An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

ON A CLASS OF p-VALENT ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY

S. M. SARANGI, M. R. KRISHNA MURTHY

1. Introduction. An analytic function f is said to be subordinated to an analytic function g (denoted $f \prec g$) if there exists an analytic function Φ with $\Phi(0) = 0$, $|\Phi(z)| < 1$ for $|z| < 1$ such that $f(z) = g[\Phi(z)]$ in [1]. Let $Q_p(A, B)$ denote the class of functions $f(z) = a_p z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} (a_p > 0)$ that are p-valent and analytic in the unit disk $E = \{z : |z| < 1\}$ and satisfy the subordination

$$f'(z)/(a_p p z^{p-1}) \prec (1+Az)/(1+Bz) \text{ where } -1 \leqslant A < B \leqslant 1.$$

Let $T_p(z_0)$ be a class of functions $f(z)$ that satisfy $f'(z_0) = p z_0^{p-1} (0 < z_0 < 1)$. For given A, B and z_0 fixed, let $R_p(A, B, z_0) = Q_p(A, B) \cap T_p(z_0)$.

In this paper we obtain a few sharp results concerning coefficient inequalities, distortion theorem and radius of convexity for the class $R_p(A, B, z_0)$. Further we show that the class $R_p(A, B, z_0)$ is closed under arithmetic mean and convex linear combinations.

The techniques adopted are of H. Silverman [3].

2. Coefficient inequalities.

Theorem 1. *A function* $f(z) = a_p z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ $(a_p > 0)$ *is in* $Q_p(A, B)$ *if and only if*

$$\sum_{n=1}^{\infty} (n+p)|a_{n+p}| \leqslant \frac{p(B-A)\, a_p}{1+B}. \tag{1}$$

Proof. Suppose (1) holds. Then

$$|f'(z)/z^{p-1} - pa_p| - |Bf'(z)/z^{p-1} - Apa_p| < 0,$$

provided

$$\left|\sum_{n=1}^{\infty} -(n+p)|a_{n+p}| z^n\right| - \left|(B-A)\, pa_p - B\sum_{n=1}^{\infty} (n+p)|a_{n+p}| z^n\right| < 0.$$

For $|z| = r < 1$ the left side of the above inequality is bounded above by

$$\sum_{n=1}^{\infty} |a_{n+p}|(n+p)r^n + B\sum_{n=1}^{\infty} |a_{n+p}|(n+p)r^n - (B-A)\, a_p p$$
$$= \sum_{n=1}^{\infty} |a_{n+p}|(n+p)(1+B)r^n - (B-A)\, a_p p$$

$$<\sum_{n=1}^{\infty}|a_{n+p}|\ (n+p)(1+B)-(B-A)a_p\ p\leqslant 0.$$

Hence $f\in Q_p[A,\ B]$. Conversely, suppose that

$$\left|\frac{\dfrac{f'(z)}{z^{p-1}}-pa_p}{\dfrac{Bf'(z)}{z^{p-1}}-Apa_p}\right|=\left|\frac{\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z^n}{(B-A)pa_p-\sum_{n=1}^{\infty}B(n+p)|a_{n+p}|z^n}\right|<1,\ z\in E.$$

Since $\operatorname{Re} z\leqslant|z|$ for all z, we have

(2) $$\operatorname{Re}\Big[\Big(\sum_{n=1}^{\infty}(n+p)\ |a_{n+p}|z^n\Big)\Big/\Big((B-A)pa_p-\sum_{n=1}^{\infty}B(n+p)|a_{n+p}|z^n\Big)\Big]<1.$$

Choose values of z on the real axis so that $f'(z)/z^{p-1}$ is real. Upon clearing the denominator of (2) and letting $z\to 1$ through real values we get

$$\sum_{n=1}^{\infty}(n+p)\ |\ a_{n+p}\ |\leqslant\frac{p(B-A)a_p}{1+B}.$$

Theorem 2. *A function* $f(z)=a_pz^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ $(a_p>0)$, $f\in T_p(z_0)$ *is in*

$R_p(A,\ B,\ z_0)$ *if and only if* $\displaystyle\sum_{n=1}^{\infty}\frac{n+p}{p}\left[\frac{1+B}{B-A}-z_0^n\right]|a_{n+p}|\leqslant 1.$

Proof. Since $\displaystyle\frac{f'(z_0)}{pz_0^{p-1}}=1=a_p-\sum_{n=1}^{\infty}\frac{n+p}{p}|a_{n+p}|z_0^n,$

the result follows by substituting

(3) $$a_p=1+\sum_{n=1}^{\infty}\frac{n+p}{p}\ |a_{n+p}|z_0^n$$

in the statement of theorem 1.

Corollary. If $f(z)\in R_p(A,\ B,\ z_0)$ *then*

$$|a_{n+p}|\leqslant\frac{p(B-A)}{(n+p)[(1+B)-(B-A)z_0^n]},$$

with equality for the functions of the form

$$f(z)=\frac{(n+p)\ (1+B)\ \ z^p-p(B-A)z^{n+p}}{(n+p)[(1+B)-(B-A)\ \ z_0^n]}.$$

3. Distortion theorem.

Theorem 3. If $f(z)=a_pz^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ is in $R_p(A,\ B,\ z_0)$

then

(4) $$|f(z)|\leqslant r^p\{(1+p)[(1+B)-(B-A)z_0]\}^{-1}[(1+p)(1+B)+rp(B-A)]$$

and

(5) $$|f'(z)|\leqslant pr^{p-1}[(1+B)-(B-A)z_0]^{-1}[(1+B)+r(B-A)].$$

Proof: From theorem 2, we have

$$\sum_{n=1}^{\infty}|a_{n+p}|\leqslant p(B-A)\ \{(1+p)[1+B-(B-A)z_0]\}^{-1}$$

and

$$\sum_{n=1}^{\infty}(n+p)|a_{n+p}|\leqslant\ p(B-A)/[(1+B)-(B-A)z_0].$$

Further we have from (3)

$$a_p=1+\sum_{n=1}^{\infty}\frac{n+p}{p}\ |a_{n+p}|z_0^n\leqslant(1+B)/[(1+B)-(B-A)z_0].$$

Therefore,

$$|f(z)|\leqslant a_pr^p+r^{p+1}\sum_{n=1}^{\infty}|a_{n+p}|$$

$$\leqslant r^p\{(1+B)/[(1+B)-(B-A)z_0]+rp\ (B-A)/[(1+p)[(1+B)-(B-A)z_0\}.$$

That is,

$$|f(z)|\leqslant r^p\{(1+p)[(1+B)-(B-A)z_0]\}^{-1}[(1+p)(1+B)+rp(B-A)].$$

Hence, (4) follows. Further,

$$|f'(z)|\leqslant pa_pr^{p-1}+r^p\sum_{n=1}^{\infty}(n+p)|a_{n+p}|.$$

Also from theorem 2, we have

$\displaystyle\sum_{n=1}^{\infty}(n+p)|a_{n+p}|\leqslant p(B-A)/[(1+B)-(B-A)z_0]$ and hence (5) follows.

The bounds are sharp since the equalities are attained for the function

$$f(z)=[(1+p)(1+B)z^p-p(B-A)z^{1+p}]/\{(1+p)[(1+B)-(B-A)z_0]\}.$$

4. Radius of convexity. Let $f(z)=a_pz^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ be analytic and p-valent in $E=\{z:|z|<1\}$. Then $f(z)$ is said to be p-valently convex if and only if $\operatorname{Re}(1+zf''(z)/f'(z))>0$ for $z\in E$.

Theorem 4. *If* $f(z)\in R_p(A,\ B,\ z_0)$ *then* $f(z)$ *is p-valently convex in the disk*

$$|z|<R_p=\inf_n\{(1+B)p/[(n+p)(B-A)]\}^{1/n}.$$

Proof. It is sufficient to show that

$$|1+zf''(z)/f'(z)-p|<p\ \text{ for }\ |z|<R_p.$$

Now,

$$|1+zf''(z)/f'(z)-p|\leqslant\sum_{n=1}^{\infty}n(n+p)|a_{n+p}||z|^n/[pa_p-\sum_{n=1}(n+p)|a_{n+p}|z|^n].$$

Therefore,

$$|1+zf''(z)/f'(z)-p|<p \text{ if}$$

$$\sum_{n=1}^{\infty} n(n+p)|a_{n+p}||z|^n<p\left[p+\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z_0^n-\sum_{n=1}^{\infty}(n+p)|a_{n+p}||z|^n\right]$$

and

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)^2|z|^n-\left(\frac{n+p}{p}\right)z_0^n\right]|a_{n+p}|<1. \tag{6}$$

From Theorem 2, we have

$$\sum_{n=1}^{\infty}\frac{n+p}{p}\left(\frac{1+B}{B-A}-z_0^n\right)|a_{n+p}|\leqslant 1.$$

Hence, (6) will be true if

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)^2|z|^n-\frac{n+p}{p}z_0^n\right]|a_{n+p}|<\sum_{n=1}^{\infty}\frac{n+p}{p}\left(\frac{1+B}{B-A}-z_0^n\right)|a_{n+p}|$$

or

$$((n+p)/p)^2|z|^n-((n+p)/p)z_0^n<(n+p)(1+B)/[p\ (B-A)]-(n+p)p\ z_0^n. \tag{7}$$

Solving (7) for $|z|$ we get

$$|z|<\{(1+B)p/[(n+p)(B-A)]\}^{1/n} \quad (n=1,\ 2,\ 3\ \ldots)$$

and hence the result follows.

5. Closure theorems.

Theorem 5. *If* $f(z)=a_pz^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ *and* $g(z)=b_pz^p-\sum_{n=1}^{\infty}|b_{n+p}|z^{n+p}$ *are in* $R_p(A,\ B,\ z_0)$ *then* $h(z)=c_pz^p-1/2\sum_{n=1}^{\infty}(|a_{n+p}|+|b_{n+p}|)z^{n+p}$, $c_p=(a_p+b_p)/2$ *is also in* $R_p(A,\ B,\ z_0)$.

Proof. Since $f(z)$ and $g(z)$ are in $R_p(A,\ B,\ z_0)$ we have,

$$\sum_{n=1}^{\infty}(n+p)[1+B-(B-A)\ z_0^n]\ |a_{n+p}|\leqslant p\ (B-A). \tag{8}$$

and

$$\sum_{n=1}^{\infty}(n+p)[1+B-(B-A)z_0^n]\ |b_{n+p}|\leqslant p\ (B-A) \tag{9}$$

From (8) and (9) we get

$$\frac{1}{2}\sum_{n=1}^{\infty}(n+p)[1+B-(B-A)z_0^n][|a_{n+p}|+|b_{n+p}|]\leqslant p\ (B-A)$$

which implies that $h(z)\in R_p(A,\ B,\ z_0)$.

Theorem 6. *If* $f_p(z)=z^p$,

$$f_{n+p}(z)=(1+B)z^p[1+B-(B-A)z_0^n]^{-1}-(B-A)pz^{n+p}\{(n+p)\ [1+B-(B-A)z_0^n]\}^{-1}$$

$(n=1,\ 2,\ 3,\ldots)$ *then* $f(z)\in R_p(A,\ B,\ z_0)$ *if and only if it can be expressed in the form*

$$f(z)=\sum_{n=0}^{\infty}\lambda_{n+p}f_{n+p}(z),\ \textit{where}\ \lambda_{n+p}\geqslant 0\ \text{and}\ \sum_{n=0}^{\infty}\lambda_{n+p}=1.$$

Proof. Let $f(z)=\sum_{n=0}^{\infty}\lambda_{n+p}f_{n+p}(z)$ where $\lambda_{n+p}\geqslant 0$ and $\sum_{n=0}^{\infty}\lambda_{n+p}=1$.

Then

$$f(z)=\lambda_p+\sum_{n=1}^{\infty}\frac{(1+B)\lambda_{n+p}}{1+B-(B-A)z_0^n}z^p-\sum_{n=1}^{\infty}\frac{p\ (B-A)\lambda_{n+p}\ z^{n+p}}{(n+p)[1+B-(B-A)z_0^n]}.$$

Note that

$$f'(z_0)/(pz_0^{p-1})=\sum_{n=0}^{\infty}\lambda_{n+p}f'_{n+p}\ (z_0)/(pz_0^{p-1})=1\ ;$$

we also have

$$\sum_{n=1}^{\infty}\lambda_{n+p}\frac{p(B-A)}{(n+p)[1+B-(B-A)z_0^n]}\ \frac{(n+p)[1+B-(B-A)z_0^n]}{p(B-A)}=$$

$$=\sum_{n=1}^{\infty}\lambda_{n+p}=1-\lambda_p\leqslant 1.$$

Hence, from theorem 2, $f(z)\in R_p(A,\ B,\ z_0)$. Conversely, let $f(z)\in R_p\ (A,\ B,\ z_0)$ Then

$$|a_{n+p}|\leqslant p(B-A)/\{(n+p)[1+B-(B-A)z_0^n]\}.$$

We may set

$$\lambda_{n+p}=(n+p)[1+B-(B-A)z_0^n]\ [p(B-A)]^{-1}\ |a_{n+p}|\quad (n=1,\ 2,\ 3,\ldots)$$

and $\lambda_p=1-\sum_{n=1}^{\infty}\lambda_{n+p}$. Then on account of (3)

$$f(z)=\sum_{n=0}^{\infty}\lambda_{n+p}f_{n+p}(z).$$

Remark. Putting $p=1$, and taking $A=2\alpha-1$, $B=\beta=1$, where $0\leqslant\alpha<1$ in the above theorems, we get the results obtained by the same authors in [2].

The authors are greatful for the suggestions of the referee.

REFERENCE

1. Pommerenke C. — *Über die subordination analytischer Funktionen.* J. für reine. U. angew. Math. 218 (1965) 159—173.
2. Sarangi S. M. and Krishnamurthy M. R. — *A subclass of univalent functions with negative coefficients.* Nepall Mathematical Science Report, Vol. 12, No. 2, 1987.
3. Silverman H. — *Univalent functions with negative coefficients.* Proc. Amer. Math. Soc. 51 (1975), 109—116.

Received 22.VII.1987
Revised 10.V.1988

Department of Mathematics
Karnatak University
Dharwad, India

Department of Mathematics
University of Agricultural Sciences
Dharwad, India

An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

A POSTSCRIPT TO „C-BINARY RELATIONS AND PAIRWISE NORMALITY"

BY

ANGEL GUTIERREZ and SALVADOR ROMAGUERA

We shall use the notation and terminology of [1]. Throughout this note, when we are concerned to T_i and T_j for a bitopological space (X, T_1, T_2), we suppose $i \neq j$ and $i, j = 1, 2$.

In a recent paper [1, theorem 2] we prove an insertion and extension theorem for pairwise normal spaces. From this theorem we derive, among other results, the following [1, corollary 2.2]: Let C be a T_1-closed and T_2-closed set in the pairwise normal space (X, T_1, T_2). Let Φ be a upper (lower) bounded function on C T_i-l. s.c. and T_j-u.s.c. on C. Then there is a real-valued function h on X T_i-l.s.c. and T_j-u.s.c. on X such that $h(x)=\Phi(x)$ for every $x \in C$.

Unfortunately, this result is not correct as we prove in the next variant of a Lane's example.

Example. Let X be the set of real numbers. Let T_1 be the discrete topology on X and let T_2 consist of the void set and complements of countable sets. (X, T_1, T_2) is a pairwise normal space. Let C be the set of all negative integer numbers and let the function Φ on C defined by $\Phi(x)=x$ for every $x \in C$. We have that C and Φ verify the hypothesis of [1, corollary 2.2] but Φ does not admit an extension T_1-l.s.c. and T_2-u.s.c. to X since X is uncountable.

In [2, proposition 11] Lane proves an extension theorem for T_i and T_j-zero sets. By using a similar technique to that in Lane's result we obtain the following modification of corollary 2.2 in [1].

Theorem. *Let C be a T_i-zero set and T_j-closed set in the pairwise normal space (X, T_1, T_2). Let Φ be a upper (lower) bounded function on C, T_i-l.s.c. and T_j-u.s.c. (T_j-l.s.c. and T_i-u.s.c.) on C. Then there is a real-valued function h on X T_i-l.s.c. and T_j-u.s.c. (T_j-l.s.c. and T_i-u.s.c.) on X such that $h(x)=\Phi(x)$ for every $x \in C$.*

Proof. Let Φ be a upper bounded function on the T_i-zero set and T_j-closed set C. Then C is T_i-closed and T_j-closed. By hypothesis, Φ is T_i-l.s.c. and T_j-u.s.c. on C. Let $\psi=\Phi/(1+|\Phi|)$. By [2, theorem 2.6] there exists a T_i-l.s.c. and T_j-u.s.c. function F on X such that $-1 \leqslant F \leqslant 1$ and $F(x)=\psi(x)$ for every $x \in C$. Since Φ is upper bounded on C it is clear that there exists a constant $\alpha<1$ satisfying $-1<\psi(x) \leqslant \alpha$ for every $x \in C$. Hence, $-1 \leqslant F(x) \leqslant \alpha<1$