Lemma 1.1, the following relations hold:

(2.2) $$B(J\tilde X, J\tilde Y)=B(\tilde X, \tilde Y),\quad B(\tilde X, J\tilde Y)=-JB(\tilde X, \tilde Y),$$

for all $\tilde X$, $\tilde Y\in\mathcal{X}(N)$, and we have the next

Theorem 2.6. *Let $\tilde M=M_1\times M_2$ be the Riemannian product of two Sasakian manifolds with the almost complex structure J defined in (1.1), and let N be a complex submanifold of $\tilde M$. Then N is totally geodesic.*

Proof. It follows from (2.2).

Remark that N is necessarily minimal, which is the result of M. Kimura [5].

Corollary 2.7. *Every complex submanifold of a Calabi-Eckmann manifold is totally geodesic.*

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RECENZII

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